Wang et al. *Journal of Inequalities and Applications* 2014, 2014:472
http://www.journalofinequalitiesandapplications.com/content/2014/1/472

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Singularity analysis for a semilinear integro-differential equation with nonlinear memory boundary

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Abstract

In this paper, we study the blow-up singularity of a semilinear parabolic equation with nonlinear memory both in the reaction term and the boundary condition. We firstly establish the local solvability for a large class of semilinear parabolic equations with various nonlocal reaction terms. Secondly, we give a complete classification for the existence of a blow-up solution and a global solution. Next, we show that under some hypotheses the blow-up can only occur on the boundary of the domain.

MSC: 35K05; 35B44

Keywords: blow-up in finite time; global existence; nonlinear memory term

1 Introduction

In this paper, we devote our attention to the singularity analysis for the semilinear equation with nonlinear memory both in the reaction term and the boundary,

$$\begin{cases} u_t = \Delta u + u^{q_1} \int_0^t u^{p_1}(x,s)\,ds, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = u^{q_2} \int_0^t u^{p_2}(x,s)\,ds, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $p_i \geq 0$, $q_i \geq 0$ $(i = 1,2)$, Ω is a bounded domain in R^N having piecewise smooth boundary $\partial\Omega$ with outward pointing unit normal ν. The initial data $u_0(x)$ is a nontrivial, nonnegative, and continuous function on $\overline{\Omega}$.

Models involving memory terms in reaction have arisen in studies of nuclear reactor dynamics [1, 2] and population dynamics [3], specifically in the case of logistic growth models involving both nondelayed and hereditary effects [4, 5]. Solvability, stabilization, and blow-up in finite time of solutions for a variety of generalizations of such models have subsequently been investigated in a number of previous works, *e.g.*, [6–11]. Particularly, the blow-up properties of semilinear parabolic equation involving memory terms in a reaction,

$$u_t = \Delta u + u^q \int_0^t u^p(x,s)\,ds, \tag{1.2}$$

coupled with a zero Dirichlet boundary condition has been completely studied (see [7–10]). Among other things, the authors obtained the following result: (1) Assume $p + q > 1$. If $q \geq 1$, then u blows up in finite time for sufficiently large u_0, and u exists globally for sufficiently small u_0. If $q \leq 1$ then u blows up in finite time for any nonnegative nontrivial u_0. (2) If $p+q \leq 1$, then u exists globally for any nonnegative u_0. Meanwhile, the authors obtained the blow-up rate in the case of $q = 0$, $p > 1$ in [8]. Furthermore, some authors extended the above works for the semilinear case (1.2) to degenerate reaction-diffusion equations involving a nonlinear memory term and obtained a corresponding blow-up analysis (see, for example, [12–15]).

Memory terms in diffusion have been studied as well, arising in models of viscoelastic forces in non-Newtonian fluids [16, 17] and resulting from a modified Fourier law applied to anisotropic, nonhomogeneous media [18].

Despite the volume of work done on models incorporating memory in reaction, diffusion, or both, there appear to be very few appearances in the literature of diffusion models in which such terms are present in the boundary flux.

Recently, Deng *et al.* did some good work on the models with flux at the boundary governed by a nonlinear memory law in [19, 20]. Particularly, the authors studied the following model that has been formulated for capillary growth in solid tumors as initiated by angiogenic growth factors in [19]:

$$\begin{cases} u_t = \Delta u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = u^q \int_0^t u^p(x,s)\,ds, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.3}$$

Their primary result is that if $0 \leq p + q \leq 1$, then every solution of (1.3) is global. On the other hand, if $p + q > 1$, then all nonnegative, nontrivial solutions blow up in finite time. Besides this, the authors proved that if $p > 1$, $q = 0$ or $p \geq 0$, $q > 1$, blow-up can occur only on the boundary.

For more related works with nonlinear memory, we refer the reader to [6, 10, 21–23] and the references therein.

Motivated by above works, we investigate the blow-up properties of problem (1.1) in this paper. According to the aforementioned works, one may expect that the blow-up result of (1.1) is a combination of (1.2) and (1.3) to some extent. In fact, we shall prove that if $p_1 + q_1 \leq 1$ and $p_2 + q_2 \leq 1$, solution exists globally for any nonnegative u_0. We also find that if $p_1 + q_1 > 1$ or $p_2 + q_2 > 1$, a blow-up singularity occurs and all nonnegative nontrivial solutions blow up in finite time. We notice that the main idea as regards the time-integral nonlocal problems is that only when the time is large, the time-integral term dominates the evolution of the solutions. Therefore, for problem (1.2), a solution still maybe exists globally even when $p + q > 1$. For our problem (1.1), however, if $p_1 + q_1 > 1$ or $p_2 + q_2 > 1$, there is no global solution. Thus, one can see that the nonlinear memory boundary plays an important role in accelerating the occurrence of a blow-up singularity.

The remaining part of this paper is organized as follows. In Section 2, we prove the local solvability of a wide class of integro-differential equations of the parabolic type involving nonlinear memory terms, which include (1.1). In Section 3, we give the comparison principle which will be used later for nonnegative solutions to (1.1). In Section 4, we establish the global existence and finite time blow-up result. In the last section, we shall investigate the blow-up set. We will prove that the blow-up may only occur on the boundary of the domain in some cases.

2 Local solvability

In this section we derive the local solvability for a large class of semilinear parabolic equations with various nonlocal reaction terms and memory boundaries, which include (1.1). Furthermore, we give the local existence theorem for (1.1), where the nonlinearity is merely locally *Hölder* continuous.

Theorem 2.1 *Assume that F, f, H, and h are all locally Lipschitz continuous functions. Consider the following problem*:

$$\begin{cases} u_t = \Delta u + F(u, \int_0^t f(u(\cdot,s))\,ds), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = H(u, \int_0^t h(u(\cdot,s))\,ds), & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

For every $u_0 \in C(\overline{\Omega})$, then there exists a $T > 0$ such that the problem has a unique classical solution $u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T))$.

Proof Set $\Sigma \equiv \{u \in L^\infty(\Omega \times (0,T)); \|u\|_{L^\infty(\Omega)} \le M\}$, where $M > M_0 \equiv \sup_{\overline{\Omega}} |u_0|$.

