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Approximation methods for solutions of generalized multi-valued mixed quasi-variational inclusion systems

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Dedicated to Professor Shih-sen Chang on his 80th birthday

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Abstract

The purpose of this paper is to introduce new approximation methods for solutions of generalized non-accretive multi-valued mixed quasi-variational inclusion systems involving (A,η)-accretive mappings in q-uniformly smooth Banach spaces and, by using the new resolvent operator technique associated with (A,η)-accretive mappings, Nadler's fixed point theorem and Liu's inequality, we prove some existence theorems of solutions for our systems by constructing the new Mann iterative algorithm. Further, we study the stability of the iterative sequence generated by the perturbed iterative algorithms. The results presented in this paper improve and generalize the corresponding results of recent works given by some authors.

Keywords: (A,η)-accretive mapping; resolvent operator technique; generalized nonlinear mixed quasi-variational inclusion system; new Mann iterative algorithm with mixed errors; convergence and stability

1 Introduction

It is well known that the ideas and techniques of the variational inequalities and variational inclusions are being applied in a variety of diverse fields of pure and applied sciences and proven to be productive and innovative. It has been shown that this theory provides the most natural, direct, simple, unified, and efficient framework for a general treatment of a wide class of linear and nonlinear problems. Correspondingly, the existence of solutions or the convergence and stability of a suitable iterative algorithm to the system of nonlinear variational inequalities or variational inclusions has also been studied by many authors, see [1–22] and the references therein.

Recently, Lan *et al.* [9] introduced a new concept of (A,η)-accretive mappings, which provides a unifying framework for maximal monotone operators, m-accretive operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, generalized m-accretive mappings, H-accretive operators, (H,η)-monotone operators, A-monotone mappings. Further, we studied some properties of (A,η)-accretive mappings and defined the resolvent operators associated with (A,η)-accretive mappings which include the existing resolvent operators as special cases. By using the new resolvent operator technique, we also developed a new perturbed iterative algorithm with errors to solve a class of nonlinear relaxed cocoercive variational inclusions with (A,η)-accretive mappings in q-uniformly smooth Banach spaces and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm. For details, we can refer to [2–5, 7, 8, 10, 11, 13, 23].

On the other hand, some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relations to Nash equilibrium problems. Huang and Fang [6] introduced a system of order complementarity problems and established some existence results for the problems by using fixed point theory. Kassay and Kolumbán [8] introduced a system of variational inequalities and proved an existence theorem by using Ky Fan's lemma. In [1], Cho *et al.* developed an iterative algorithm to approximate the solution of a system of nonlinear variational inequalities by using the classical resolvent operator technique. By using the resolvent operator technique associated with an (H,η)-monotone operator, Fang *et al.* [3] further studied the approximating solution of a system of variational inclusions in Hilbert spaces. Very recently, Guan and Hu [22] introduced and studied a system of generalized variational inclusions involving a new monotone mapping in Banach spaces. Furthermore, by using the concept of (A,η)-accretive mappings and the new resolvent operator technique associated with (A,η)-accretive mappings, Lan [15] introduced and studied a system of general mixed quasi-variational inclusions involving (A,η)-accretive mappings in Banach spaces, and construct a new perturbed iterative algorithm with mixed errors for this system of nonlinear (A,η)-accretive variational inclusions in q-uniformly smooth Banach spaces. Kazmi *et al.* [16] considered the convergence and stability of an iterative algorithm for a system of generalized implicit variational-like inclusions in Banach spaces. Suwannawit and Petrot [17] studied the existence of solutions and the stability of iterative algorithm for a system of random set-valued variational inclusion problems involving (A,m,η)-generalized monotone operators. Because stability is one of optimization theory, it is not surprising to see a number of papers dealing with the study of convergence and stability to investigate various important themes. For other related works, we refer to [10, 12, 18, 19, 24] and the references therein.

Motivated and inspired by the above works, in this paper, we consider the following *system of generalized non-accretive multi-valued mixed quasi-variational inclusions*:

Find $(x,y)\in\mathbb{B}_1\times\mathbb{B}_2$, $u\in F(x)$, and $v\in G(y)$ such that

$$\begin{cases} 0\in N_1(x,v)+M_1(x,x),\\ 0\in N_2(u,y)+M_2(y,y),\end{cases} \tag{1.1}$$

where $\mathbb{B}_i$ is a real Banach space, $N_i:\mathbb{B}_1\times\mathbb{B}_2\to\mathbb{B}_i$, $A_i:\mathbb{B}_i\to\mathbb{B}_i$, and $\eta_i:\mathbb{B}_i\times\mathbb{B}_i\to\mathbb{B}_i$ are single-valued mappings, and $F:\mathbb{B}_1\to 2^{\mathbb{B}_1}$ and $G:\mathbb{B}_2\to 2^{\mathbb{B}_2}$ are multi-valued mappings, $M_i:\mathbb{B}_i\times\mathbb{B}_i\to 2^{\mathbb{B}_i}$ is an any nonlinear mapping such that $M_i(\cdot,t):\mathbb{B}_i\to 2^{\mathbb{B}_i}$ is an (A_i,η_i)-accretive mapping for all $t\in\mathbb{B}_i$ and $i=1,2$.

We remark that, for suitable choices of N_i, M_i, A_i, η_i, F, G, and $\mathbb{B}_i$ for $i=1,2$, it is easy to see that the problem (1.1) includes a number (systems) of quasi-variational inclusions, generalized quasi-variational inclusions, quasi-variational inequalities, implicit quasi-variational inequalities studied by many authors as special cases. See, for example, [1–22] and the following examples:

Example 1.1 If $F : \mathbb{B}_1 \to \mathbb{B}_1$ and $G : \mathbb{B}_2 \to \mathbb{B}_2$ are two single-valued mappings, then, from the problem (1.1), we have the following problem: Find $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$ such that

$$\begin{cases} 0 \in N_1(x, G(y)) + M_1(x, x), \\ 0 \in N_2(F(x), y) + M_2(y, y). \end{cases} \tag{1.2}$$

Example 1.2 In (1.2), for any $(a, b) \in \mathbb{B}_1 \times \mathbb{B}_2$, if $N_1(x, G(y)) = E_1(f(x), y) - a$ and $N_2(x, G(y)) = E_2(x, g(y)) - b$, where $f : \mathbb{B}_1 \to \mathbb{B}_1$ and $g : \mathbb{B}_2 \to \mathbb{B}_2$ are two single-valued mappings, then the problem (1.2) reduces to finding $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$ such that

$$\begin{cases} a \in E_1(f(x), y) + M_1(x, x), \\ b \in E_2(x, g(y)) + M_2(y, y). \end{cases} \tag{1.3}$$

The problem (1.3) is called a *system of mixed quasi-variational inclusion problems*, which was studied by Lan [15].

