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Sharp bounds for Neuman means in terms of one-parameter family of bivariate means

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Abstract

We present the best possible parameters $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \in [0,1]$ such that the double inequalities $G_{p_1}(a,b) < S_{HA}(a,b) < G_{q_1}(a,b)$, $Q_{p_2}(a,b) < S_{CA}(a,b) < Q_{q_2}(a,b)$, $H_{p_3}(a,b) < S_{AH}(a,b) < H_{q_3}(a,b)$, $C_{p_4}(a,b) < S_{AC}(a,b) < C_{q_4}(a,b)$ hold for all $a, b > 0$ with $a \neq b$, where S_{HA}, S_{CA}, S_{AH}, S_{AC} are the Neuman means, and G_p, Q_p, H_p, C_p are the one-parameter means.

MSC: 26E60

Keywords: Neuman means; one-parameter mean; harmonic mean; geometric mean; arithmetic mean; quadratic mean; contraharmonic mean

1 Introduction

Let $a, b > 0$ with $a \neq b$. Then the Schwab-Borchardt mean $SB(a,b)$ [1–3], and the Neuman means $S_{HA}(a,b)$, $S_{AH}(a,b)$, $S_{CA}(a,b)$, and $S_{AC}(a,b)$ [4, 5] of a and b are given by

$$SB(a,b) = \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} \quad (a<b), \qquad SB(a,b) = \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} \quad (a>b),$$
$$S_{HA}(a,b) = SB\big[H(a,b), A(a,b)\big], \qquad S_{AH}(a,b) = SB\big[A(a,b), H(a,b)\big],$$
$$S_{CA}(a,b) = SB\big[C(a,b), A(a,b)\big], \qquad S_{AC}(a,b) = SB\big[A(a,b), C(a,b)\big],$$

respectively. Here, $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are, respectively, the inverse cosine and inverse hyperbolic cosine functions, and $H(a,b) = 2ab/(a+b)$, $A(a,b) = (a+b)/2$, and $C(a,b) = (a^2+b^2)/(a+b)$ are, respectively, the classical harmonic, arithmetic, and contraharmonic means of a and b.

Let $v = (a-b)/(a+b) \in (-1,1)$, and $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0, \log(2+\sqrt{3}))$, and $s \in (0,\pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$ and $\cosh(r) = 1/\cosh(s) = 1 + v^2$. Then the following explicit formulas were found by Neuman [4]:

$$S_{AH}(a,b) = A(a,b)\frac{\tanh(p)}{p}, \qquad S_{HA}(a,b) = A(a,b)\frac{\sin(q)}{q}, \tag{1.1}$$

$$S_{CA}(a,b) = A(a,b)\frac{\sinh(r)}{r}, \qquad S_{AC}(a,b) = A(a,b)\frac{\tan(s)}{s}. \tag{1.2}$$

Let $p \in [0,1]$ and N be a bivariate symmetric mean. Then the one-parameter bivariate mean $N_p(a,b)$ was defined by Neuman [6] as follows:

$$N_p(a,b) = N\left[\frac{(1+p)}{2}a + \frac{(1-p)}{2}b, \frac{(1+p)}{2}b + \frac{(1-p)}{2}a\right]. \tag{1.3}$$

Recently, the Neuman means S_{AH}, S_{HA}, S_{CA}, and S_{AC}, and the one-parameter bivariate mean N_p have been the subject of intensive research. He *et al.* [7] found the greatest values $\alpha_1, \alpha_2 \in [0,1/2]$, and $\alpha_3, \alpha_4 \in [1/2,1]$, and the least values $\beta_1, \beta_2 \in [0,1/2]$, and $\beta_3, \beta_4 \in [1/2,1]$ such that the double inequalities

$$\begin{aligned}
&H\big[\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a\big] < S_{AH}(a,b) < H\big[\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a\big],\\
&H\big[\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a\big] < S_{HA}(a,b) < H\big[\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a\big],\\
&C\big[\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\big] < S_{CA}(a,b) < C\big[\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\big],\\
&C\big[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\big] < S_{AC}(a,b) < C\big[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\big]
\end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$.