Given $u \in L^\infty(\Omega \times (0,T))$, define

$$\begin{aligned} \Phi[u](x,t) \equiv{}& \int_\Omega G_N(x,y,t,0)u_0(y)\,dy + \int_0^t \int_\Omega G_N(x,y,t,\tau) F\left(u, \int_0^\tau f\big(u(\cdot,s)\big)\,ds\right) dy\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} G_N(x,y,t,\tau) H\left(u, \int_0^\tau h\big(u(\cdot,s)\big)\,ds\right) dS_y\,d\tau. \end{aligned}$$

Here $G_N(x,y,t,\tau)$ is the Green's function for the heat equation with homogeneous Neumann boundary condition. (We refer the readers to [24, 25] and [26] for its construction and properties.) As F, f, H, and h are all locally Lipschitz continuous functions, we may assume that, for all $M > 0$, there exists $L = L(M)$, $l = l(M)$, $K = K(M)$, $k = k(M)$ such that, for all $a_1, a_2, b_1, b_2 \in R$ with $|a_2|, |a_1|, |b_1|, |b_2| \le M$, the functions F, f satisfy

$$\big|F(a_1,b_1) - F(a_2,b_2)\big| \le L\big(|a_1 - a_2| + |b_1 - b_2|\big), \qquad \big|f(a_1) - f(a_2)\big| \le l\big(|a_1 - a_2|\big),$$

and H, h satisfy

$$\big|H(a_1,b_1) - H(a_2,b_2)\big| \le K\big(|a_1 - a_2| + |b_1 - b_2|\big), \qquad \big|h(a_1) - h(a_2)\big| \le k\big(|a_1 - a_2|\big).$$

Noticing that $\int_\Omega G_N(x,y,t,0)\,dy = 1$ and $\sup_{x \in \overline{\Omega}, 0 \le \tau \le t} \int_0^\tau \int_{\partial\Omega} G_N(x,y,\tau,\eta)\,dS_y\,d\eta \le C_0 t^{1/2}$ for constants $\varepsilon_0, C_0 > 0$ with $t < \varepsilon_0$ [26], we have

$$\begin{aligned} \big\|\Phi[u]\big\|_{L^\infty(\Omega)} \le{}& M_0 + C_1 \int_0^t \left(F(0,0) + L\left(\|u\| + \int_0^\tau \big(f(0) + l\|u\|\big)\,d\sigma\right)\right) d\tau \\ &+ C_0 t^{1/2} \left(H(0,0) + K\left(\|u\| + \int_0^\tau \big(h(0) + k\|u\|\big)\,d\sigma\right)\right) \end{aligned}$$

$$\begin{aligned}&\le M_0 + C_1 T\bigl(F(0,0) + LM + L\bigl(f(0) + lM\bigr)T\bigr)\\&\quad + C_0 T^{1/2}\bigl(H(0,0) + KM + K\bigl(h(0) + kM\bigr)T\bigr),\end{aligned}$$

which implies $\|\Phi[u]\|_{L^\infty(\Omega)} \le M$ if T is small enough.

So, Φ is a mapping from Σ into itself.

Similarly, we obtain

$$\begin{aligned}\bigl\|\Phi[u] - \Phi[v]\bigr\|_{L^\infty(\Omega)} &\le \left\|\int_0^t \int_\Omega G_N(x,y,t,\tau)\left(F\left(u, \int_0^\tau f\bigl(u(\cdot,s)\bigr)\,ds\right)\right.\right.\\&\quad \left.\left.- F\left(v, \int_0^\tau f\bigl(v(\cdot,s)\bigr)\,ds\right)\right)dy\,d\tau\right\|\\&\quad + \left\|\int_0^t \int_{\partial\Omega} G_N(x,y,t,\tau)\left(H\left(u, \int_0^\tau h\bigl(u(\cdot,s)\bigr)\,ds\right)\right.\right.\\&\quad \left.\left.- H\left(v, \int_0^\tau h\bigl(v(\cdot,s)\bigr)\,ds\right)\right)dS_y\,d\tau\right\|\\&\le \bigl(TL(1 + Tl) + KC_0 T^{1/2}(1 + Tk)\bigr)\|u - v\|.\end{aligned}$$

Therefore, if $T > 0$ is small enough, then we see that Φ is a strict contraction. Thus, Φ has a unique fixed point by Banach's fixed point theorem. This implies that, for any $u_0 \in C(\overline{\Omega})$, there exists a unique local solution $u \in L^\infty((0,T) \times \Omega)$ in the integral sense for $T > 0$ small enough.

Concerning the regularity, we can see that the corresponding solution u is automatically in $C^{2,1}(\overline{\Omega} \times (0,T))$ from the standard bootstrap argument. On the other hand, the continuity of the solution $u \in C(\overline{\Omega} \times [0,T])$ follows from (1.1) itself (see [27] for details). □

Of course, when $\min\{p_1, q_1, p_2, q_2\} < 1$ in problem (1.1), the above local well-posedness does not apply to (1.1). However, for problem (1.1), we still have the following local existence theorem.

Theorem 2.2 *For every nonnegative nontrivial* $u_0 \in C(\overline{\Omega})$, *there exists a* $T > 0$ *such that problem* (1.1) *has a unique nonnegative classical solution* $u \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T))$.

Proof We may only give the proof for the case that $p_i, q_i < 1$ ($i = 1,2$). Let

$$f_{in}(z) = \begin{cases} z^{p_i}, & z > \frac{1}{n},\\ (p_i n^{1-p_i} z + \frac{1-p_i}{n^{p_i}})_+, & \text{otherwise},\end{cases}$$

and, similarly,

$$g_{in}(z) = \begin{cases} z^{q_i}, & z > \frac{1}{n},\\ (q_i n^{1-q_i} z + \frac{1-q_i}{n^{q_i}})_+, & \text{otherwise}.\end{cases}$$

Note that $f_i(n)$, $g_i(n)$ ($i = 1,2$) are monotone decreasing with respect to n and

$$f_{in}(z) \to \begin{cases} z^{p_i}, & z > 0,\\ 0, & z \le 0,\end{cases} \qquad g_{in}(z) \to \begin{cases} z^{q_i}, & z > 0,\\ 0, & z \le 0,\end{cases} \qquad \text{as } n \to \infty.$$

For any fixed n, $f_i(n)$, $g_i(n)$ $(i = 1,2)$ are non-decreasing, locally Lipschitz functions of z.

Let (u_n) be a sequence of solutions such that

$$\begin{cases} (u_n)_t = \Delta u_n + g_{1n}(u_n)\int_0^t f_{1n}(u_n)(s)\,ds, & x\in\Omega, t>0,\\ \frac{\partial u_n}{\partial\nu} = g_{2n}(u_n)\int_0^t f_{2n}(u_n)(s)\,ds, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{2.2}$$

Then we have a unique classical approximated solution u_n by Theorem 2.1. Since $f_{in}(0)$, $g_{in}(0)\ge 0$, by the maximum principle we know that $u_n \ge 0$, and by the comparison theorem (see Lemma 3.3) we see that u_n is monotone decreasing. Hence, there exists a bounded nonnegative function $u = \lim_{n\to\infty} u_n$, which corresponds to the continuous solution of (1.1). On the other hand, we get the additional regularity of u from the standard argument. When u_0 is nontrivial, the uniqueness follows from the strong maximum principle. □

3 Comparison principle

In order to use the super-sub-solution technique, we next introduce the definition of the super- and the sub-solution and the comparison principle for problem (1.1).