Example 1.3 If $F = G = I$, $M_i(\cdot, t) = M_i(\cdot)$ for all $t \in \mathbb{B}_i$ and $M_2(\cdot, t) = M_2(\cdot)$ for all $t \in \mathbb{B}_2$, then the problem (1.2) is equivalent to the problem of finding $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$ such that

$$\begin{cases} 0 \in N_1(x, y) + M_1(x), \\ 0 \in N_2(x, y) + M_2(y), \end{cases} \tag{1.4}$$

which is studied by Fang *et al.* [3] and Verma [10] when M_i is A-monotone and (H, η)-monotone for $i = 1, 2$, respectively. Some special cases of the problem (1.4) can be found in [1, 8, 12–14] and the references therein.

Moreover, in this paper, by using the new resolvent operator technique associated with (A, η)-accretive mappings, Nadler's fixed point theorem and Liu's inequality, we prove some existence theorems of solutions for our systems by constructing the new Mann iterative algorithm. Further, we study the stability of the iterative sequence generated by the perturbed iterative algorithms. The results presented in this paper improve and generalize the corresponding results of recent works given by some authors.

2 Preliminaries

Let $\mathbb{B}$ be a real Banach space with the dual space $\mathbb{B}^*$, $\langle \cdot, \cdot \rangle$ be the dual pair between $\mathbb{B}$ and $\mathbb{B}^*$, $2^{\mathbb{B}}$ denote the family of all the nonempty subsets of $\mathbb{B}$ and $CB(\mathbb{B})$ denote the family of all nonempty closed bounded subsets of $\mathbb{B}$. The *generalized duality mapping* $J_q : \mathbb{B} \to 2^{\mathbb{B}^*}$ is defined by

$$J_q(x) = \{ f^* \in \mathbb{B}^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \}$$

for all $x \in \mathbb{B}$, where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if $\mathbb{B}$ is strictly convex.

In the sequel, we always suppose that $\mathbb{B}$ is a real Banach space such that J_q is single-valued and $\mathcal{H}$ is a Hilbert space. If $\mathbb{B} = \mathcal{H}$, then J_2 becomes the identity mapping on $\mathcal{H}$.

The *modulus of smoothness* of $\mathbb{B}$ is the function $\varrho_{\mathbb{B}} : [0,\infty) \to [0,\infty)$ defined by

$$\varrho_{\mathbb{B}}(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

(1) A Banach space $\mathbb{B}$ is said to be *uniformly smooth* if

$$\lim_{t\to 0}\frac{\varrho_{\mathbb{B}}(t)}{t} = 0.$$

(2) $\mathbb{B}$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\varrho_{\mathbb{B}}(t) \leq ct^q$$

for all $q > 1$.

Note that J_q is single-valued if $\mathbb{B}$ is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [25] proved the following result.

Lemma 2.1 *Let $q > 1$ be a given real number and $\mathbb{B}$ be a real uniformly smooth Banach space. Then $\mathbb{B}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that, for all $x, y \in \mathbb{B}$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

In the sequel, we give some concept and lemmas for our main results later.

Definition 2.1 Let $\mathbb{B}$ be a q-uniformly smooth Banach space and $f, A : \mathbb{B} \to \mathbb{B}$ be two single-valued mappings. T is said to be:

(1) *accretive* if

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq 0$$

for all $x, y \in \mathbb{B}$;

(2) *strictly accretive* if T is accretive and

$$\langle f(x) - f(y), J_q(x-y)\rangle = 0$$

if and only if $x = y$;

(3) *r-strongly accretive* if there exists a constant $r > 0$ such that

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq r\|x-y\|^q$$

for all $x, y \in \mathbb{B}$;

(4) *γ-strongly accretive* with respect to A if there exists a constant $\gamma > 0$ such that

$$\langle f(x) - f(y), J_q\big(A(x) - A(y)\big)\rangle \geq \gamma\|x-y\|^q$$

for all $x, y \in \mathbb{B}$;

(5) *m-relaxed cocoercive* with respect to A if there exists a constant $m > 0$ such that

$$\langle f(x) - f(y), J_q(A(x) - A(y)) \rangle \geq -m \| f(x) - f(y) \|^q$$

for all $x, y \in \mathbb{B}$;

(6) (α, ξ)*-relaxed cocoercive* with respect to A if there exist constants $\alpha, \xi > 0$ such that

$$\langle f(x) - f(y), J_q(A(x) - A(y)) \rangle \geq -\alpha \| f(x) - f(y) \|^q + \xi \|x - y\|^q$$

for all $x, y \in \mathbb{B}$;

(7) *s-Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\| f(x) - f(y) \| \leq s \|x - y\|$$

for all $x, y \in \mathbb{B}$.

Remark 2.1 When $\mathbb{B} = \mathcal{H}$, (1)-(4) of Definition 2.1 reduce to the definitions of monotonicity, strict monotonicity, strong monotonicity, and strong monotonicity with respect to A, respectively (see [2, 3]).

Definition 2.2 A multi-valued mapping $F : \mathbb{B} \to 2^{\mathbb{B}}$ is said to be ζ-$\hat{\mathbf{H}}$*-Lipschitz continuous* if there exists a constant $\zeta > 0$ such that

$$\hat{\mathbf{H}}(F(x), F(y)) \leq \zeta \|x - y\|$$

for all $x, y \in \mathbb{B}$, where $\hat{\mathbf{H}} : 2^{\mathbb{B}} \times 2^{\mathbb{B}} \to (-\infty, +\infty) \cup \{+\infty\}$ is the *Hausdorff metric*, *i.e.*,

$$\hat{\mathbf{H}}(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{x \in B} \inf_{y \in A} \|x - y\| \right\}$$

for all $A, B \in 2^{\mathbb{B}}$.

Definition 2.3 A single-valued mapping $\eta : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is said to be τ*-Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\| \eta(x, y) \| \leq \tau \|x - y\|$$

for all $x, y \in \mathbb{B}$.

