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Statistical approximation of modified Schurer-type q-Bernstein Kantorovich operators

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Abstract

New modified Schurer-type q-Bernstein Kantorovich operators are introduced. The local theorem and statistical Korovkin-type approximation properties of these operators are investigated. Furthermore, the rate of approximation is examined in terms of the modulus of continuity and the elements of Lipschitz class functions.

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1 Introduction

In 1987, Lupaş [1] introduced a q-analogue of Bernstein operators, and in 1997 another q-generalization of the Bernstein polynomials was introduced by Phillips [2]. After that generalizations of the Bernstein polynomials based on the q-integers attracted a lot of interest and were studied widely by a number of authors. Some new generalizations of well-known positive linear operators based on q-integers were introduced and studied by several authors (*e.g.*, see [3–6]). On the other hand, the study of the statistical convergence for sequences of positive operators was attempted by Gadjiev and Orhan [7]. Very recently, the statistical approximation properties have also been investigated for q-analogue polynomials. For instance, in [8] q-Bleimann, Butzer and Hahn operators; in [9] Kantorovich-type q-Bernstein operators; in [10] a q-analogue of MKZ operators; in [11] Kantorovich-type q-Szász-Mirakjan operators; in [12] Kantorovich-type q-Bernstein-Stancu operators were introduced and their statistical approximation properties were studied.

The paper is organized as follows. In Section 2, we introduce a new modification of Schurer-type q-Bernstein Kantorovich operators and evaluate the moments of these operators. In Section 3 we study local convergence properties in terms of the first and the second modulus of continuity. In Section 4, we obtain their statistical approximation properties with the help of the Korovkin-type theorem proved by Gadjiev and Orhan. Furthermore, in Section 5, we compute the degree of convergence of the approximation process in terms of the modulus of continuity and the Lipschitz class functions.

2 Construction of the operators

Some definitions and notations regarding the concept of q-calculus can be found in [5]. Let $\alpha, \beta, p \in \mathbb{N}^0$ (the set of all nonnegative integers) be such that $0 \le \alpha \le \beta$. We introduce a new modification of Schurer-type q-Bernstein Kantorovich operators $K_{n,q}^{(\alpha,\beta)}(f;x) : C[0,1+p] \to C[0,1]$ as follows:

$$K_{n,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 f\left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right) d_q t, \tag{2.1}$$

where $x \in [0,1]$ and $\overline{p}_{n,k}(q;x) = \left[{n+p \atop k}\right]_q x^k \prod_{s=0}^{n+p-k-1}(1-q^s x)$. It is clear that $K_{n,q}^{(\alpha,\beta)}(f;x)$ is a linear and positive operator. When $\alpha = \beta = 0$, it reduces to the Schurer-type q-Bernstein Kantorovich operators (see [13])

$$K_n^p(f;q;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n+p-k-1} (1-q^s x) \int_0^1 f\left(\frac{t}{[n+1]_q} + \frac{q[k]_q}{[n+1]_q}\right) d_q t.$$

In order to investigate the approximation properties of $K_{n,q}^{(\alpha,\beta)}$, we need the following lemmas.

Lemma 2.1 ([14]) *For the generalized q-Schurer-Stancu operators*

$$\left(S_{n,p,q}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n+p-k-1} (1-q^s x) f\left(\frac{[k+\alpha]_q}{[n+\beta]_q}\right), \quad x \in [0,1],$$

the following properties hold:

$$\left(S_{n,p,q}^{(\alpha,\beta)} 1\right)(x) = 1, \tag{2.2}$$

$$\left(S_{n,p,q}^{(\alpha,\beta)} t\right)(x) = \frac{q^\alpha [n+p]_q}{[n+\beta]_q} x + \frac{[\alpha]_q}{[n+\beta]_q}, \tag{2.3}$$

$$\left(S_{n,p,q}^{(\alpha,\beta)} t^2\right)(x) = \frac{[n+p]_q [n+p-1]_q}{[n+\beta]_q^2} q^{2\alpha+1} x^2 + \frac{[n+p]_q q^\alpha}{[n+\beta]_q^2} \left(2[\alpha]_q + q^\alpha\right) x + \frac{[\alpha]_q^2}{[n+\beta]_q^2}. \tag{2.4}$$