Definition 3.1 A function $\overline{u}$ is called the super-solution of problem (1.1) if $\overline{u}(x,t)\in C^{2,1}(\overline{\Omega}\times[0,T))$ and satisfies

$$\begin{cases} \overline{u}_t \ge \Delta\overline{u} + \overline{u}^{q_1}\int_0^t \overline{u}^{p_1}(x,s)\,ds, & x\in\Omega, t>0,\\ \frac{\partial\overline{u}}{\partial\nu} \ge \overline{u}^{q_2}\int_0^t \overline{u}^{p_2}(x,s)\,ds, & x\in\partial\Omega, t>0,\\ \overline{u}(x,0)\ge u_0(x), & x\in\overline{\Omega}. \end{cases} \tag{3.1}$$

Similarly, we can obtain the definition of sub-solution of problem (1.1) by all inequalities revised.

Lemma 3.1 *Suppose that $w(x,t)\in C^{2,1}(\Omega\times(0,T))\cap C(\overline{\Omega}\times[0,T])$ and satisfies*

$$\begin{cases} w_t \ge \Delta w + a_1 w + a_2\int_0^t a_3(x,s)w(x,s)\,ds, & x\in\Omega, t>0,\\ \frac{\partial w}{\partial\nu} \ge a_4 w + a_5\int_0^t a_6(x,s)w(x,s)\,ds, & x\in\partial\Omega, t>0,\\ w(x,0)\ge 0, & x\in\overline{\Omega}, \end{cases} \tag{3.2}$$

where $a_i(x,t)$ $(i=1,2,\dots,6)$ are bounded functions and $a_i\ge 0$ $(i=2,3,5,6)$. Then $w\ge 0$.

Proof Let $\xi(x)$ be a positive smooth function and satisfies $\frac{\partial\xi}{\partial\nu}\ge\alpha\xi$, $\alpha>0$ is a constant to be determined later. Let $W(x,t)=e^{-\lambda t}\xi(x)w(x,t)$, where $\lambda>0$ is a constant to be determined. Then, if we choose $\alpha > \|a_4\|_\infty + T\|a_5\|_\infty\|a_6\|_\infty$, $\lambda > \|\frac{\Delta\xi}{\xi}\|_\infty + \|a_1\|_\infty + T\|a_2\|_\infty\|a_3\|_\infty$, then

$$\begin{cases} W_t \ge \Delta W + \frac{2\nabla\xi}{\xi}\cdot\nabla W + (\frac{\Delta\xi}{\xi}+a_1-\lambda)W \\ \qquad + a_2 e^{-\lambda t}\int_0^t a_3 e^{\lambda s}W(x,s)\,ds, & x\in\Omega, t>0,\\ \frac{\partial W}{\partial\nu} \ge (a_4 - \frac{1}{\xi}\cdot\frac{\partial\xi}{\partial\nu})W + a_5 e^{-\lambda t}\int_0^t a_6 e^{\lambda s}W(x,s)\,ds, & x\in\partial\Omega, t>0,\\ W(x,0)\ge 0, & x\in\overline{\Omega}. \end{cases} \tag{3.3}$$

Suppose that $W(x,t)$ attains negative minimum at (x_0,t_0). If $x_0 \in \Omega$, then $W_t \le 0$, $\Delta W \ge 0$, $\nabla W = 0$ at (x_0,t_0). On the other hand, we know from the first inequality of (3.3) that

$$W_t \ge \Delta W + \frac{2\nabla\xi}{\xi}\cdot\nabla W + \left(\frac{\Delta\xi}{\xi} + a_1 + Ta_2a_3 - \lambda\right)W(x_0,t_0) > 0, \quad x\in\Omega, t>0.$$

This is a contradiction.

If $x_0 \in \partial\Omega$, then at (x_0,t_0),

$$\frac{\partial W}{\partial\nu} \ge -W(x_0,t_0)\left(\frac{1}{\xi}\cdot\frac{\partial\xi}{\partial\nu} - a_4 - a_5e^{-\lambda t}\int_0^t a_6e^{\lambda s}\,ds\right) > 0,$$

which contradict to $\frac{\partial W}{\partial\nu}|_{(x_0,t_0)} \le 0$. Therefore, for any $(x,t)\in\overline{\Omega}\times[0,T]$, we have $W(x,t) \ge 0$. The same for $w(x,t)$. □

Lemma 3.2 *Suppose that* $p_i \ge 1$, $q_i \ge 1$ $(i=1,2)$. *If* $\overline{u}$ *and* $\underline{u}$ *are the nonnegative super-solution and sub-solution of* (1.1), *respectively, then* $\overline{u} \ge \underline{u}$ *in* $\overline{\Omega}\times[0,T)$.

Proof Let $w = \overline{u} - \underline{u}$. It is easy to verify that w satisfies (3.2), where a_i $(i=1,2,\ldots,6)$ are such that

$$\begin{aligned}
&\left[\overline{u}^{q_1}(x,t) - \underline{u}^{q_1}(x,t)\right]\int_0^t \overline{u}^{p_1}(x,s)\,ds \equiv a_1(x,t)\left[\overline{u}(x,t)-\underline{u}(x,t)\right],\\
&a_2(x,t) \equiv \overline{u}^{q_1}(x,t),\\
&\overline{u}^{p_1}(x,t) - \underline{u}^{p_1}(x,t) \equiv a_3(x,t)\left[\overline{u}(x,t)-\underline{u}(x,t)\right],\\
&\left[\overline{u}^{q_2}(x,t) - \underline{u}^{q_2}(x,t)\right]\int_0^t \overline{u}^{p_2}(x,s)\,ds \equiv a_4(x,t)\left[\overline{u}(x,t)-\underline{u}(x,t)\right],\\
&a_5(x,t) \equiv \overline{u}^{q_2}(x,t),\\
&\overline{u}^{p_2}(x,t) - \underline{u}^{p_2}(x,t) \equiv a_6(x,t)\left[\overline{u}(x,t)-\underline{u}(x,t)\right].
\end{aligned}$$

From Lemma 3.1, we know $\overline{u} \ge \underline{u}$. □

Remark 3.1 From the above proof, we know when $p_i < 1$ or $q_i < 1$ $(i=1,2)$ and there exists some $\delta > 0$ such that $\overline{u} \ge \delta > 0$, $\underline{u} \ge 0$, the functions a_i $(i=1,2,\ldots,6)$ are still bounded. Therefore the conclusion of the lemma is valid in this case.