In [4, 5], Neuman proved that the inequalities

$$\begin{aligned}
&H(a,b) < S_{AH}(a,b) < L(a,b) < S_{HA}(a,b) < P(a,b),\\
&T(a,b) < S_{CA}(a,b) < Q(a,b) < S_{AC}(a,b) < C(a,b),\\
&H^{1/3}(a,b)A^{2/3}(a,b) < S_{HA}(a,b) < \frac{1}{3}H(a,b) + \frac{2}{3}A(a,b),\\
&C^{1/3}(a,b)A^{2/3}(a,b) < S_{CA}(a,b) < \frac{1}{3}C(a,b) + \frac{2}{3}A(a,b),\\
&A^{1/3}(a,b)H^{2/3}(a,b) < S_{AH}(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}H(a,b),\\
&A^{1/3}(a,b)C^{2/3}(a,b) < S_{AC}(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}C(a,b)
\end{aligned} \tag{1.4}$$

hold for all $a,b > 0$ with $a \neq b$, where $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ are, respectively, the logarithmic, first Seiffert, quadratic, and second Seiffert means of a and b.

Qian and Chu [8] proved that the double inequalities

$$\begin{aligned}
&\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < S_{HA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b),\\
&\alpha_2 A(a,b) + (1-\alpha_2)Q(a,b) < S_{CA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)Q(a,b)
\end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$, $\beta_1 \geq \pi/2$, $\alpha_2 \geq 1/3$, and $\beta_2 \leq [\sqrt{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})] = 0.2394\cdots$, where $G(a,b) = \sqrt{ab}$ is the geometric mean of a and b.

In [9], the authors proved that the double inequalities

$$\begin{aligned}
&\alpha_1\left[\frac{H(a,b)}{3}+\frac{2A(a,b)}{3}\right]+(1-\alpha_1)H^{1/3}(a,b)A^{2/3}(a,b)<S_{HA}(a,b)\\
&\quad<\beta_1\left[\frac{H(a,b)}{3}+\frac{2A(a,b)}{3}\right]+(1-\beta_1)H^{1/3}(a,b)A^{2/3}(a,b),\\
&\alpha_2\left[\frac{C(a,b)}{3}+\frac{2A(a,b)}{3}\right]+(1-\alpha_2)C^{1/3}(a,b)A^{2/3}(a,b)<S_{CA}(a,b)\\
&\quad<\beta_2\left[\frac{C(a,b)}{3}+\frac{2A(a,b)}{3}\right]+(1-\beta_2)C^{1/3}(a,b)A^{2/3}(a,b),\\
&\alpha_3\left[\frac{A(a,b)}{3}+\frac{2H(a,b)}{3}\right]+(1-\alpha_3)A^{1/3}(a,b)H^{2/3}(a,b)<S_{AH}(a,b)\\
&\quad<\beta_3\left[\frac{A(a,b)}{3}+\frac{2H(a,b)}{3}\right]+(1-\beta_3)A^{1/3}(a,b)H^{2/3}(a,b),\\
&\alpha_4\left[\frac{A(a,b)}{3}+\frac{2C(a,b)}{3}\right]+(1-\alpha_4)A^{1/3}(a,b)C^{2/3}(a,b)<S_{AC}(a,b)\\
&\quad<\beta_4\left[\frac{A(a,b)}{3}+\frac{2C(a,b)}{3}\right]+(1-\beta_4)A^{1/3}(a,b)C^{2/3}(a,b)
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 4/5$, $\beta_1\geq 3/\pi$, $\alpha_2\leq 3[\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})]=0.7528\cdots$, $\beta_2\geq 4/5$, $\alpha_3\leq 0$, $\beta_3\geq 4/5$, $\alpha_4\leq 4/5$, and $\beta_4\geq 3(3\sqrt{3}-\sqrt[3]{4}\pi)/[(5-3\sqrt[3]{4})\pi]=0.8400\cdots$.