Lemma 2.2 *For* $K_{n,q}^{(\alpha,\beta)}(t^i;x)$, $i = 0,1,2$, *we have*

$$K_{n,q}^{(\alpha,\beta)}(1;x) = 1, \tag{2.5}$$

$$K_{n,q}^{(\alpha,\beta)}(t;x) = \frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} x + \frac{1}{[n+1+\beta]_q} \left(\frac{1}{[2]_q} + q[\alpha]_q\right), \tag{2.6}$$

$$\begin{aligned} K_{n,q}^{(\alpha,\beta)}\left(t^2;x\right) = {} & \frac{[n+p]_q [n+p-1]_q}{[n+1+\beta]_q^2} q^{2\alpha+3} x^2 \\ & + \frac{[n+p]_q}{[n+1+\beta]_q^2} \left(\frac{2}{[2]_q} q^{\alpha+1} + q^{2+\alpha}\left(2[\alpha]_q + q^\alpha\right)\right) x \\ & + \frac{1}{[n+1+\beta]_q^2} \left(\frac{1}{[3]_q} + \frac{2q[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2\right). \end{aligned} \tag{2.7}$$

Proof It is obvious that

$$\int_0^1 1\, d_q t = 1, \qquad \int_0^1 t\, d_q t = \frac{1}{[2]_q}, \qquad \int_0^1 t^2\, d_q t = \frac{1}{[3]_q}, \quad 0 < q < 1.$$

For $i = 0$, since $\sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) = 1$, so (2.5) holds.

For $i = 1$, we get

$$\begin{aligned} K_{n,q}^{(\alpha,\beta)}(t;x) &= \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 \left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} \right) d_q t \\ &= \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 \frac{t}{[n+1+\beta]_q}\, d_q t + \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\, d_q t \\ &= \frac{1}{[2]_q[n+1+\beta]_q} + \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{q[k+\alpha]_q}{[n+1+\beta]_q}. \end{aligned}$$

Using (2.3), we have

$$\begin{aligned} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{q[k+\alpha]_q}{[n+1+\beta]_q} &= \frac{q[n+\beta]_q}{[n+1+\beta]_q} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q}{[n+\beta]_q} \\ &= \frac{q[n+\beta]_q}{[n+1+\beta]_q} \bigl(S_{n,p,q}^{(\alpha,\beta)} t\bigr)(x). \end{aligned}$$

So

$$K_{n,q}^{(\alpha,\beta)}(t;x) = \frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} x + \frac{1}{[n+1+\beta]_q} \left(\frac{1}{[2]_q} + q[\alpha]_q \right).$$

For $i = 2$,

$$\begin{aligned} &\int_0^1 \left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} \right)^2 d_q t \\ &\quad = \frac{1}{[n+1+\beta]_q^2} \left(\int_0^1 t^2\, d_q t + 2q[k+\alpha]_q \int_0^1 t\, d_q t + q^2 [k+\alpha]_q^2 \int_0^1 1\, d_q t \right) \\ &\quad = \frac{1}{[n+1+\beta]_q^2} \left(\frac{1}{[3]_q} + \frac{2q[k+\alpha]_q}{[2]_q} + q^2 [k+\alpha]_q^2 \right), \end{aligned}$$

we obtain

$$\begin{aligned} &K_{n,q}^{(\alpha,\beta)}\bigl(t^2;x\bigr) \\ &\quad = \frac{1}{[n+1+\beta]_q^2} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \left(\frac{1}{[3]_q} + \frac{2q[k+\alpha]_q}{[2]_q} + q^2 [k+\alpha]_q^2 \right) \\ &\quad = \frac{1}{[3]_q[n+1+\beta]_q^2} + \frac{[n+\beta]_q}{[n+1+\beta]_q^2} \frac{2q}{[2]_q} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q}{[n+\beta]_q} \\ &\qquad + \frac{q^2[n+\beta]_q^2}{[n+1+\beta]_q^2} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q^2}{[n+\beta]_q^2} \\ &\quad = \frac{1}{[3]_q[n+1+\beta]_q^2} + \frac{[n+\beta]_q}{[n+1+\beta]_q^2} \frac{2q}{[2]_q} \bigl(S_{n,p,q}^{(\alpha,\beta)} t\bigr)(x) + \frac{q^2[n+\beta]_q^2}{[n+1+\beta]_q^2} \bigl(S_{n,p,q}^{(\alpha,\beta)} t^2\bigr)(x). \end{aligned}$$