Using Lemma 3.1, we could obtain another version of the comparison theorem, which is useful in the proof of the local existence of the solution.

Lemma 3.3 *Let* f_i, g_i $(i=1,2)$ *be non-decreasing locally Lipschitz functions. Suppose that* $u,v \in C^{2,1}(\Omega\times(0,T))\cap C(\overline{\Omega}\times[0,T])$ *such that*

$$\begin{cases}
u_t - \Delta u - g_1(u)\int_0^t f_1(u)(s)\,ds \ge v_t - \Delta v - g_1(v)\int_0^t f_1(v)(s)\,ds \ge 0, & x\in\Omega, t>0,\\
\frac{\partial u}{\partial\nu} - g_2(u)\int_0^t f_2(u)(s)\,ds \ge \frac{\partial v}{\partial\nu} - g_2(v)\int_0^t f_2(v)(s)\,ds \ge 0, & x\in\partial\Omega, t>0,\\
u(x,0) = u_0(x) \ge v(x,0) = v_0(x) \ge 0, & x\in\Omega.
\end{cases}$$

Then $u(x,t) \ge v(x,t) \ge 0$ *in* $\overline{\Omega}\times[0,T)$.

Proof The proof is similar to that of Theorem 3.3 in [7] by using Lemma 3.1. We only give a sketch of the proof.

We only need to prove $u(x,t) \geq v(x,t)$ in $\overline{\Omega} \times [0,T)$. Set $w(x,t) = u(x,t) - v(x,t)$. Let $a_i(x,t)$ $(i = 1,2,\ldots,6)$ be continuous functions defined by

$$a_1(x,t) = \int_0^t f_1(u)\,ds \times \begin{cases} \frac{g_1(u)-g_1(v)}{u-v}, & u \neq v, \\ g_1'(u), & u = v, \end{cases}$$

$$a_2(x,t) = g_1(v),$$

$$a_3(x,t) = \begin{cases} \frac{f_1(u)-f_1(v)}{u-v}, & u \neq v, \\ f_1'(u), & u = v, \end{cases}$$

$$a_4(x,t) = \int_0^t f_2(u)\,ds \times \begin{cases} \frac{g_2(u)-g_2(v)}{u-v}, & u \neq v, \\ g_2'(u), & u = v, \end{cases}$$

$$a_5(x,t) = g_2(v),$$

$$a_6(x,t) = \begin{cases} \frac{f_2(u)-f_2(v)}{u-v}, & u \neq v, \\ f_2'(u), & u = v. \end{cases}$$

Then $w(x,t)$ satisfies the condition of the lemma. Thus, $w(x,t) \geq 0$ in $\overline{\Omega} \times [0,T)$. □

4 Global existence and finite time blow-up

In this section, we shall determine when the solution of problem (1.1) exists globally or blows up in finite time. We first establish the global existence result.

Theorem 4.1 *If $p_1 + q_1 \leq 1$ and $p_2 + q_2 \leq 1$, then the solution of problem* (1.1) *exists globally for any nonnegative u_0.*

Proof We shall construct a super-solution of (1.1). From [28], we know that there exists a function $\varphi(x) \in C^2(\overline{\Omega})$ satisfying

$$0 < \varphi(x) \leq 1, \quad x \in \Omega, \qquad \nabla\varphi \cdot \nu \geq 1, \quad x \in \partial\Omega.$$

Denote $m_1 = \max_{\overline{\Omega}} |\nabla\varphi|$ and $m_2 = \max_{\overline{\Omega}} |\Delta\varphi|$. Define

$$\overline{u} = Me^{\delta t + \varphi},$$

where

$$M = \max\{\|u_0\|_{L^\infty(\Omega)}, 1\}, \qquad \delta = \max\{m_1^2 + m_2 + 1, 1/p_1, 1/p_2\}.$$

Direct calculations show that

$$\begin{aligned} &\overline{u}_t - \Delta\overline{u} - \overline{u}^{q_1} \int_0^t \overline{u}^{p_1}(x,s)\,ds \\ &\quad = Me^{\delta t+\varphi}\left[\delta - |\nabla\varphi|^2 - \Delta\varphi - M^{p_1+q_1-1}\left(e^{\delta t+\varphi}\right)^{q_1-1} \int_0^t \left(e^{\delta s+\varphi}\right)^{p_1} ds\right] \end{aligned}$$

$$\begin{aligned}
&\geq Me^{\delta t+\varphi}\big[\delta - m_1^2 - m_2 - \big(Me^{\delta t+\varphi}\big)^{p_1+q_1-1}/(\delta p_1)\big] \\
&\geq Me^{\delta t+\varphi}\big[\delta - m_1^2 - m_2 - 1/(\delta p_1)\big] \\
&\geq 0, \\
&\frac{\partial \overline{u}}{\partial \nu} - \overline{u}^{q_2}\int_0^t \overline{u}^{p_2}(x,s)\,ds \\
&= Me^{\delta t+\varphi}\left[\frac{\partial \varphi}{\partial \nu} - M^{p_2+q_2-1}\big(e^{\delta t+\varphi}\big)^{q_2-1}\int_0^t \big(e^{\delta s+\varphi}\big)^{p_2}\,ds\right] \\
&\geq Me^{\delta t+\varphi}\big[1 - \big(Me^{\delta t+\varphi}\big)^{p_2+q_2-1}/(\delta p_2)\big] \\
&\geq 0.
\end{aligned}$$

We notice that

$$\overline{u}(x,0) = Me^{\varphi} \geq M \geq u_0(x).$$

Therefore, $\overline{u}$ is a super-solution of problem (1.1). From the comparison principle, we know the solution of (1.1) exists globally. □

In the remainder of this section, we shall establish the finite time blow-up result of problem (1.1). We have the following theorem.

Theorem 4.2 *If $p_1 + q_1 > 1$ or $p_2 + q_2 > 1$, then all nonnegative solutions of* (1.1) *blow up in finite time*.

To prove this theorem, we need to consider first the following problem:

$$\begin{cases} u_t = \Delta u + c\int_0^t u^\lambda(x,s)\,ds, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{4.1}$$

Lemma 4.1 *For any positive constant c, if $\lambda > 1$, then all nonnegative solutions of* (4.1) *blow up in finite time*.