Let $p,p_i,q_i,\alpha_j,\beta_j\in[0,1]$ $(i,j=1,2,\ldots,8)$. Then Neuman [6, 10] proved that the inequalities

$$\begin{aligned}
&H_{p_1}(a,b)<P(a,b)<H_{q_1}(a,b),\qquad G_{p_2}(a,b)<P(a,b)<G_{q_2}(a,b),\\
&Q_{p_3}(a,b)<T(a,b)<Q_{q_3}(a,b),\qquad C_{p_4}(a,b)<T(a,b)<C_{q_4}(a,b),\\
&Q_{p_5}(a,b)<M(a,b)<Q_{q_5}(a,b),\qquad C_{p_6}(a,b)<M(a,b)<C_{q_6}(a,b),\\
&H_{p_7}(a,b)<L(a,b)<H_{q_7}(a,b),\qquad G_{p_8}(a,b)<L(a,b)<G_{q_8}(a,b),\\
&\alpha_1A(a,b)+(1-\alpha_1)G_p(a,b)<P_p(a,b)<\beta_1A(a,b)+(1-\beta_1)G_p(a,b),\\
&\alpha_2Q_p(a,b)+(1-\alpha_2)A(a,b)<T_p(a,b)<\beta_2Q_p(a,b)+(1-\beta_2)A(a,b),\\
&\alpha_3Q_p(a,b)+(1-\alpha_3)A(a,b)<M_p(a,b)<\beta_3Q_p(a,b)+(1-\beta_3)A(a,b),\\
&\alpha_4A(a,b)+(1-\alpha_4)G_p(a,b)<L_p(a,b)<\beta_4A(a,b)+(1-\beta_4)G_p(a,b),\\
&A^{\alpha_5}(a,b)G_p^{1-\alpha_5}(a,b)<P_p(a,b)<A^{\beta_5}(a,b)G_p^{1-\beta_5}(a,b),\\
&Q_p^{\alpha_6}(a,b)A^{1-\alpha_6}(a,b)<T_p(a,b)<Q_p^{\beta_6}(a,b)A^{1-\beta_6}(a,b),\\
&Q_p^{\alpha_7}(a,b)A^{1-\alpha_7}(a,b)<M_p(a,b)<Q_p^{\beta_7}(a,b)A^{1-\beta_7}(a,b),\\
&A^{\alpha_8}(a,b)G_p^{1-\alpha_8}(a,b)<L_p(a,b)<A^{\beta_8}(a,b)G_p^{1-\beta_8}(a,b),
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p_1\geq\sqrt{1-2/\pi}$, $q_1\leq\sqrt{6}/6$, $p_2\geq\sqrt{1-4/\pi^2}$, $q_2\leq\sqrt{3}/3$, $p_3\leq\sqrt{16/\pi^2-1}$, $q_3\geq\sqrt{6}/3$, $p_4\leq\sqrt{4/\pi-1}$, $q_4\geq\sqrt{3}/3$, $p_5\leq\sqrt{1/\log^2(1+\sqrt{2})-1}$,