Using (2.3) and (2.4), by a simple calculation we can get the stated result (2.7). □

Lemma 2.3 *From Lemma* 2.2, *we have*

$$\mu_{n,q}^{p}(x) := K_{n,q}^{(\alpha,\beta)}(t-x;x) = \left(\frac{q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q} - 1\right)x + \frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right) \tag{2.8}$$

and

$$\begin{aligned}\delta_{n,q}^{p}(x) &:= K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\\ &= \left(\frac{[n+p]_q[n+p-1]_q q^{2\alpha+3}}{[n+1+\beta]_q^2} - \frac{2q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q} + 1\right)x^2\\ &\quad + \left(\frac{[n+p]_q}{[n+1+\beta]_q^2}\left(\frac{2}{[2]_q}q^{\alpha+1} + q^{2+\alpha}\left(2[\alpha]_q + q^{\alpha}\right)\right)\right.\\ &\quad \left. - \frac{2}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right)\right)x\\ &\quad + \frac{1}{[n+1+\beta]_q^2}\left(\frac{1}{[3]_q} + \frac{2q[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2\right).\end{aligned} \tag{2.9}$$

3 Local approximation

Now, we consider a sequence $q = q_n$ satisfying the following two expressions:

$$\lim_{n\to\infty} q_n = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0. \tag{3.1}$$

By the Korovkin theorem, we can state the following theorem.

Theorem 3.1 *Let* $K_{n,q_n}^{(\alpha,\beta)}(f;x)$ *be a sequence satisfying* (3.1) *for* $0 < q_n < 1$. *Then, for any function* $f \in C[0,p+1]$, *the following equality*

$$\lim_{n\to\infty} \left\| K_{n,q_n}^{(\alpha,\beta)}(f;\cdot) - f \right\|_{C[0,1]} = 0$$

is satisfied.

Proof We know that $K_{n,q}^{(\alpha,\beta)}(f;x)$ is linear positive. By Lemma 2.2, if we choose the sequence $q = q_n$ satisfying conditions (3.1), and using the equality

$$[n+\alpha]_{q_n} = [n]_{q_n} + q_n^n[\alpha]_{q_n}, \qquad [n+1+\beta]_{q_n} = [n]_{q_n} + q_n^n[\beta+1]_{q_n}, \tag{3.2}$$

we have

$$K_{n,q_n}^{(\alpha,\beta)}(q;x) \rightrightarrows 1, \qquad K_{n,q_n}^{(\alpha,\beta)}(t;x) \rightrightarrows x, \qquad K_{n,q_n}^{(\alpha,\beta)}\left(t^2;x\right) \rightrightarrows x^2$$

as $n \to \infty$. Because of the linearity and positivity of $K_{n,q_n}^{(\alpha,\beta)}(f;x)$, the proof is complete by the classical Korovkin theorem. □

Consider the following K-functional:

$$K_2\left(f,\delta^2\right) := \inf\left\{\|f-g\| + \delta^2 \left\|g''\right\| : g \in C^2[0,p+1]\right\}, \quad \delta \geq 0,$$

where

$$C^2[0,p+1] := \{g : g, g', g'' \in C[0,p+1]\}.$$

Then, in view of a known result [15], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta^2) \le C\omega_2(f,\delta), \tag{3.3}$$

where

$$\omega_2(f,\delta) := \sup_{0<h\le\delta} \sup_{x\pm h\in[0,p+1]} \bigl|f(x-h) - 2f(x) + f(x+h)\bigr|$$

is the second modulus of smoothness of $f \in C[0,p+1]$.

Let $f \in C[0,p+1]$, for any $\delta > 0$, the usual modulus of continuity for f is defined as $\omega(f,\delta) = \sup_{0<h\le\delta} \sup_{x,x+h\in[0,p+1]} |f(x+h) - f(x)|$.

We next present the following local theorem of the operators $K_{n,q}^{(\alpha,\beta)}(f;x)$ in terms of the first and the second modulus of continuity of the function $f \in C[0,p+1]$.