Proof From now on, we use c_i ($i = 1, 2, \ldots$) to denote various positive constants. Let $G_N(x,y,t,\tau)$ be the Green's function for the heat equation with homogeneous Neumann boundary condition. Then we have the following representation formulas for the solution of (4.1):

$$\begin{aligned} u(x,t) = {}& \int_\Omega G_N(x,y,t,0)u_0(y)\,dy \\ &+ c\int_0^t\int_\Omega G_N(x,y,t,\tau)\int_0^\tau u^\lambda(y,s)\,ds\,dy\,d\tau. \end{aligned} \tag{4.2}$$

As is well known, the Green's function G_N satisfies (see, *e.g.*, [20])

$$c_1 \leq \int_\Omega G_N(x,y,t,\tau)\,dx \leq c_2, \quad y \in \overline{\Omega}, 0 \leq \tau < t < T. \tag{4.3}$$

By (4.2), (4.3), and Jensen's inequality, we have

$$\begin{aligned}\int_\Omega u(x,t)\,dx &= \int_\Omega\int_\Omega G_N(x,y,t,0)u_0(y)\,dy\,dx \\ &\quad + c\int_\Omega\left(\int_0^t\int_\Omega G_N(x,y,t,\tau)\int_0^\tau u^\lambda(y,s)\,ds\,dy\,d\tau\right)dx \\ &\geq c_1\int_\Omega u_0(y)\,dy + cc_1\int_0^t\int_\Omega\int_0^\tau u^\lambda(y,s)\,ds\,dy\,d\tau \\ &\geq c_4 + c_3\int_0^t \tau^{1-\lambda}\left(\int_0^\tau\int_\Omega u(y,s)\,dy\,ds\right)^\lambda d\tau. \end{aligned} \tag{4.4}$$

Denote

$$F(t) = \int_0^t\int_\Omega u(x,\tau)\,dx\,d\tau, \quad t > 0.$$

Then it follows from (4.4) that

$$F'(t) \geq c_4 + c_3\int_0^t \tau^{1-\lambda}F^\lambda(\tau)\,d\tau, \quad t > 0.$$

Integrating this inequality from 0 to t, we obtain

$$\begin{aligned}F(t) &\geq c_4 t + c_3\int_0^t\int_0^\tau \zeta^{1-\lambda}F^\lambda(\zeta)\,d\zeta\,d\tau \\ &\geq c_4 t + c_3\int_0^t (t-\zeta)\zeta^{1-\lambda}F^\lambda(\zeta)\,d\zeta \\ &\geq c_4 t + c_3 t^{1-\lambda}\int_0^t (t-\zeta)F^\lambda(\zeta)\,d\zeta.\end{aligned}$$

Assume to the contrary that (4.1) has a global solution u. Then, for any $T > 0$, we have

$$F(t) \geq c_4 T + c_5 T^{1-\lambda}\int_T^t (t-\zeta)F^\lambda(\zeta)\,d\zeta, \quad T \leq t \leq 2T.$$

Thus, $F(t) \geq H(t)$ for any $T < t \leq 2T$, where

$$H(t) = c_4 T + c_5 T^{1-\lambda}\int_T^t (t-\zeta)H^\lambda(\zeta)\,d\zeta, \quad T \leq t \leq 2T.$$

$H(t)$ satisfies

$$\begin{cases} H''(t) = c_5 T^{1-\lambda}H^\lambda(t), & T < t < 2T, \\ H(T) = c_4 T, \qquad H'(T) = 0. \end{cases} \tag{4.5}$$

Multiplying the first equality in (4.5) by $H'(t)$ and integrating over (T,t), we obtain

$$H'(T) = c_6 T^{(1-\lambda)/2}\left[H^{\lambda+1}(t) - H^{\lambda+1}(T)\right]^{1/2}.$$

Integrating this equality again from T to $2T$, one can get

$$\begin{aligned}
c_6 T^{(3-\lambda)/2} &= \int_T^{2T} \bigl[H^{\lambda+1}(t) - H^{\lambda+1}(T)\bigr]^{-1/2} H'(t)\,dt \\
&= \int_{H(T)}^{H(2T)} \bigl[z^{\lambda+1} - H^{\lambda+1}(T)\bigr]^{-1/2}\,dz \\
&\leq (\lambda+1)^{-1/2} H^{-\lambda/2}(T) \int_{H(T)}^{2H(T)} \bigl[z - H(T)\bigr]^{-1/2}\,dz + 2^{(\lambda+1)/2} \int_{2H(T)}^{\infty} z^{-(\lambda+1)/2}\,dz \\
&= 2\bigl[(\lambda+1)^{-1/2} + 2(\lambda-1)^{-1}\bigr] c_4^{(1-\lambda)/2} T^{(1-\lambda)/2},
\end{aligned}$$

that is,

$$T \leq 2\bigl[(\lambda+1)^{-1/2} + 2(\lambda-1)^{-1}\bigr] c_4^{(1-\lambda)/2}/c_6. \tag{4.6}$$

This leads to a contradiction when T is large enough. Therefore, problem (4.1) has no global solution. □

Remark 4.1 We can also easily reach our conclusion by using the comparison principle. In fact, the solution of the corresponding Dirichlet problem,

$$\begin{cases}
u_t = \Delta u + c\int_0^t u^{\lambda}(x,s)\,ds, & x \in \Omega, t > 0, \\
u = 0, & x \in \partial\Omega, t > 0, \\
u(x,0) = u_0(x), & x \in \overline{\Omega},
\end{cases}$$

is the sub-solution of (4.1). We know from Theorem 5.1 of [8] and Theorem 3.3 of [29] that if $\lambda > 1$ the solution of this first initial boundary value problem blows up in finite time for any nonnegative nontrivial initial data. However, applying the representation formula of the Neumann problem (4.1), we gave a completely different approach here.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2 When $p_1 + q_1 > 1$, we first consider the following problem,

$$\begin{cases}
\phi_t = \Delta\phi + \phi^{q_1}\int_0^t \phi^{p_1}(x,s)\,ds, & x \in \Omega, t > 0, \\
\frac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega, t > 0, \\
\phi(x,0) = \phi_0(x), & x \in \overline{\Omega}.
\end{cases} \tag{4.7}$$

If $q_1 < 1$, we know $\phi(x,t) > 0$ ($x \in \overline{\Omega}$, $t > 0$) by the maximum principle. Let

$$z = M\phi^{1-q_1},$$

where $M = \|\phi_0\|_\infty^{q_1}$, then z satisfies

$$\begin{cases}
z_t \geq \Delta z + M^{(1-p_1-q_1)/(1-q_1)}(1-q_1)\int_0^t z^{p_1/(1-q_1)}(x,\tau)\,d\tau, & x \in \Omega, t > 0, \\
\frac{\partial z}{\partial\nu} = 0, & x \in \partial\Omega, t > 0, \\
z(x,0) \geq \phi_0(x), & x \in \overline{\Omega}.
\end{cases}$$

So z is a super-solution of problem (4.1). As $\frac{p_1}{1-q_1} > 1$, Lemma 4.1 shows that z blows up in finite time, so does ϕ.