Definition 2.4 Let $\mathbb{B}$ be a q-uniformly smooth Banach space, $\eta : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ and $H : \mathbb{B} \to \mathbb{B}$ be single-valued mappings. Then set-valued mapping $M : \mathbb{B} \to 2^{\mathbb{B}}$ is said to be:

(1) *accretive* if

$$\langle u - v, J_q(x - y) \rangle \geq 0$$

for all $x, y \in \mathbb{B}$, $u \in M(x)$, and $v \in M(y)$;

(2) *η-accretive* if

$$\langle u-v, J_q(\eta(x,y))\rangle \geq 0$$

for all $x, y \in \mathbb{B}$, $u \in M(x)$, and $v \in M(y)$;

(3) *strictly η-accretive* if M is η-accretive and equality holds if and only if $x = y$;

(4) *r-strongly η-accretive* if there exists a constant $r > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq r\|x-y\|^q$$

for all $x, y \in \mathbb{B}$, $u \in M(x)$, and $v \in M(y)$;

(5) α-relaxed η-accretive if there exists a constant $\alpha > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \geq -\alpha\|x-y\|^q$$

for all $x, y \in \mathbb{B}$, $u \in M(x)$, and $v \in M(y)$;

(6) *m-accretive* if M is accretive and $(I + \rho M)(\mathbb{B}) = \mathbb{B}$ for all $\rho > 0$, where I denotes the identity operator on $\mathbb{B}$;

(7) *generalized m-accretive* if M is η-accretive and $(I + \rho M)(\mathbb{B}) = \mathbb{B}$ for all $\rho > 0$;

(8) *H-accretive* if M is accretive and $(H + \rho M)(\mathbb{B}) = \mathbb{B}$ for all $\rho > 0$;

(9) *(H, η)-accretive* if M is η-accretive and $(H + \rho M)(\mathbb{B}) = \mathbb{B}$ for every $\rho > 0$.

In a similar way, we can define strictly η-accretivity and strongly η-accretivity of the single-valued mapping $A : \mathbb{B} \to \mathbb{B}$.

Definition 2.5 The mapping $N : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is said to be *ϵ-Lipschitz continuous* with respect to the first argument if there exists a constant $\epsilon > 0$ such that

$$\|N(x,\cdot) - N(y,\cdot)\| \leq \epsilon\|x-y\|$$

for all $x, y \in \mathbb{B}$.

In a similar way, we can define the Lipschitz continuity of the mapping $N(\cdot,\cdot)$ with respect to the second argument.

Definition 2.6 Let $A : \mathbb{B} \to \mathbb{B}$, $\eta : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be two single-valued mappings. Then a multi-valued mapping $M : \mathbb{B} \to 2^{\mathbb{B}}$ is said to be *(A, η)-accretive* if

(1) M is m-relaxed η-accretive;

(2) $(A + \rho M)(\mathbb{B}) = \mathbb{B}$ for all $\rho > 0$.

Lemma 2.2 ([9]) *Let $\mathbb{B}$ be a q-uniformly smooth Banach space and $\eta : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be τ-Lipschitz continuous, $A : \mathbb{B} \to \mathbb{B}$ be a r-strongly η-accretive mapping and $M : \mathbb{B} \to 2^{\mathbb{B}}$ be an (A, η)-accretive mapping. Then the resolvent operator $R^{\rho,A}_{\eta,M} : \mathbb{B} \to \mathbb{B}$ defined by*

$$R^{\rho,A}_{\eta,M}(u) = (A + \rho M)^{-1}(u)$$

for all $u \in \mathbb{B}$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,

$$\left\|R_{\eta,M}^{\rho,A}(x) - R_{\eta,M}^{\rho,A}(y)\right\| \le \frac{\tau^{q-1}}{r-\rho m}\|x-y\|$$

for all $x, y \in \mathbb{B}$, where $\rho \in (0, \frac{r}{m})$ is a constant.

3 Approximation methods and main results

In this section, by using the resolvent operator technique associated with (A,η)-accretive mappings, we introduce the new Mann iterative algorithm with mixed errors for solving the system (1.1) of generalized nonlinear mixed quasi-variational inclusion in Banach spaces and prove the convergence and stability of the iterative sequence generated by the Mann iterative algorithm.

Definition 3.1 Let S be a self-mapping of $\mathbb{B}$, $x_0 \in X$ and let $\{x_n\}$ be an iterative sequence in $\mathbb{B}$ defined by $x_{n+1} = h(S, x_n)$ for all $n \ge 0$. Suppose that $\{x \in \mathbb{B} : Sx = x\} \ne \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of S. Let $\{\upsilon_n\}$ be a sequence in $\mathbb{B}$ and let $\epsilon_n = \|\upsilon_{n+1} - h(S, \upsilon_n)\|$. If $\lim \epsilon_n = 0$ implies that $\upsilon_n \to x^*$, then the iterative sequence $\{x_n\}$ defined by $x_{n+1} = h(S, x_n)$ for all $n \ge 0$ is said to be *S-stable* or *stable* with respect to S.

Lemma 3.1 ([26]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \le (1-t_n)a_n + b_n t_n + c_n$$

for all $n \ge n_0$, where $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

The solvability of the problem (1.1) depends on the equivalence between (1.1) and the problem of finding the fixed point of the associated generalized resolvent operator. From Definition 2.6, we can obtain the following.

Lemma 3.2 *For $i = 1, 2$, let A_i, η_i, M_i, N_i, F, and G be the same as in the problem* (1.1). *Then the following statements are mutually equivalent:*

(1) *An element $(x, y, u, v) \in \mathbb{B}_1 \times \mathbb{B}_2 \times \mathbb{B}_1 \times \mathbb{B}_2$ is a solution to the problem* (1.1).

(2) *There exist $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$, $u \in F(x)$, and $v \in G(y)$ such that*

$$\begin{aligned} x &= R_{\eta_1, M_1(\cdot,x)}^{\lambda, A_1}\bigl[A_1(x) - \lambda N_1(x,v)\bigr], \\ y &= R_{\eta_2, M_2(\cdot,y)}^{\rho, A_2}\bigl[A_2(y) - \rho N_2(u,y)\bigr], \end{aligned} \tag{3.1}$$

where $R_{\eta_1, M_1(\cdot,x)}^{\lambda, A_1} = (A_1 + \lambda M_1(\cdot, x))^{-1}$, $R_{\eta_2, M_2(\cdot,y)}^{\rho, A_2} = (A_2 + \rho M_2(\cdot, y))^{-1}$, and $\lambda > 0$, $\rho > 0$ are two constants.