$q_5 \geq \sqrt{3}/3$, $p_6 \leq \sqrt{1/\log(1+\sqrt{2})-1}$, $q_6 \geq \sqrt{6}/6$, $p_7 = 1$, $q_7 \leq \sqrt{3}/3$, $p_8 = 1$, $q_8 \leq \sqrt{6}/3$, $\alpha_1 \leq 2/\pi$, $\beta_1 \geq 2/3$, $\alpha_2 \leq (4-\pi)/[(\sqrt{2}-1)\pi]$, $\beta_2 \geq 2/3$, $\alpha_3 \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]$, $\beta_3 \geq 1/3$, $\alpha_4 = 0$, $\beta_4 \geq 1/3$, $\alpha_5 \leq 2/3$, $\beta_5 = 1$, $\alpha_6 \leq 2/3$, $\beta_6 \geq (4\log 2 - 2\log\pi)/\log 2$, $\alpha_7 \leq 1/3$, $\beta_7 \geq -\log[\log(1+\sqrt{2})]/\log[\cosh(\log(1+\sqrt{2}))]$, $\alpha_8 \leq 1/3$, $\beta_8 = 1$, where $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$ is the Neuman-Sándor mean of a and b.

The main purpose of this paper is to present the best possible parameters $p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$ on the interval $[0,1]$ such that the double inequalities

$$G_{p_1}(a,b) < S_{HA}(a,b) < G_{q_1}(a,b), \qquad Q_{p_2}(a,b) < S_{CA}(a,b) < Q_{q_2}(a,b),$$
$$H_{p_3}(a,b) < S_{AH}(a,b) < H_{q_3}(a,b), \qquad C_{p_4}(a,b) < S_{AC}(a,b) < C_{q_4}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

2 Main results

Theorem 2.1 *Let $p_1, q_1 \in [0,1]$. Then the double inequality*

$$G_{p_1}(a,b) < S_{HA}(a,b) < G_{q_1}(a,b) \tag{2.1}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p_1 \geq \sqrt{6}/3$ and $q_1 \leq \sqrt{1-4/\pi^2}$.

Proof Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt{1-\lambda^2}$ and $p \in [0,1]$. Then $v, \lambda, x \in (0,1)$, and (1.1) and (1.3) lead to

$$\begin{aligned} S_{HA}(a,b) - G_p(a,b) &= A(a,b)\left[\frac{\lambda}{\arcsin(\lambda)} - \sqrt{1-p^2\bigl(1-\sqrt{1-\lambda^2}\bigr)}\right] \\ &= \frac{A(a,b)\sqrt{1-p^2(1-\sqrt{1-\lambda^2})}}{\arcsin(\lambda)}F(x), \end{aligned} \tag{2.2}$$

where

$$F(x) = \frac{\sqrt{1-x^2}}{\sqrt{1-p^2(1-x)}} - \arcsin\bigl(\sqrt{1-x^2}\bigr), \tag{2.3}$$

$$F(0) = \frac{1}{\sqrt{1-p^2}} - \frac{\pi}{2}, \qquad F(1) = 0, \tag{2.4}$$

$$F'(x) = -\frac{(1-x)f(x)}{2\sqrt{1-x^2}(p^2x+1-p^2)^{3/2}[2(p^2x+1-p^2)^{3/2}+p^2x+2(1-p^2)x+p^2]}, \tag{2.5}$$

where

$$\begin{aligned} f(x) &= -p^4x^3 + \bigl(4p^6+3p^4-4p^2\bigr)x^2 \\ &\quad + \bigl(-8p^6+9p^4+4p^2-4\bigr)x + \bigl(4p^6-11p^4+12p^2-4\bigr), \end{aligned} \tag{2.6}$$

$$f'(x) = -3p^4x^2 + 2\bigl(4p^6+3p^4-4p^2\bigr)x + \bigl(-8p^6+9p^4+4p^2-4\bigr). \tag{2.7}$$

We divide the discussion into two cases.

Case 1 $p = \sqrt{6}/3$. Then (2.6) becomes

$$f(x) = \frac{4}{27}(1-x)\bigl(3x^2+4x+2\bigr). \tag{2.8}$$

From (2.5) and (2.8) we clearly see that $F(x)$ is strictly decreasing on $[0,1]$, then (2.4) leads to the conclusion that

$$F(x) > 0 \tag{2.9}$$

for all $x \in (0,1)$.