Theorem 3.2 *Let $f \in C[0,p+1]$, there exists an absolute constant $C > 0$ such that*

$$\bigl|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)\bigr| \le C\omega_2\Bigl(f, \sqrt{a_{n,q}^p(x)}\Bigr) + \omega\bigl(f, b_{n,q}^p(x)\bigr),$$

where

$$\begin{aligned} a_{n,q}^p(x) &= \left(\frac{q^{2\alpha+2}[n+p]_q^2}{[n+1+\beta]_q^2} + \frac{[n+p]_q[n+p-1]_q}{[n+1+\beta]_q^2}q^{2\alpha+3} - \frac{4q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q} + 2\right)x^2 \\ &\quad + \left(\frac{[n+p]_q}{[n+1+\beta]_q^2}\left(\frac{4q^{\alpha+1}}{[2]_q} + 4q^{\alpha+2}[\alpha]_q + q^{\alpha+2}\right) - \frac{4}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right)\right)x \\ &\quad + \frac{1}{[n+1+\beta]_q^2}\left(2q^2[\alpha]_q^2 + \frac{4q[\alpha]_q}{[2]_q} + \frac{1}{[2]_q^2} + \frac{1}{[3]_q}\right) \end{aligned}$$

and

$$b_{n,q}^p(x) = \left|\left(\frac{q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q} - 1\right)x + \frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right)\right|.$$

Proof Let

$$\overline{K}_{n,q}^{(\alpha,\beta)}(f;x) := K_{n,q}^{(\alpha,\beta)}(f;x) - f\bigl(\xi_{n,q}^p(x)\bigr) + f(x),$$

where $f \in C[0,p+1]$ and

$$\xi_{n,q}^p(x) = \frac{[n+p]_q}{[n+1+\beta]_q}q^{\alpha+1}x + \frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right).$$

Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)\,ds$$

for $g \in C^2[0,p+1]$, we have

$$\overline{K}_{n,q}^{(\alpha,\beta)}(g;x) = g(x) + K_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)\,ds;x\right) - \int_x^{\xi_{n,q}^p(x)} \left(\xi_{n,q}^p(x)-s\right)g''(s)\,ds.$$

Hence

$$\begin{aligned}
&\left|\overline{K}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)\right| \\
&\quad\le K_{n,q}^{(\alpha,\beta)}\left(\left|\int_x^t |t-s|\cdot\left|g''(s)\right|ds\right|;x\right) + \left|\int_x^{\xi_{n,q}^p(x)} \left|\xi_{n,q}^p(x)-s\right|\cdot\left|g''(s)\right|ds\right| \\
&\quad\le \left\|g''\right\|K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) + \left\|g''\right\|\left(\xi_{n,q}^p(x)-x\right)^2.
\end{aligned}$$

Observe that

$$K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right) + \left(\xi_{n,q}^p(x)-x\right)^2 = a_{n,q}^p(x),$$

we obtain

$$\left|\overline{K}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)\right| \le a_{n,q}^p(x)\left\|g''\right\|. \tag{3.4}$$

Using (3.4) and the uniform boundedness of $K_{n,q}^{(\alpha,\beta)}$, we get

$$\begin{aligned}
\left|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)\right| &\le \left|\overline{K}_{n,q}^{(\alpha,\beta)}(f-g;x)\right| + \left|\overline{K}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)\right| \\
&\quad + \left|f(x)-g(x)\right| + \left|f\left(\xi_{n,q}^p(x)\right) - f(x)\right| \\
&\le 4\|f-g\| + a_{n,q}^p(x)\left\|g''\right\| + \omega\left(f, b_{n,q}^p(x)\right).
\end{aligned}$$

Taking the infimum on the right-hand side over all $g \in C^2[0,p+1]$, we obtain

$$\left|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)\right| \le CK_2\left(f, \sqrt{a_{n,q}^p(x)}\right) + \omega\left(f, b_{n,q}^p(x)\right),$$

which together with (3.3) gives the proof of the theorem. □

4 Korovkin-type statistical approximation properties

Further on, let us recall the concept of statistical convergence which was introduced by Fast [16].