Now we consider the case of $q_1 \geq 1$. If $p_1 = 0$, the solution of the following ODE:

$$v'(t) = t \cdot v^{q_1}, \quad t > 0,$$
$$v(0) = \min_{x\in\overline{\Omega}} \phi_0(x),$$

is the sub-solution of problem (4.7). The fact that $p_1 + q_1 > 1$ shows that $q_1 > 1$, which leads to the finite time blow-up of this ODE. So does the solution of (4.7).

If $q_1 \geq 1$ and $p_1 \neq 0$, for any $\mu > 0$ there exists a constant $c_\mu > 0$ such that the solution to (4.7) satisfies

$$\phi(x,t) \geq c_\mu, \quad x \in \partial\Omega, t \geq \mu > 0. \tag{4.8}$$

Here we used the fact that the solution of the heat equation with homogeneous Neumann boundary condition is a sub-solution of problem (4.7).

Let $q_1 = \alpha + \gamma$, where $\gamma < 1$ and $\gamma + p_1 > 1$, then we have

$$\phi^{q_1} = \phi^{\alpha+\gamma} \geq c_\mu^\alpha \phi^\gamma, \quad x \in \partial\Omega, t \geq \mu > 0.$$

Next we consider the following problem:

$$\begin{cases} \underline{u}_t = \Delta\underline{u} + c_\mu^\alpha \underline{u}^\gamma \int_\mu^t \underline{u}^{p_1}(x,s)\,ds, & x \in \Omega, t > \mu, \\ \frac{\partial\underline{u}}{\partial\nu} = 0, & x \in \partial\Omega, t > \mu, \\ \underline{u}(x,\mu) \leq u(x,\mu), & x \in \overline{\Omega}. \end{cases} \tag{4.9}$$

Proceeding analogously to the proof of Lemma 4.1 and the case $q_1 < 1$, we can show that $\underline{u}$ blows up in finite time. Since $\underline{u}$ is a super-solution of (4.7), ϕ cannot exist globally. The solution to problem (1.1) is a super-solution of (4.7).

Therefore, when $p_1 + q_1 > 1$, all solutions of (1.1) blow up in finite time.

When $p_2 + q_2 > 1$, we know from [19] that solutions of

$$\begin{cases} u_t = \Delta u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial\nu} = u^{q_2} \int_0^t u^{p_2}(x,s)\,ds, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{4.10}$$

blow up in finite time. The comparison principle tells that all solutions of (1.1) blow up in finite time. □

5 Blow-up set

Examples in [30] indicate that the blow-up may occur in the interior of the domain for some semilinear parabolic equation if the heat supply through the boundary is fast enough. However, we shall prove in this section that the blow-up of problem (1.1) will occur only at the boundary of the domain in some cases. This implies that the diffusion term is the dominating term in the interior of the domain for these cases. To this end, we need a lemma as follows. We will prove this lemma by the idea introduced in [31]. However, we need some careful modification due to the appearance of the nonlinear memory reaction term.

Lemma 5.1 *Suppose that the function $u(x,t)$ is continuous on the domain $\overline{\Omega} \times [0,T)$ and satisfies*

$$u_t = \Delta u + u^{q_1} \int_0^t u^{p_1}(x,s)\,ds, \quad x \in \Omega, 0 < t < T,$$

$$u \leq \frac{C}{(T-t)^q}, \quad x \in \partial\Omega, 0 < t < T$$

for some $0 < q < \frac{2}{p_1+q_1-1}$ $(p_1 + q_1 > 1)$. Then, for any $\Omega' \subset\subset \Omega$,

$$\sup\{u(x,t); (x,t) \in \Omega' \times [0,T)\} < \infty.$$

Proof We will prove this lemma in a similar way to Theorem 4.1 of [31].

Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ and

$$v(x) = d^2(x) \quad \text{for } x \in \overline{N_\varepsilon(\partial\Omega)},$$

where $N_\varepsilon(\partial\Omega) = \{x \in \Omega, d(x) < \varepsilon\}$. By approximating the domain from inside if necessary, we may assume without loss of generality that $\partial\Omega$ is smooth, say C^2. Henceforth, the function $v(x)$ is in $C^2(\overline{N_\varepsilon(\partial\Omega)})$ if ε is small enough. As stated in Theorem 4.1 of [31], we could extend $v(x)$ to a function on $\overline{\Omega}$ such that $v \in C^2(\overline{\Omega})$, $v \geq c_0 > 0$ on $\overline{\Omega \setminus N_{\varepsilon_0}(\partial\Omega)}$ and

$$\Delta v - \frac{(q+1)|\nabla v|^2}{v} \geq -C^* \quad \text{on } \overline{\Omega}$$

for some small enough ε_0 and $C^* > 0$.

Set

$$w(x,t) = \frac{C_1}{[v(x) + l(T-t)]^q}.$$

Denote $S(x,t) = v(x) + l(T-t)$, then there exist $M_1 > 0$, $M_2 > 0$ such that $M_1 \leq S(x,t) \leq M_2$ for any $(x,t) \in \Omega' \times [0,T)$ $(\Omega' \subset\subset \Omega)$. If $qp_1 > 1$, then direct calculation shows that

$$\begin{aligned}
&w_t - \Delta w - w^{q_1} \int_0^t w^{p_1}(x,s)\,ds \\
&\quad = C_1 q\big[v(x) + l(T-t)\big]^{-q-1}\left(l + \Delta v - \frac{q+1}{v(x) + l(T-t)}|\nabla v|^2\right) \\
&\qquad - \frac{C_1^{p_1+q_1}}{l(qp_1-1)} S^{-q(p_1+q_1)+1} + \frac{C_1^{p_1+q_1}}{l(qp_1-1)}\big(v(x) + lT\big)^{1-qp_1} S^{-qq_1} \\
&\quad \geq C_1 q S^{-q-1}\left\{l - C^* - \frac{{C_1}^{p_1+q_1-1}}{l(qp_1-1)} S^{-q(p_1+q_1-1)+2}\left[1 - \left(\frac{S}{v(x) + lT}\right)^{qp_1-1}\right]\right\} \\
&\quad \geq C_1 q S^{-q-1}\left(l - C^* - \frac{{C_1}^{p_1+q_1-1}}{ql(qp_1-1)} M_2^{-q(p_1+q_1-1)+2}\right).
\end{aligned}$$