Therefore,

$$S_{HA}(a,b) > G_{\sqrt{6}/3}(a,b) \tag{2.10}$$

for all $a,b > 0$ with $a \neq b$ follows from (2.2) and (2.9).

Case 2 $p = \sqrt{1-4/\pi^2}$. Then numerical computations lead to

$$4p^6 + 3p^4 - 4p^2 = \frac{3\pi^6 - 56\pi^4 + 240\pi^2 - 256}{\pi^6} < 0, \tag{2.11}$$

$$-8p^6 + 9p^4 + 4p^2 - 4 = \frac{\pi^6 + 8\pi^4 - 240\pi^2 + 512}{\pi^6} < 0, \tag{2.12}$$

$$f(0) = 4p^6 - 11p^4 + 12p^2 - 4 = \frac{\pi^6 - 8\pi^4 + 16\pi^2 - 256}{\pi^6} > 0, \tag{2.13}$$

$$f(1) = 4\bigl(3p^2 - 2\bigr) = -\frac{4(12-\pi^2)}{\pi^2} < 0. \tag{2.14}$$

It follows from (2.7) and (2.11) together with (2.12) that $f(x)$ is strictly decreasing on $[0,1]$. Then inequalities (2.13) and (2.14) together with (2.5) lead to the conclusion that there exists $\lambda_1 \in (0,1)$ such that $F(x)$ is strictly decreasing on $[0,\lambda_1]$ and strictly increasing on $[\lambda_1,1]$.

Note that inequality (2.4) becomes

$$F(0) = F(1) = 0. \tag{2.15}$$

From (2.2), (2.15), and the piecewise monotonicity of $F(x)$ we clearly see that the inequality

$$S_{HA}(a,b) < G_{\sqrt{1-4/\pi^2}}(a,b) \tag{2.16}$$

holds for all $a,b > 0$ with $a \neq b$.

Note that

$$\lim_{\lambda\to 0^+} \frac{\sqrt{\arcsin^2(\lambda) - \lambda^2}}{\arcsin(\lambda)\sqrt{1-\sqrt{1-\lambda^2}}} = \frac{\sqrt{6}}{3}, \tag{2.17}$$

$$\lim_{\lambda\to 1} \frac{\sqrt{\arcsin^2(\lambda) - \lambda^2}}{\arcsin(\lambda)\sqrt{1-\sqrt{1-\lambda^2}}} = \sqrt{1-\frac{4}{\pi^2}}. \tag{2.18}$$

Therefore, Theorem 2.1 follows from (2.10) and (2.16)-(2.18) together with the fact that inequality (2.1) is equivalent to the inequality (2.19) as follows:

$$q_1 < \frac{\sqrt{\arcsin^2(\lambda) - \lambda^2}}{\arcsin(\lambda)\sqrt{1-\sqrt{1-\lambda^2}}} < p_1. \tag{2.19}$$

□

Theorem 2.2 *Let $p_2, q_2 \in [0,1]$. Then the double inequality*

$$Q_{p_2}(a,b) < S_{CA}(a,b) < Q_{q_2}(a,b) \tag{2.20}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p_2 \leq \sqrt{6}/3$ and $q_2 \geq \sqrt{3/\log^2(2+\sqrt{3}) - 1} = 0.8542\cdots$.

Proof Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\mu = v\sqrt{2+v^2}$, $x = \sqrt{1+\mu^2}$, and $p \in [0,1]$. Then $v \in (0,1)$, $\mu \in (0,\sqrt{3})$, $x \in (1,2)$, and (1.2) and (1.3) lead to

$$\begin{aligned} S_{CA}(a,b) - Q_p(a,b) &= A(a,b)\left[\frac{\mu}{\sinh^{-1}(\mu)} - \sqrt{1+p^2\left(\sqrt{1+\mu^2}-1\right)}\right] \\ &= \frac{A(a,b)\sqrt{1+p^2(\sqrt{1+\mu^2}-1)}}{\sinh^{-1}(\mu)} G(x), \end{aligned} \tag{2.21}$$