Let the set $K \in N$ and $K_n = \{k \le n : k \in K\}$, the natural density of K is defined by $\delta(K) := \lim_{n\to\infty} \frac{1}{n}|K_n|$ if the limit exists (see [17]), where $|K_n|$ denotes the cardinality of the set K_n.

A sequence $x = x_k$ is called statistically convergent to a number L if for every $\varepsilon > 0$, $\delta\{k \in N : |x_k - L| \ge \varepsilon\} = 0$. This convergence is denoted as $st\text{-}\lim_k x_k = L$. It is known that any convergent sequence is statistically convergent, but not conversely. Details can be found in [18].

In approximation theory by linear positive operators, the concept of statistical convergence was used by Gadjiev and Orhan [7]. They proved the following Bohman-Korovkin-type approximation theorem for statistical convergence.

Theorem 4.1 ([7]) *If the sequence of linear positive operators* $A_n : C[a,b] \to C[a,b]$ *satisfies the conditions*

$$st\text{-}\lim_n \left\| A_n(e_i;\cdot) - e_i \right\|_{C[a,b]} = 0$$

for $e_i(t) = t^i$, $i = 0,1,2$, *then, for any* $f \in C[a,b]$,

$$st\text{-}\lim_n \left\| A_n(f;\cdot) - f \right\|_{C[a,b]} = 0.$$

In this section, we establish the following Korovkin-type statistical approximation theorems.

Theorem 4.2 *Let* $q = q_n$, $0 < q_n < 1$, *be a sequence satisfying the following conditions*:

$$st\text{-}\lim_n q_n = 1, \qquad st\text{-}\lim_n q_n^n = a \quad (a < 1) \quad and \quad st\text{-}\lim_n \frac{1}{[n]_{q_n}} = 0, \tag{4.1}$$

then, for $f \in C[0,p+1]$, *we have*

$$st\text{-}\lim_n \left\| K_{n,q_n}^{(\alpha,\beta)}(f;\cdot) - f \right\|_{C[0,1]} = 0.$$

Proof From Theorem 4.1, it is enough to prove that $st\text{-}\lim_n \| K_{n,q_n}^{(\alpha,\beta)}(e_i;\cdot) - e_i \|_{C[0,1]} = 0$ for $e_i = t^i$, $i = 0,1,2$.

By (2.5), we can easily get

$$st\text{-}\lim_n \left\| K_{n,q_n}^{(\alpha,\beta)}(e_0;\cdot) - e_0 \right\|_{C[0,1]} = 0. \tag{4.2}$$

From equality (2.8) and (3.2) we have

$$\begin{aligned} &\left\| K_{n,q_n}^{(\alpha,\beta)}(e_1;\cdot) - e_1 \right\|_{C[0,1]} \\ &\quad \leq \left| \frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1 \right| + \frac{1}{[n+1+\beta]_{q_n}} \left(\frac{1}{[2]_{q_n}} + q_n[\alpha]_{q_n} \right) \\ &\quad \leq \left| \frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1 \right| + \frac{1+\alpha}{[n]_{q_n}}. \end{aligned} \tag{4.3}$$

Now, for given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned} U &= \left\{ k : \left\| K_{n,q_k}^{(\alpha,\beta)}(e_1;\cdot) - e_1 \right\|_{C[0,1]} \geq \varepsilon \right\}, \\ U_1 &= \left\{ k : \frac{q_k^{\alpha+1}[n+p]_{q_k}}{[n+1+\beta]_{q_k}} - 1 \geq \frac{\varepsilon}{2} \right\}, \\ U_2 &= \left\{ k : \frac{1+\alpha}{[n]_{q_k}} \geq \frac{\varepsilon}{2} \right\}. \end{aligned}$$

From (4.3), one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\begin{aligned}
&\delta\left\{k \le n : \left\| K_{n,q_k}^{(\alpha,\beta)}(e_1;\cdot) - e_1 \right\|_{C[0,1]} \ge \varepsilon\right\} \\
&\quad \le \delta\left\{k \le n : \frac{q_k^{\alpha+1}[n+p]_{q_k}}{[n+1+\beta]_{q_k}} - 1 \ge \frac{\varepsilon}{2}\right\} + \delta\left\{k \le n : \frac{1+\alpha}{[n]_{q_k}} \ge \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