Choose $l = \frac{C^*+[(C^*)^2+4C_1^{p_1+q_1-1}(M_2^{-q(p_1+q_1-1)+2}/(qp_1-1))]^{\frac{1}{2}}}{2}$, then $w_t - \Delta w - w^{q_1}\int_0^t w^{p_1}(x,s)\,ds \ge 0$ for any $(x,t) \in \Omega' \times (0,T)$. Moreover, as $q < \frac{2}{p_1+q_1-1}$, we can choose C_1 to be large enough so that $w(x,0) \ge u(x,0)$ and $w(x,t) \ge u(x,t)$ for $x \in \partial\Omega'$. Then the comparison principle implies that $w(x,t) \ge u(x,t)$, and

$$\sup\{u(x,t);(x,t) \in \Omega' \times [0,T)\} \le C_1 \sup\left\{\frac{1}{v^q(x)}; x \in \Omega'\right\} < \infty.$$

When $qp_1 < 1$,

$$\begin{aligned} & w_t - \Delta w - w^{q_1}\int_0^t w^{p_1}(x,s)\,ds \\ & \quad \ge C_1 q S^{-q-1}\left\{l - C^* - \frac{C_1^{p_1+q_1-1}}{l(qp_1-1)}S^{-q(p_1+q_1-1)+2}\left[1 - \left(\frac{S}{v(x)+lT}\right)^{qp_1-1}\right]\right\} \\ & \quad \ge C_1 q S^{-q-1}\left(l - C^* - \frac{C_1^{p_1+q_1-1}}{ql(1-qp_1)}M_2^{-q(p_1+q_1-1)+2}\left(\frac{D_1}{D_2+lT}\right)^{qp_1-1}\right), \end{aligned}$$

where $D_1 = \min_{\overline{\Omega}'} v(x)$, $D_2 = \max_{\overline{\Omega}'} v(x)$.

When $qp_1 = 1$,

$$\begin{aligned} & w_t - \Delta w - w^{q_1}\int_0^t w^{p_1}(x,s)\,ds \\ & \quad \ge C_1 q S^{-q-1}\left\{l - C^* - \frac{C_1^{p_1+q_1-1}}{ql}D^{q(1-q_1)+1}\ln\left(1 + \frac{l}{M_1}T\right)\right\}. \end{aligned}$$

By a similar way used in the case of $qp_1 > 1$; we could choose suitable l and C_1 such that $w_t - \Delta w - w^{q_1}\int_0^t w^{p_1}(x,s)\,ds \ge 0$ for any $(x,t) \in \Omega' \times (0,T)$. Moreover, $w(x,0) \ge u(x,0)$ and $w(x,t) \ge u(x,t)$ for $x \in \partial\Omega'$.

Therefore, when $q < \frac{2}{p_1+q_1-1}$, we get

$$\sup\{u(x,t);(x,t) \in \Omega' \times [0,T)\} < \infty$$

from the comparison principle. □

Now we are ready to give our main result as regards the blow-up set. To this end, we need the following hypothesis.

(H1) $\Delta u_0 \ge 0$, *i.e.* $u_t(x,0) \ge 0$ for all $x \in \Omega$.

Theorem 5.1 *If* $p_1 + q_1 > 1$ *and one of the following cases occurs, the blow-up of problem* (1.1) *occurs only at the domain of the boundary*:

(i) $q_2 = 0$, $p_2 > p_1 + q_1$, *and* (H1) *is valid,*

(ii) $p_1 = 0$,

(iii) $q_2 > \frac{p_1+q_1+1}{2}$.

Proof $G_N(x,y,t,\tau)$ is the Green's function for the heat equation with a homogeneous Neumann boundary condition. Then we have the following representation formulas for the solution of problem (1.1):

$$u(x,t) = \int_\Omega G_N(x,y,t,0)u_0(y)\,dy + \int_0^t \int_{\partial\Omega} G_N(x,y,t,\tau)u^{q_2}(y,\tau)\int_0^\tau u^{p_2}(y,s)\,ds\,dS_y\,d\tau$$
$$+ \int_0^t \int_\Omega G_N(x,y,t,\tau)u^{q_1}(y,\tau)\int_0^\tau u^{p_1}(y,s)\,ds\,dy\,d\tau. \tag{5.1}$$

Next, we consider three cases, respectively.

Case 1 ($q_2 = 0$, $p_2 > p_1 + q_1$). Set

$$J(t) = \int_0^t \int_0^\tau \int_{\partial\Omega} u^{p_2}(y,s)\,dS_y\,ds\,d\tau + \int_0^t \int_\Omega u^{q_1}(y,\tau)\int_0^\tau u^{p_1}(y,s)\,ds\,dy\,d\tau.$$

Then

$$J''(t) = \int_{\partial\Omega} u^{p_2}(y,t)\,dS_y + \int_\Omega \left[q_1 u^{q_1-1}(y,t)u_t(y,t)\int_0^t u^{p_1}(y,s)\,ds + u^{p_1+q_1}(y,t)\right]dy.$$

We have $u_t \geq 0$ from (H1) and the comparison principle of the usual parabolic equation (see, *e.g.*, [8]). From [26], we know

$$\int_{\partial\Omega} G_N(x,y,t,\tau)\,dS_x \geq c_7, \quad y \in \overline{\Omega}, 0 \leq \tau < t < T. \tag{5.2}$$

Therefore, (5.1) and Jensen's inequality show that

$$\begin{aligned} J''(t) &\geq \int_{\partial\Omega} u^{p_2}(y,t)\,dS_y \\ &\geq c_7^{p_2}|\partial\Omega|^{1-p_2}J^{p_2}(t). \end{aligned}$$

Multiplying both sides of the above inequality by $J'(t)$ and integrating over $(0,t)$, we obtain

$$J'(t) \geq c_8 J^{(p_2+1)/2}(t).$$

Integrating this inequality over $(0,t)$, one can get

$$\int_{J(t)}^\infty s^{-(p_2+1)/2}\,ds \geq c_9(T-t),$$

that is,

$$J(t) \leq c_{10}(T-t)^{-2/(p_2-1)}, \quad 0 \leq t < T. \tag{5.3}$$

We now take an arbitrary $\Omega' \subset\subset \Omega$ with $\operatorname{dist}(\partial\Omega,\Omega') = \varepsilon > 0$. For this Ω', we then take $\Omega'' \subset\subset \Omega$ such that $\Omega' \subset\subset \Omega''$, $\operatorname{dist}(\partial\Omega'',\Omega') \geq \frac{\varepsilon}{3}$ and $\operatorname{dist}(\partial\Omega,\Omega'') \geq \frac{\varepsilon}{3}$. It is well known that, for any $\varepsilon > 0$,

$$0 \leq G_N(x,y,t,\tau) \leq C_\varepsilon \quad \text{for } |x-y| \geq \varepsilon/3, x,y \in \overline{\Omega}, 0 < \tau < t < T, \tag{5.4}$$

where C_ε is a positive constant depending on ε. Then by (5.1), (5.4),

$$\max_{\overline{\Omega}''} u(x,t) \le c_{11} + C_\varepsilon J(t) \le c_{12}(T-t)^{-2/(p_2-1)}.$$

As $p_2 > p_1 + q_1$, Lemma 5.1 implies that the blow-up occurs only at the boundary.