(3) *For any $\lambda > 0$ and $\rho > 0$, the mapping $T_{\lambda,\rho} : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1 \times \mathbb{B}_2$ defined by*

$$T_{\lambda,\rho}(x,y) = \bigl(P_\lambda(x,y), Q_\rho(x,y)\bigr)$$

for all $(x,y) \in \mathbb{B}_1 \times \mathbb{B}_2$ *has a fixed point* $(x,y,u,v) \in \mathbb{B}_1 \times \mathbb{B}_2 \times \mathbb{B}_1 \times \mathbb{B}_2$, *where mappings* $P_\lambda : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1$ *and* $Q_\rho : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_2$ *are defined by*

$$P_\lambda(x,y) = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x)}\big[A_1(x) - \lambda N_1(x,v)\big]$$

for all $v \in G(y)$ *and*

$$Q_\rho(x,y) = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y)}\big[A_2(y) - \rho N_2(u,y)\big]$$

for all $u \in F(x)$, *respectively.*

This fixed point formulation allows us to construct the following perturbed iterative algorithm with mixed errors.

Algorithm 3.1 Step 1. For any $(x_0,y_0) \in \mathbb{B}_1 \times \mathbb{B}_2$, define the iterative sequence $\{(x_n,y_n)\}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_n)}(z_n) + \alpha_n d_n + e_n, \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_n)}(w_n) + \alpha_n f_n + h_n, \\ z_n = A_1(x_n) - \lambda N_1(x_n,v_n), \\ w_n = A_2(y_n) - \rho N_2(u_n,y_n) \end{cases} \tag{3.2}$$

for all $n \geq 0$, where $u_n \in F(x_n)$, $v_n \in G(x_n)$, and $\lambda, \rho > 0$ are constants.

Step 2. Choose the sequences $\{\alpha_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ such that, for all $n \geq 0$, $\{\alpha_n\}$ is a sequence in $(0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{d_n\}, \{e_n\} \subset \mathbb{B}_1$, and $\{f_n\}, \{h_n\} \subset \mathbb{B}_2$ are the sequences of errors and satisfy the following conditions:

(a) $d_n = d'_n + d''_n$ and $f_n = f'_n + f''_n$, where $\{d'_n\}, \{d''_n\} \subset \mathbb{B}_1$ and $\{f'_n\}, \{f''_n\} \subset \mathbb{B}_2$;

(b) $\lim_{n\to\infty} \|d'_n\| = 0$ and $\lim_{n\to\infty} \|f'_n\| = 0$;

(c) $\sum_{n=0}^{\infty} \|d''_n\| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\sum_{n=0}^{\infty} \|f''_n\| < \infty$ and $\sum_{n=0}^{\infty} \|h_n\| < \infty$.

Step 3. If the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{\alpha_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ satisfy (3.2) to sufficient accuracy, then go to Step 4. Otherwise, set $n := n+1$ and return to Step 1.

Step 4. Let $\{(\varphi_n, \psi_n)\}$ be any sequence in $\mathbb{B}_1 \times \mathbb{B}_2$ and define a sequence $\{(\epsilon_n, \varepsilon_n)\}$ in $\mathbb{R} \times \mathbb{R}$ by

$$\begin{cases} \epsilon_n = \|\varphi_{n+1} - \{(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n\}\|, \\ \varepsilon_n = \|\psi_{n+1} - \{(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n\}\|, \\ s_n = A_1(\varphi_n) - \lambda N_1(\varphi_n, \varpi_n), \\ t_n = A_2(\psi_n) - \rho N_2(\chi_n, \psi_n), \end{cases} \tag{3.3}$$

where $\chi_n \in F(\varphi_n)$ and $\varpi_n \in G(\psi_n)$.

Step 5. If the sequences $\{\epsilon_n\}$, $\{\varepsilon_n\}$, $\{\varphi_{n+1}\}$, $\{\psi_{n+1}\}$, $\{s_n\}$, $\{t_n\}$, $\{\alpha_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ satisfy (3.3) to sufficient accuracy, the stop here. otherwise, set $n := n+1$ and return to Step 2.

Now, we show the existence of solutions of the problem (1.1) and prove the convergence and stability of Algorithm 3.1.

Theorem 3.1 *For $i = 1, 2$, let $\mathbb{B}_i$ be a q_i-uniformly smooth Banach space with $q_i > 1$, $\eta_i : \mathbb{B}_i \times \mathbb{B}_i \to \mathbb{B}_i$ be τ_i-Lipschitz continuous, $A_i : \mathbb{B}_i \to \mathbb{B}_i$ be r_i-strongly η_i-accretive and σ_i-Lipschitz continuous, $F : \mathbb{B}_1 \to CB(\mathbb{B}_1)$ be k-$\hat{\mathbf{H}}$-Lipschitz continuous, $G : \mathbb{B}_2 \to CB(\mathbb{B}_2)$ be κ-$\hat{\mathbf{H}}$-Lipschitz continuous, $M_1 : \mathbb{B}_1 \times \mathbb{B}_1 \to 2^{\mathbb{B}_1}$ be (A_1, η_1)-accretive in the first variable and $M_2 : \mathbb{B}_2 \times \mathbb{B}_2 \to 2^{\mathbb{B}_2}$ be (A_2, η_2)-accretive in the first variable. Suppose that $N_1 : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1$ is (π_1, ι_1)-relaxed cocoercive with respect to A_1, δ_1-Lipschitz continuous in the first argument, β_2-Lipschitz continuous in the second variable and $N_2 : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_2$ is (π_2, ι_2)-relaxed cocoercive with respect to A_2, δ_2-Lipschitz continuous in the second argument and β_1-Lipschitz continuous in the first variable. If*

$$\begin{aligned} &\left\| R^{\lambda, A_1}_{\eta_1, M_1(\cdot, x)}(z) - R^{\lambda, A_1}_{\eta_1, M_1(\cdot, y)}(z) \right\| \le \nu_1 \|x - y\|, \\ &\left\| R^{\rho, A_2}_{\eta_2, M_2(\cdot, x)}(z) - R^{\rho, A_2}_{\eta_2, M_2(\cdot, y)}(z) \right\| \le \nu_2 \|x - y\| \end{aligned} \tag{3.4}$$

for all $(x, y, z) \in \mathbb{B}_2 \times \mathbb{B}_2 \times \mathbb{B}_2$ and there exist constants $\lambda \in (0, r_1/m_1)$ and $\rho \in (0, r_2/m_2)$ such that