By (4.1) it is clear that

$$st\text{-}\lim_n \left(\frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1 \right) = 0$$

and

$$st\text{-}\lim_n \frac{1+\alpha}{[n]_{q_n}} = 0.$$

So we have

$$st\text{-}\lim_n \left\| K_{n,q_n}^{(\alpha,\beta)}(e_1;\cdot) - e_1 \right\|_{C[0,1]} = 0. \tag{4.4}$$

Finally, in view of (2.7), one can write

$$\begin{aligned}
&\left\| K_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} \\
&\quad \le \left| \frac{[n+p]_{q_n}[n+p-1]_{q_n}}{[n+1+\beta]_{q_n}^2} q_n^{2\alpha+3} - 1 \right| \\
&\qquad + \frac{[n+p]_{q_n}}{[n+1+\beta]_{q_n}^2} \left(\frac{2}{[2]_{q_n}} q_n^{\alpha+1} + q_n^{2+\alpha}\left(2[\alpha]_{q_n} + q_n^{\alpha}\right) \right) \\
&\qquad + \frac{1}{[n+1+\beta]_{q_n}^2} \left(\frac{1}{[3]_{q_n}} + \frac{2q_n[\alpha]_{q_n}}{[2]_{q_n}} + q_n^2[\alpha]_{q_n}^2 \right).
\end{aligned}$$

Using (3.2),

$$\begin{aligned}
&\frac{2}{[2]_{q_n}} q_n^{\alpha+1} + q_n^{2+\alpha}\left(2[\alpha]_{q_n} + q_n^{\alpha}\right) \le 3 + 2\alpha, \\
&\frac{1}{[3]_{q_n}} + \frac{2q_n[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2 \le (1+\alpha)^2,
\end{aligned}$$

and

$$q_n[n+p-1]_{q_n} = [n+p]_{q_n} - 1,$$

we can write

$$\begin{aligned}
&\left\| K_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} \\
&\quad \le \frac{[n+p]_{q_n} q_n^{2\alpha+2}}{[n+1+\beta]_{q_n}^2} + \left| \frac{q_n^{2\alpha+2}[n+p]_{q_n}^2}{[n+1+\beta]_{q_n}^2} - 1 \right| + \frac{(3+2\alpha)[n+p]_{q_n}}{[n+1+\beta]_{q_n}^2} + \frac{(1+\alpha)^2}{[n+1+\beta]_{q_n}^2}
\end{aligned}$$

$$\leq \left| \frac{q_n^{2\alpha+2}[n+p]_{q_n}^2}{[n+1+\beta]_{q_n}^2} - 1 \right| + \frac{(4+2\alpha)}{[n]_{q_n}} \left(1 + \frac{[p]_{q_n}}{[n]_{q_n}} \right) + \frac{(1+\alpha)^2}{[n]_{q_n}^2}$$
$$=: \theta_n + \gamma_n + \eta_n.$$

Then, from (4.1), we have

$$st\text{-}\lim_n \theta_n = st\text{-}\lim_n \gamma_n = st\text{-}\lim_n \eta_n = 0. \tag{4.5}$$

Here, for given $\varepsilon > 0$, let us define the following sets:

$$T = \left\{ k : \left\| K_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} \geq \varepsilon \right\},$$
$$T_1 = \left\{ k : \theta_k \geq \frac{\varepsilon}{3} \right\}, \qquad T_2 = \left\{ k : \gamma_k \geq \frac{\varepsilon}{3} \right\}, \qquad T_3 = \left\{ k : \eta_k \geq \frac{\varepsilon}{3} \right\}.$$

It is clear that $T \subseteq T_1 \cup T_2 \cup T_3$. So we get

$$\delta\left\{ k \leq n : \left\| K_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} \geq \varepsilon \right\}$$
$$\leq \delta\left\{ k \leq n : \theta_k \geq \frac{\varepsilon}{3} \right\} + \delta\left\{ k \leq n : \gamma_k \geq \frac{\varepsilon}{3} \right\} + \delta\left\{ k \leq n : \eta_k \geq \frac{\varepsilon}{3} \right\}.$$