where

$$G(x) = \frac{\sqrt{x^2-1}}{\sqrt{1+p^2(x-1)}} - \sinh^{-1}\left(\sqrt{x^2-1}\right),$$

$$G(1) = 0, \qquad G(2) = \frac{\sqrt{3}}{\sqrt{1+p^2}} - \log(2+\sqrt{3}), \tag{2.22}$$

$$G'(x) = -\frac{(x-1)f(x)}{2\sqrt{x^2-1}(p^2x+1-p^2)^{3/2}[p^2x^2+2(p^2x+1-p^2)^{3/2}+2(1-p^2)x+p^2]}, \tag{2.23}$$

where $f(x)$ is defined by (2.6).

We divide the discussion into two cases.

Case 1 $p = \sqrt{6}/3$. Then it follows from (2.6) that

$$f(x) = -\frac{4}{27}(x-1)\left(3x^2+4x+2\right) < 0 \tag{2.24}$$

for all $x \in (1,2)$.

Therefore,

$$S_{CA}(a,b) > Q_{\sqrt{6}/3}(a,b) \tag{2.25}$$

for all $a,b > 0$ with $a \neq b$ follows easily from (2.21)-(2.24).

Case 2 $p = \sqrt{3/\log^2(2+\sqrt{3}) - 1}$. Then numerical computations lead to

$$4p^6 + 3p^4 - 4p^2 = 0.2329\cdots > 0, \tag{2.26}$$

$$-8p^6 + 9p^4 + 4p^2 - 4 = 0.6027\cdots > 0, \tag{2.27}$$

$$3p^4 - p^2 - 1 = -0.1322\cdots < 0, \tag{2.28}$$

$$f(1) = 4\left(3p^2 - 2\right) = 0.7567\cdots > 0, \tag{2.29}$$

$$f(2) = 4p^6 + 11p^4 + 4p^2 - 12 = -1.669\cdots < 0. \tag{2.30}$$

It follows from (2.7) and (2.26)-(2.28) that

$$\begin{aligned} f'(x) &< -3p^4x^2 + 2\left(4p^6 + 3p^4 - 4p^2\right)x^2 + \left(-8p^6 + 9p^4 + 4p^2 - 4\right)x^2 \\ &= 4\left(3p^4 - p^2 - 1\right)x^2 < 0 \end{aligned} \tag{2.31}$$

for $x \in (1,2)$.

Equation (2.23) and inequalities (2.29)-(2.31) lead to the conclusion that there exists $\lambda_2 \in (1,2)$ such that $G(x)$ is strictly decreasing on $[0,\lambda_2]$ and strictly increasing on $[\lambda_2,1]$.

Note that (2.22) becomes

$$G(1) = G(2) = 0. \tag{2.32}$$

Therefore,

$$S_{CA}(a,b) < Q_{\sqrt{3/\log^2(2+\sqrt{3})-1}}(a,b) \tag{2.33}$$

for all $a,b > 0$ with $a \neq b$ follows from (2.21) and (2.32) together with the piecewise monotonicity of $G(x)$.

Note that

$$\lim_{\mu\to 0^+} \frac{\sqrt{\mu^2 - [\sinh^{-1}(\mu)]^2}}{\sinh^{-1}(\mu)\sqrt{\sqrt{1+\mu^2}-1}} = \frac{\sqrt{6}}{3}, \tag{2.34}$$

$$\lim_{\mu\to \sqrt{3}} \frac{\sqrt{\mu^2 - [\sinh^{-1}(\mu)]^2}}{\sinh^{-1}(\mu)\sqrt{\sqrt{1+\mu^2}-1}} = \sqrt{\frac{3}{\log^2(2+\sqrt{3})} - 1}. \tag{2.35}$$