$$\begin{cases} \frac{\tau_1^{q_1 - 1} \sqrt[q_1]{\sigma_1^{q_1} - q_1 \lambda \iota_1 + q_1 \lambda \pi_1 \delta_1^{q_1} + c_{q_1} \lambda^{q_1} \delta_1^{q_1}}}{r_1 - \lambda m_1} + \frac{\rho k \beta_1 \tau_2^{q_2 - 1}}{r_2 - \rho m_2} + \nu_1 < 1, \\ \frac{\tau_2^{q_2 - 1} \sqrt[q_2]{\sigma_2^{q_2} - q_2 \rho \iota_2 + q_2 \rho \pi_2 \delta_2^{q_2} + c_{q_2} \rho^{q_2} \delta_2^{q_2}}}{r_2 - \rho m_2} + \frac{\lambda \kappa \beta_2 \tau_1^{q_1 - 1}}{r_1 - \lambda m_1} + \nu_2 < 1, \end{cases} \tag{3.5}$$

where c_{q_1}, c_{q_2} are the constants as in Lemma 2.1, then

(1) *the problem (1.1) has a solution (x^*, y^*, u^*, v^*);*
(2) *the iterative sequence $\{(x_n, y_n, u_n, v_n)\}$ generated by Algorithm 3.1 converges strongly to the solution (x^*, y^*, u^*, v^*);*
(3) *if, in addition, there exists $\alpha > 0$ such that $\alpha_n \ge \alpha$ for all $n \ge 0$, then*

$$\lim_{n \to \infty} (\varphi_n, \psi_n, \chi_n, \varpi_n) = (x^*, y^*, u^*, v^*) \quad \Longleftrightarrow \quad \lim_{n \to \infty} (\epsilon_n, \varepsilon_n) = (0, 0),$$

where $(\epsilon_n, \varepsilon_n)$ is defined by (3.3).

Proof For any $\lambda > 0$ and $\rho > 0$, define $P_\lambda : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1$ and $Q_\rho : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_2$ by

$$\begin{cases} P_\lambda(x, y) = R^{\lambda, A_1}_{\eta_1, M_1(\cdot, x)}[A_1(x) - \lambda N_1(x, v)], \\ Q_\rho(x, y) = R^{\rho, A_2}_{\eta_2, M_2(\cdot, y)}[A_2(y) - \rho N_2(u, y)] \end{cases} \tag{3.6}$$

for all $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$, $u \in F(x)$ and $v \in G(v)$. Now, define the norm $\| \cdot \|_*$ on $\mathbb{B}_1 \times \mathbb{B}_2$ by

$$\left\| (x, y) \right\|_* = \|x\| + \|y\|$$

for all $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$. It is easy to see that $(\mathbb{B}_1 \times \mathbb{B}_2, \| \cdot \|_*)$ is a Banach space (see [4]). By (3.6), for any $\lambda > 0$ and $\rho > 0$, define $T_{\lambda, \rho} : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1 \times \mathbb{B}_2$ by

$$T_{\lambda, \rho}(x, y) = \big(P_\lambda(x, y), Q_\rho(x, y)\big)$$

for all $(x, y) \in \mathbb{B}_1 \times \mathbb{B}_2$.

Now, we prove that $T_{\lambda,\rho}$ is a contractive mapping. In fact, for any $(x_i, y_i) \in \mathbb{B}_1 \times \mathbb{B}_2$ and $i = 1, 2$, there exist $u_1 \in F(x_1)$ and $v_1 \in G(y_1)$ such that

$$\begin{cases} P_\lambda(x_1, y_1) = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_1)}[A_1(x_1) - \lambda N_1(x_1, v_1)], \\ Q_\rho(x_1, y_1) = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_1)}[A_2(y_1) - \rho N_2(u_1, y_1)]. \end{cases} \tag{3.7}$$

Since $F(x_2) \in CB(\mathbb{B}_1)$ and $G(y_2) \in CB(\mathbb{B}_2)$, it follows from Nadler's result [27] that there exist $u_2 \in F(x_2)$ and $v_2 \in G(y_2)$ such that

$$\|u_1 - u_2\| \le \hat{\mathbf{H}}\big\|F(x_1) - F(x_2)\big\|, \qquad \|v_1 - v_2\| \le \hat{\mathbf{H}}\big\|G(y_1) - G(y_2)\big\|. \tag{3.8}$$

Setting

$$\begin{cases} P_\lambda(x_2, y_2) = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_2)}[A_1(x_2) - \lambda N_1(x_2, v_2)], \\ Q_\rho(x_2, y_2) = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_2)}[A_2(y_2) - \rho N_2(u_2, y_2)]. \end{cases} \tag{3.9}$$

Thus it follows from (3.4), (3.6), (3.9), and Lemma 2.2 that

$$\begin{aligned} &\big\|P_\lambda(x_1, y_1) - P_\lambda(x_2, y_2)\big\| \\ &\quad \le \big\|R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_1)}\big[A_1(x_1) - \lambda N_1(x_1, v_1)\big] - R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_2)}\big[A_1(x_1) - \lambda N_1(x_1, v_1)\big]\big\| \\ &\qquad + \big\|R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_2)}\big[A_1(x_1) - \lambda N_1(x_1, v_1)\big] - R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_2)}\big[A_1(x_2) - \lambda N_1(x_2, v_2)\big]\big\| \\ &\quad \le \frac{\tau_1^{q_1-1}}{r_1 - \lambda m_1}\big\|A_1(x_1) - A_1(x_2) - \lambda\big(N_1(x_1, v_1) - N_1(x_2, v_1)\big)\big\| \\ &\qquad + \nu_1\|x_1 - x_2\| + \frac{\lambda\tau_1^{q_1-1}}{r_1 - \lambda m_1}\big\|N_1(x_2, v_1) - N_1(x_2, v_2)\big\| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\big\|Q_\rho(x_1, y_1) - Q_\rho(x_2, y_2)\big\| \\ &\quad \le \frac{\tau_2^{q_2-1}}{r_2 - \rho m_2}\big\|A_2(y_1) - A_2(y_2) - \rho\big(N_2(u_1, y_1) - N_2(u_1, y_2)\big)\big\| \\ &\qquad + \nu_2\|y_1 - y_2\| + \frac{\rho\tau_2^{q_2-1}}{r_2 - \rho m_2}\big\|N_2(u_1, y_2) - N_2(u_2, y_2)\big\|. \end{aligned} \tag{3.11}$$