By condition (4.5), we have

$$\delta\left\{ k \leq n : \left\| K_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} \geq \varepsilon \right\} = 0,$$

which implies that

$$st\text{-}\lim_n \left\| K_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2 \right\|_{C[0,1]} = 0. \tag{4.6}$$

In view of (4.2), (4.4) and (4.6), the proof is complete. □

5 Rates of convergence

Let $f \in C[0,p+1]$ for any $t \in [0,p+1]$ and $x \in [0,1]$. Then we have $|f(t)-f(x)| \leq \omega(f,|t-x|)$, so for any $\delta > 0$, we get

$$\omega(f,|t-x|) \leq \begin{cases} \omega(f,\delta), & |t-x| < \delta, \\ \omega(f,\frac{(t-x)^2}{\delta}), & |t-x| \geq \delta. \end{cases}$$

Owing to $\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$ for $\lambda > 0$, it is obvious that we have

$$\left|f(t) - f(x)\right| \leq \left(1 + \delta^{-2}(t-x)^2\right)\omega(f,\delta) \tag{5.1}$$

for any $t \in [0,p+1]$, $x \in [0,1]$ and $\delta > 0$.

Now, we give the convergence rate of $K_{n,q}^{(\alpha,\beta)}(f;x)$ to the function $f \in C[0,p+1]$ in terms of the modulus of continuity.

Theorem 5.1 *Let $q = q_n$, $0 < q_n < 1$, be a sequence satisfying* (4.1), *then for any function $f \in C[0,p+1]$, $x \in [0,1]$, we have*

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \le 2\omega\Bigl(f, \sqrt{\delta_{n,q_n}^p(x)}\Bigr),$$

where $\delta_{n,q_n}^p(x)$ is given by (2.9).

Proof Using the linearity and positivity of the operator $K_{n,q}^{(\alpha,\beta)}(f;x)$ and inequality (5.1), for any $f \in C[0,p+1]$ and $x \in [0,1]$, we get

$$\begin{aligned}\left|K_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\le K_{n,q}^{(\alpha,\beta)}\bigl(\left|f(t) - f(x)\right|;x\bigr) \\ &\le \bigl(1 + \delta^{-2}K_{n,q}^{(\alpha,\beta)}\bigl((t-x)^2;x\bigr)\bigr)\omega(f,\delta). \end{aligned} \tag{5.2}$$

Take $q = q_n$, $0 < q_n < 1$, be a sequence satisfying condition (4.1) and choose $\delta = \sqrt{\delta_{n,q_n}^p(x)}$ in (5.2), the desired result follows immediately. □

Finally, we give the rate of convergence of $K_{n,q}^{(\alpha,\beta)}(f;x)$ with the help of functions of the Lipschitz class. We recall a function $f \in \mathrm{Lip}_M(\lambda)$ on $[0,p+1]$ if the inequality

$$\left|f(t) - f(x)\right| \le M|t-x|^\lambda, \quad t,x \in [0,p+1] \tag{5.3}$$

holds.

Theorem 5.2 *Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0,p+1]$, $0 < \lambda \le 1$. Let $q = q_n$, $0 < q_n < 1$ be a sequence satisfying the conditions given in* (4.1). *If we take $\delta_{n,q_n}^p(x)$ as in* (2.9), *then we have*

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \le M\bigl(\delta_{n,q_n}^p(x)\bigr)^{\lambda/2}, \quad x \in [0,1].$$

Proof Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0,p+1]$, $0 < \lambda \le 1$. Since $K_{n,q_n}^{(\alpha,\beta)}(f;x)$ is linear and positive, by using (5.3), we have

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \le K_{n,q_n}^{(\alpha,\beta)}\bigl(\left|f(t) - f(x)\right|;x\bigr) \le K_{n,q_n}^{(\alpha,\beta)}\bigl(|t-x|^\lambda;x\bigr).$$

If we take $p' = \frac{2}{\lambda}$, $q' = \frac{2}{2-\lambda}$ and apply the Hölder inequality, then we obtain

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \le M\bigl(K_{n,q_n}^{(\alpha,\beta)}\bigl((t-x)^2;x\bigr)\bigr)^{\lambda/2} \le M\bigl(\delta_{n,q_n}^p(x)\bigr)^{\lambda/2}.$$

□

Competing interests
The author declares that they have no competing interests.

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