In the remaining two cases, we set

$$\begin{aligned}\tilde{J}(t) = {}& \int_\sigma^t \int_{\partial\Omega} u^{q_2}(y,\tau) \int_\sigma^\tau u^{p_2}(y,s)\,ds\,dS_y\,d\tau \\ &+ \int_\sigma^t \int_\Omega u^{q_1}(y,\tau) \int_\sigma^\tau u^{p_1}(y,s)\,ds\,dy\,d\tau,\end{aligned} \tag{5.5}$$

where $0 < \sigma < T/4$. In view of Case 1 and Lemma 5.1, it suffices to prove that $\tilde{J}(t)$ satisfies a similar estimate to (5.3). Precisely, if we can prove that

$$\tilde{J}(t) \le \tilde{c}(T-t)^{-\beta} \quad \text{for } T/2 \le t < T$$

for some positive constants $\tilde{c}$, β, then the blow-up can only occur on the boundary $\partial\Omega$.

Case 2 ($p_1 = 0$). $p_1 + q_1 > 1$ and $p_1 = 0$ show that $q_1 > 1$. From (4.3), (5.1), and Jensen's inequality, we can obtain that

$$\begin{aligned}\tilde{J}'(t) &= \int_{\partial\Omega} u^{q_2}(y,t) \int_\sigma^t u^{p_2}(y,s)\,ds\,dS_y + \int_\sigma^t \int_\Omega u^{q_1}(y,t)\,dy\,ds \\ &\ge (t-\sigma)\int_\Omega u^{q_1}(y,t)\,dy \\ &\ge \frac{T}{4}|\Omega|^{1-q_1}\left[\int_\Omega u(y,t)\,dy\right]^{q_1} \\ &\ge \frac{T}{4}c_1^{q_1}|\Omega|^{1-q_1}\tilde{J}^{q_1}(t).\end{aligned}$$

Integrating this inequality over (t,T), we have

$$\int_{\tilde{J}(t)}^\infty s^{-q_1}\,ds \ge \tilde{c}_1(T-t),$$

that is,

$$\tilde{J}(t) \le \tilde{c}_2(T-t)^{-1/(q_1-1)}, \quad T/2 \le t < T.$$

Case 3 ($q_2 > \frac{p_1+q_1+1}{2}$). As $q_2 > \frac{p_1+q_1+1}{2} > 1$, we know from [19] that, for some $\beta > 0$, there exists $c_\beta > 0$ such that, for $y \in \partial\Omega$, $t \in [T/2, T)$,

$$\int_\sigma^t u^{p_2}(y,s)\,ds \ge c_\beta^{p_2}(T-\sigma) \ge c_\beta^{p_2}T/4.$$

Denote $\tilde{c}_3 = c_\beta^{p_2}T/4$, then

$$\int_\sigma^t u^{p_2}(y,s)\,ds \ge \tilde{c}_3. \tag{5.6}$$

Using (5.1), (5.2), (5.6), and Jensen's inequality, one can easily get

$$\begin{aligned}\int_{\partial\Omega} u(x,t)\,dS_x &\geq \int_{\partial\Omega}\left(\int_0^t\int_{\partial\Omega} G_N(x,y,t,\tau)u^{q_2}(y,\tau)\int_0^\tau u^{p_2}(y,s)\,ds\,dS_y\,d\tau\right)dS_x \\ &\quad + \int_{\partial\Omega}\left(\int_0^t\int_{\Omega} G_N(x,y,t,\tau)u^{q_1}(y,\tau)\int_0^\tau u^{p_1}(y,s)\,ds\,dy\,d\tau\right)dS_x \\ &\geq C_0\int_0^t\int_{\partial\Omega} G_N(x,y,t,\tau)u^{q_2}(y,\tau)\int_0^\tau u^{p_2}(y,s)\,ds\,dS_y\,d\tau \\ &\quad + C_0\int_0^t\int_{\Omega} G_N(x,y,t,\tau)u^{q_1}(y,\tau)\int_0^\tau u^{p_1}(y,s)\,ds\,dy\,d\tau \\ &\geq C_0\tilde{J}(t).\end{aligned}$$

From this we can obtain

$$\tilde{J}'(t) \geq \tilde{c}_3\int_{\partial\Omega} u^{q_2}(y,t)\,dS_y \geq \tilde{c}_3 c_{14}^{q_2}|\partial\Omega|^{1-q_2}\tilde{J}^{q_2}(t).$$

Integrating over $(0,T)$, we know that

$$\int_{\tilde{J}(t)}^{\infty} s^{-q_2}\,ds \geq \tilde{c}_4(T-t),$$

that is,

$$\tilde{J}(t) \leq \tilde{c}_5(T-t)^{-1/(q_2-1)}, \quad T/2 \leq t < T.$$

Since $q_2 > \frac{p_1+q_1+1}{2}$, we can conclude that $u(x,t)$ blows up only on the boundary in a similar way to Case 1 by Lemma 5.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YW carried out most of the studies and wrote the manuscript. JC did some work in Section 3 to Section 5. CH participated in the study of blow-up set. All authors read and approved the final manuscript.

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Acknowledgements
The authors are very grateful to the referees for their valuable comments, which greatly improved the manuscript. This work is supported by Scientific Research Found of Sichuan Provincial Education Department (14ZA0119), Xihua University Young Scholars Training Program, and Applied Basic Research Project of Sichuan province (2013JY0178).

Received: 4 April 2014 Accepted: 10 November 2014 Published: 26 Nov 2014

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10.1186/1029-242X-2014-472
Cite this article as: Wang et al.: **Singularity analysis for a semilinear integro-differential equation with nonlinear memory boundary.** *Journal of Inequalities and Applications* 2014, 2014:472