By the assumptions, (3.8), and Lemma 2.1, we have

$$\begin{aligned} &\big\|A_1(x_1) - A_1(x_2) - \lambda\big(N_1(x_1, v_1) - N_1(x_2, v_1)\big)\big\|^{q_1} \\ &\quad \le \big\|A_1(x_1) - A_1(x_2)\big\|^{q_1} - q_1\lambda\big\langle N_1(x_1, v_1) - N_1(x_2, v_1), J_{q_1}\big(A_1(x_1) - A_1(x_2)\big)\big\rangle \\ &\qquad + \lambda^{q_1}c_{q_1}\big\|N_1(x_1, v_1) - N_1(x_2, v_1)\big\|^{q_1} \\ &\quad \le \big(\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}\big)\|x_1 - x_2\|^{q_1}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\big\|A_2(y_1) - A_2(y_2) - \rho\big(N_2(u_1, y_1) - N_2(u_1, y_2)\big)\big\|^{q_2} \\ &\quad \le \big(\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}\big)\|y_1 - y_2\|^{q_2}, \end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\|N_1(x_2, v_1) - N_1(x_2, v_2)\| \\
&\quad \le \beta_2 \|v_1 - v_2\| \le \beta_2 \hat{\mathbf{H}} \|G(y_1) - G(y_2)\| \le \kappa \beta_2 \|y_1 - y_2\|, \qquad (3.14)\\
&\|N_2(u_1, y_2) - N_2(u_2, y_2)\| \\
&\quad \le \beta_1 \|u_1 - u_2\| \le \beta_1 \hat{\mathbf{H}} \|F(x_1) - F(x_2)\| \le k \beta_1 \|x_1 - x_2\|. \qquad (3.15)
\end{aligned}$$

Combining (3.10)-(3.15), we infer

$$\begin{cases}
\|P_\lambda(x_1, y_1) - P_\lambda(x_2, y_2)\| \\
\quad \le (\nu_1 + \frac{\tau_1^{q_1-1}}{r_1-\lambda m_1} \sqrt[q_1]{\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}})\|x_1 - x_2\| \\
\qquad + \frac{\lambda\kappa\beta_2\tau_1^{q_1-1}}{r_1-\lambda m_1}\|y_1 - y_2\|, \\
\|Q_\rho(x_1, y_1) - Q_\rho(x_2, y_2)\| \\
\quad \le (\nu_2 + \frac{\tau_2^{q_2-1}}{r_2-\rho m_2} \sqrt[q_2]{\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}})\|y_1 - y_2\| \\
\qquad + \frac{\rho k\beta_1\tau_2^{q_2-1}}{r_2-\rho m_2}\|x_1 - x_2\|.
\end{cases} \qquad (3.16)$$

It follows from (3.16) that

$$\begin{aligned}
&\|P_\lambda(x_1, y_1) - P_\lambda(x_2, y_2)\| + \|Q_\rho(x_1, y_1) - Q_\rho(x_2, y_2)\| \\
&\quad \le \theta(\|x_1 - x_2\| + \|y_1 - y_2\|), \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
\theta = \max\biggl\{ &\frac{\tau_1^{q_1-1}}{r_1 - \lambda m_1} \sqrt[q_1]{\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}} + \frac{\rho k\beta_1\tau_2^{q_2-1}}{r_2 - \rho m_2} + \nu_1, \\
&\frac{\tau_2^{q_2-1}}{r_2 - \rho m_2} \sqrt[q_2]{\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}} + \frac{\lambda\kappa\beta_2\tau_1^{q_1-1}}{r_1 - \lambda m_1} + \nu_2 \biggr\}.
\end{aligned}$$

By (3.5), we know that $0 \le \theta < 1$ and it follows from (3.17) that

$$\|T_{\lambda,\rho}(x_1, y_1) - T_{\lambda,\rho}(x_2, y_2)\|_* \le \theta \|(x_1, y_1) - (x_2, y_2)\|_*.$$

This proves that $T_{\lambda,\rho} : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1 \times \mathbb{B}_2$ is a contraction mapping. Thus, from Nadler's fixed point theorem [27], it follows that there exist $(x^*, y^*) \in \mathbb{B}_1 \times \mathbb{B}_2$, $u^* \in F(x^*)$ and $v^* \in G(y^*)$ such that

$$T_{\lambda,\rho}(x^*, y^*) = (x^*, y^*),$$

that is,

$$x^* = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x^*)}[A_1(x^*) - \lambda N_1(x^*, v^*)], \qquad y^* = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y^*)}[A_2(y^*) - \rho N_2(u^*, y^*)].$$

Hence, by Lemma 3.2, (x^*, y^*) is a solution of the problem (1.1).

Next, for any $u^* \in F(x^*)$ and $v^* \in G(y^*)$, let

$$z^* = A_1(x^*) - \lambda N_1(x^*, v^*), \qquad w^* = A_2(y^*) - \rho N_2(u^*, y^*),$$

$$x^* = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x^*)}\bigl(z^*\bigr), \qquad y^* = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y^*)}\bigl(w^*\bigr).$$

Then, by (3.2) and the proof of (3.16), it follows that

$$\begin{aligned}
\|z_n - z^*\| &= \|A_1(x_n) - A_1(x^*) - \lambda(N_1(x_n,v_n) - N_1(x^*,v^*))\| \\
&\le \sqrt[q_1]{\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}}\,\|x_n - x^*\| + \lambda\kappa\beta_2\|y_n - y^*\|, \\
\|w_n - w^*\| &= \|A_2(y_n) - A_2(y^*) - \rho(N_2(x_n,y_n) - N_2(x^*,y^*))\| \\
&\le \sqrt[q_2]{\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}}\,\|y_n - y^*\| + \rho k\beta_1\|x_n - x^*\|,
\end{aligned}$$

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(\|d_n'\| + \|d_n''\|) + \|e_n\| + \alpha_n\bigl\|R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_n)}(z_n) - x^*\bigr\| \\
&\quad\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(\|d_n'\| + \|d_n''\|) + \|e_n\| \\
&\qquad + \alpha_n\bigl\|R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_n)}(z_n) - R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x^*)}(z_n)\bigr\| + \alpha_n\bigl\|R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x^*)}(z_n) - R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x^*)}(z^*)\bigr\| \\
&\quad\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\nu_1\|x_n - x^*\| + \alpha_n\frac{\tau_1^{q_1-1}}{r_1 - \lambda m_1}\|z_n - z^*\| \\
&\qquad + \alpha_n\|d_n'\| + (\|d_n''\| + \|e_n\|),
\end{aligned}$$