Therefore, Theorem 2.2 follows from (2.25) and (2.33)-(2.35) together with the fact that inequality (2.20) is equivalent to the inequality (2.36) as follows:

$$p_2 < \frac{\sqrt{\mu^2 - [\sinh^{-1}(\mu)]^2}}{\sinh^{-1}(\mu)\sqrt{\sqrt{1+\mu^2}-1}} < q_2. \tag{2.36}$$

□

Theorem 2.3 *Let $p_3, q_3 \in [0,1]$. Then the double inequality*

$$H_{p_3}(a,b) < S_{AH}(a,b) < H_{q_3}(a,b) \tag{2.37}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p_3 = 1$ and $q_3 \leq \sqrt{6}/3$.

Proof Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt{1-\lambda^2}$ and $p \in [0,1]$. Then $v, \lambda, x \in (0,1)$, and (1.1) and (1.3) lead to

$$\begin{aligned} S_{AH}(a,b) - H_p(a,b) &= A(a,b)\left[\frac{\lambda}{\tanh^{-1}(\lambda)} + p^2\left(1-\sqrt{1-\lambda^2}\right) - 1\right] \\ &= \frac{A(a,b)[1-p^2(1-\sqrt{1-\lambda^2})]}{\tanh^{-1}(\lambda)} H(x), \end{aligned} \tag{2.38}$$

where

$$H(x) = \frac{\sqrt{1-x^2}}{p^2x + (1-p^2)} - \tanh^{-1}\left(\sqrt{1-x^2}\right),$$

$$H(1) = 0, \tag{2.39}$$

$$H'(x) = -\frac{1-x}{x\sqrt{1-x^2}[p^2x+(1-p^2)]^2} g(x), \tag{2.40}$$

where

$$g(x) = \left(p^4 + p^2 - 1\right)x - p^4 + 2p^2 - 1. \tag{2.41}$$

We divide the discussion into two cases.

Case 1 $p = \sqrt{6}/3$. Then (2.41) leads to

$$g(x) = -\frac{1}{9}(1-x) < 0 \tag{2.42}$$

for $x \in (0,1)$.

Therefore,

$$S_{AH}(a,b) < H_{\sqrt{6}/3}(a,b) \tag{2.43}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (2.38)-(2.40) and (2.42).

Case 2 $p = 1$. Then it follows from (1.3) and (1.4) that

$$S_{AH}(a,b) > H(a,b) = H_1(a,b) \tag{2.44}$$

for all $a, b > 0$ with $a \neq b$.

Note that

$$\lim_{\lambda\to 0^+} \sqrt{\frac{\tanh^{-1}(\lambda) - \lambda}{\tanh^{-1}(\lambda)(1-\sqrt{1-\lambda^2})}} = \frac{\sqrt{6}}{3}, \tag{2.45}$$

$$\lim_{\lambda\to 1} \sqrt{\frac{\tanh^{-1}(\lambda) - \lambda}{\tanh^{-1}(\lambda)(1-\sqrt{1-\lambda^2})}} = 1. \tag{2.46}$$

Therefore, Theorem 2.3 follows from (2.43)-(2.46) and the fact that inequality (2.37) is equivalent to

$$q_3 < \sqrt{\frac{\tanh^{-1}(\lambda) - \lambda}{\tanh^{-1}(\lambda)(1-\sqrt{1-\lambda^2})}} < p_3.$$

□

Theorem 2.4 *Let* $p_4, q_4 \in [0,1]$. *Then the double inequality*

$$C_{p_4}(a,b) < S_{AC}(a,b) < C_{q_4}(a,b) \tag{2.47}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $p_4 \leq \sqrt{3\sqrt{3}/\pi - 1}$ *and* $q_4 \geq \sqrt{6}/3$.