and

$$\begin{aligned}
&\|y_{n+1} - y^*\| \\
&\quad\le (1-\alpha_n)\|y_n - y^*\| + \alpha_n(\|f_n'\| + \|f_n''\|) + \|h_n\| + \alpha_n\bigl\|R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_n)}(w_n) - y^*\bigr\| \\
&\quad\le (1-\alpha_n)\|y_n - y^*\| + \alpha_n(\|f_n'\| + \|f_n''\|) + \|h_n\| \\
&\qquad + \alpha_n\bigl\|R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_n)}(w_n) - R^{\rho,A_2}_{\eta_2,M_2(\cdot,y^*)}(w_n)\bigr\| + \alpha_n\bigl\|R^{\rho,A_2}_{\eta_2,M_2(\cdot,y^*)}(w_n) - R^{\rho,A_2}_{\eta_2,M_2(\cdot,y^*)}(w^*)\bigr\| \\
&\quad\le (1-\alpha_n)\|y_n - y^*\| + \alpha_n\nu_2\|y_n - y^*\| + \alpha_n\frac{\tau_2^{q_2-1}}{r_2 - \rho m_2}\|w_n - w^*\| \\
&\qquad + \alpha_n\|f_n'\| + (\|f_n''\| + \|h_n\|).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad\le (1-\alpha_n)(\|x_n - x^*\| + \|y_n - y^*\|) \\
&\qquad + \alpha_n\biggl[\nu_1 + \frac{\rho k\beta_1\tau_2^{q_2-1}}{r_2 - \rho m_2} + \frac{\tau_1^{q_1-1}}{r_1 - \lambda m_1}\sqrt[q_1]{\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}}\biggr]\|x_n - x^*\| \\
&\qquad + \alpha_n\biggl[\nu_2 + \frac{\lambda\kappa\beta_2\tau_1^{q_1-1}}{r_1 - \lambda m_1} + \frac{\tau_2^{q_2-1}}{r_2 - \rho m_2}\sqrt[q_2]{\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}}\biggr] \\
&\qquad \times \|y_n - y^*\| + \alpha_n(\|f_n'\| + \|d_n'\|) + (\|f_n''\| + \|h_n\| + \|d_n''\| + \|e_n\|) \\
&\quad\le [1 - \alpha_n(1-\theta)](\|x_n - x^*\| + \|y_n - y^*\|) \\
&\qquad + \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|f_n'\| + \|d_n'\|) + (\|f_n''\| + \|h_n\| + \|d_n''\| + \|e_n\|).
\end{aligned} \tag{3.18}$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, it follows from Lemma 3.1, (3.5), and (3.18) that

$$\|x_n - x^*\| + \|y_n - y^*\| \to 0$$

as $n \to \infty$. Further, by $u_n \in F(x_n)$, $u^* \in F(x^*)$, $v_n \in G(y_n)$, $v^* \in G(y^*)$, and the $\hat{\mathbf{H}}$-Lipschitz continuity of F and G, we obtain

$$\|u_n - u^*\| \le \hat{\mathbf{H}}\|F(x_n) - F(x^*)\| \le k\|x_n - x^*\|$$

and

$$\|v_n - v^*\| \le \hat{\mathbf{H}}\|G(y_n) - G(y^*)\| \le \kappa\|y_n - y^*\|.$$

Thus we know that the sequence $\{(x_n, y_n, u_n, v_n)\}$ converges to a solution (x^*, y^*, u^*, v^*) of the problem (1.1).

Now, we prove the conclusion (3). By (3.3), we know

$$\begin{cases} \|\varphi_{n+1} - x^*\| \le \|(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n - x^*\| + \epsilon_n, \\ \|\psi_{n+1} - y^*\| \le \|(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n - y^*\| + \varepsilon_n. \end{cases} \tag{3.19}$$

As in the proof of the inequality (3.18), we have

$$\begin{aligned} &\|(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n - x^*\| \\ &\quad + \|(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n - y^*\| \\ &\quad \le [1 - \alpha_n(1-\theta)](\|\varphi_n - x^*\| + \|\psi_n - y^*\|) \\ &\qquad + \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|d'_n\| + \|f'_n\|) + (\|d''_n\| + \|e_n\| + \|f''_n\| + \|h_n\|). \end{aligned} \tag{3.20}$$

Since $0 < \alpha \le \alpha_n$, it follows from (3.17) and (3.18) that

$$\begin{aligned} &\|\varphi_{n+1} - x^*\| + \|\psi_{n+1} - y^*\| \\ &\quad \le [1 - \alpha_n(1-\theta)](\|\varphi_n - x^*\| + \|\psi_n - y^*\|) \\ &\qquad + \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}\left(\|d'_n\| + \|f'_n\| + \frac{\epsilon_n + \varepsilon_n}{\alpha}\right) + (\|d''_n\| + \|f''_n\| + \|e_n\| + \|h_n\|). \end{aligned}$$

Suppose that $\lim_{n\to\infty}(\epsilon_n, \varepsilon_n) = (0,0)$. Then, from $\sum_{n=0}^{\infty}\alpha_n = \infty$ and Lemma 3.1, it follows that

$$\|\varphi_n - x^*\| + \|\psi_n - y^*\| \to 0$$

as $n \to \infty$. Further, from $\chi_n \in F(\varphi_n)$, $u^* \in F(x^*)$, $\varpi_n \in G(\psi_n)$, $v^* \in G(y^*)$, and the $\hat{\mathbf{H}}$-Lipschitz continuity of F and G, we have

$$\|\chi_n - u^*\| \le \hat{\mathbf{H}}\|F(\varphi_n) - F(x^*)\| \le k\|\varphi_n - x^*\|$$

and

$$\|\varpi_n - v^*\| \leq \hat{\mathbf{H}}\|G(\psi_n) - G(y^*)\| \leq \kappa\|\psi_n - y^*\|.$$

Hence we know that $\lim_{n\to\infty}(\varphi_n, \psi_n, \chi_n, \varpi_n) = (x^*, y^*, u^*, v^*)$.