Proof Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\mu = v\sqrt{2+v^2}$, $x = \sqrt{1+\mu^2}$, and $p \in [0,1]$. Then $v \in (0,1)$, $\mu \in (0,\sqrt{3})$, $x \in (1,2)$, and (1.2) and (1.3) lead to

$$\begin{aligned} S_{AC}(a,b) - C_p(a,b) &= A(a,b)\left[\frac{\mu}{\arctan(\mu)} - p^2\left(\sqrt{1+\mu^2}-1\right) - 1\right] \\ &= \frac{A(a,b)[1+p^2(\sqrt{1+\mu^2}-1)]}{\arctan(\mu)} J(x), \end{aligned} \tag{2.48}$$

where

$$J(x) = \frac{\sqrt{x^2-1}}{p^2x + (1-p^2)} - \arctan\left(\sqrt{x^2-1}\right),$$

$$J(1) = 0, \qquad J(2) = \frac{\sqrt{3}}{p^2+1} - \frac{\pi}{3}, \tag{2.49}$$

$$J'(x) = -\frac{x-1}{x\sqrt{x^2-1}[p^2x + (1-p^2)]^2} g(x), \tag{2.50}$$

where $g(x)$ is defined by (2.41).

We divide the discussion into two cases.

Case 1 $p = \sqrt{6}/3$. Then (2.41) leads to

$$g(x) = \frac{1}{9}(x-1) > 0 \tag{2.51}$$

for $x \in (1,2)$.

Therefore,

$$S_{AC}(a,b) < C_{\sqrt{6}/3}(a,b) \tag{2.52}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (2.48)-(2.51).

Case 2 $p = \sqrt{3\sqrt{3}/\pi - 1}$. Then numerical computations lead to

$$p^4 + p^2 - 1 = \frac{27 - \pi^2 - 3\sqrt{3}\pi}{\pi^2} > 0, \tag{2.53}$$

$$g(1) = 3p^2 - 2 = \frac{9\sqrt{3} - 5\pi}{\pi} < 0, \tag{2.54}$$

$$g(2) = p^4 + 4p^2 - 3 = \frac{27 - 6\pi^2 + 6\sqrt{3}\pi}{\pi^2} > 0. \tag{2.55}$$

From (2.41) and (2.50) together with (2.53)-(2.55) we clearly see that there exists $\lambda_3 \in (1,2)$ such that $J(x)$ is strictly increasing on $[1,\lambda_3]$ and strictly decreasing on $[\lambda_3, 2]$.

Note that (2.49) becomes

$$J(1) = J(2) = 0. \tag{2.56}$$

It follows from (2.56) and the piecewise monotonicity of $J(x)$ that

$$J(x) > 0 \tag{2.57}$$

for all $x \in (1, 2)$.

Therefore,

$$S_{AC}(a,b) > C_{\sqrt{3\sqrt{3}/\pi - 1}}(a,b) \tag{2.58}$$

for all $a, b > 0$ with $a \neq b$ follows from (2.48) and (2.58).

Note that

$$\lim_{\mu \to 0^+} \sqrt{\frac{\mu - \arctan(\mu)}{\arctan(\mu)(\sqrt{1+\mu^2} - 1)}} = \frac{\sqrt{6}}{3}, \tag{2.59}$$

$$\lim_{\mu \to 1} \sqrt{\frac{\mu - \arctan(\mu)}{\arctan(\mu)(\sqrt{1+\mu^2} - 1)}} = \sqrt{\frac{3\sqrt{3}}{\pi} - 1}. \tag{2.60}$$

Therefore, Theorem 2.4 follows from (2.52) and (2.58)-(2.60) together with the fact that inequality (2.47) is equivalent to

$$p_4 < \sqrt{\frac{\mu - \arctan(\mu)}{\arctan(\mu)(\sqrt{1+\mu^2} - 1)}} < q_4. \qquad \square$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Z-HS provided the main idea and carried out the proof of Theorem 2.1. W-MQ carried out the proof of Theorem 2.2. Y-MC carried out the proof of Theorems 2.3 and 2.4. All authors read and approved the final manuscript.

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