Conversely, if $\lim_{n\to\infty}(\varphi_n, \psi_n, \chi_n, \varpi_n) = (x^*, y^*, u^*, v^*)$, then we have

$$\begin{aligned}
\epsilon_n &= \bigl\|\varphi_{n+1} - \bigl\{(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n\bigr\}\bigr\| \\
&\leq \|\varphi_{n+1} - x^*\| + \bigl\|(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n - x^*\bigr\|, \\
\varepsilon_n &= \bigl\|\psi_{n+1} - \bigl\{(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n\bigr\}\bigr\| \\
&\leq \|\psi_{n+1} - y^*\| + \bigl\|(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n - y^*\bigr\|
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_n + \varepsilon_n \leq{}& \|\varphi_{n+1} - x^*\| + \|\psi_{n+1} - y^*\| + \bigl[1 - \alpha_n(1-\theta)\bigr]\bigl(\|\varphi_n - x^*\| + \|\psi_n - y^*\|\bigr) \\
&+ \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}\bigl(\|d'_n\| + \|f'_n\|\bigr) + \bigl(\|d''_n\| + \|f''_n\| + \|e_n\| + \|h_n\|\bigr) \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Remark 3.1 If $\mathbb{B}_1$ and $\mathbb{B}_2$ are both 2-uniformly smooth Banach space and $0 < \lambda = \rho < \min\{r_1/m_1, r_2/m_2\}$ is a constant such that

$$\begin{cases}
|\rho - \frac{\iota_1+\pi_1\delta_1^2-r_1(1-\nu_1)(m_1(1-\nu_1)+k\beta_1\tau_1)}{c_2\delta_1^2\tau_1^2-(m_1(1-\nu_1)+k\beta_1\tau_1)^2}| \\
\quad < \frac{\sqrt{[\iota_1+\pi_1\delta_1^2-r_1(1-\nu_1)(m_1(1-\nu_1)+k\beta_1\tau_1)]^2-[\sigma_1^2\tau_1^2-r_1^2(1-\nu_1^2)][c_2\delta_1^2\tau_1^2-(m_1(1-\nu_1)+k\beta_1\tau_1)^2]}}{c_2\delta_1^2\tau_1^2-(m_1(1-\nu_1)+k\beta_1\tau_1)^2}, \\
|\rho - \frac{\iota_2+\pi_2\delta_2^2-r_2(1-\nu_2)(m_2(1-\nu_2)+\kappa\beta_2\tau_2}{c_2\delta_2^2\tau_2^2-(m_2(1-\nu_2)+\kappa\beta_2\tau_2)^2}| \\
\quad < \frac{\sqrt{[\iota_2+\pi_2\delta_2^2-r_2(1-\nu_2)(m_2(1-\nu_2)+\kappa\beta_2\tau_2)]^2-[\sigma_2^2\tau_2^2-r_2^2(1-\nu_2^2)][c_2\delta_2^2\tau_2^2-(m_2(1-\nu_2)+\kappa\beta_2\tau_2)^2]}}{c_2\delta_2^2\tau_2^2-(m_2(1-\nu_2)+\kappa\beta_2\tau_2)^2}, \\
\iota_1 + \pi_1\delta_1^2 - r_1(1-\nu_1)(m_1(1-\nu_1) + k\beta_1\tau_1) \\
\quad > \sqrt{[\sigma_1^2\tau_1^2 - r_1^2(1-\nu_1^2)][c_2\delta_1^2\tau_1^2 - (m_1(1-\nu_1) + k\beta_1\tau_1)^2]}, \\
\iota_2 + \pi_2\delta_2^2 - r_2(1-\nu_2)(m_2(1-\nu_2) + \kappa\beta_2\tau_2) \\
\quad > \sqrt{[\sigma_2^2\tau_2^2 - r_2^2(1-\nu_2^2)][c_2\delta_2^2\tau_2^2 - (m_2(1-\nu_2) + \kappa\beta_2\tau_2)^2]}, \\
c_2\delta_1^2\tau_1^2 > (m_1(1-\nu_1) + k\beta_1\tau_1)^2, \qquad c_2\delta_2^2\tau_2^2 > (m_2(1-\nu_2) + \kappa\beta_2\tau_2)^2,
\end{cases}$$

then (3.5) holds. We note that Hilbert space and L_p (or l_p) ($2 \leq p < \infty$) spaces are 2-uniformly smooth Banach spaces.

From Theorem 3.1, we have the following results.

Corollary 3.1 *For $i = 1, 2$, let η_i, A_i, M_i, F, G, and $\mathbb{B}_i$ be the same as in Theorem* 3.1. *Suppose that $N_1 : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1$ is ι_1-strong accretive with respect to A_1, δ_1-Lipschitz continuous in the first argument, β_2-Lipschitz continuous in the second variable and $N_2 : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_2$ is ι_2- with respect to A_2, δ_2-Lipschitz continuous in the second argument and β_1-Lipschitz continuous in the first variable. If condition* (3.5) *in Theorem* 3.1 *holds, and there exist* *constants* $\lambda \in (0, r_1/m_1)$ *and* $\rho \in (0, r_2/m_2)$ *such that*

$$\begin{cases} \frac{\tau_1^{q_1-1}\sqrt[q_1]{\sigma_1^{q_1}-q_1\lambda\iota_1+c_{q_1}\lambda^{q_1}\delta_1^{q_1}}}{r_1-\lambda m_1} + \frac{\rho k\beta_1\tau_2^{q_2-1}}{r_2-\rho m_2} + \nu_1 < 1, \\ \frac{\tau_2^{q_2-1}\sqrt[q_2]{\sigma_2^{q_2}-q_2\rho\iota_2+c_{q_2}\rho^{q_2}\delta_2^{q_2}}}{r_2-\rho m_2} + \frac{\lambda\kappa\beta_2\tau_1^{q_1-1}}{r_1-\lambda m_1} + \nu_2 < 1, \end{cases}$$

where c_{q_1}, c_{q_2} *are the constants as in Lemma* 2.1, *then the iterative sequence* $\{(x_n, y_n, u_n, v_n)\}$ *generated by Algorithm* 3.1 *converges strongly to a solution* (x^*, y^*, u^*, v^*) *of the problem* (1.1). *Moreover, if, in addition, there exists* $\alpha > 0$ *such that* $\alpha_n \geq \alpha$ *for all* $n \geq 0$, *then*

$$\lim_{n\to\infty}(\varphi_n, \psi_n, \chi_n, \varpi_n) = (x^*, y^*, u^*, v^*) \quad \Longleftrightarrow \quad \lim_{n\to\infty}(\epsilon_n, \varepsilon_n) = (0, 0),$$

where $(\epsilon_n, \varepsilon_n)$ *is defined by* (3.3).

Corollary 3.2 *For* $i = 1, 2$, *let* η_i, A_i, M_i, N_i, *and* $\mathbb{B}_i$ *be the same as in Theorem* 3.1, *and* $F : \mathbb{B}_1 \to \mathbb{B}_1$ *be k-Lipschitz continuous and* $G : \mathbb{B}_2 \to \mathbb{B}_2$ *be* κ*-Lipschitz continuous. Assume that for any* $(x_0, y_0) \in \mathbb{B}_1 \times \mathbb{B}_2$, *the iterative sequence* $\{(x_n, y_n)\}$ *is generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_n)}(z_n) + \alpha_n d_n + e_n, \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_n)}(w_n) + \alpha_n f_n + h_n, \\ z_n = A_1(x_n) - \lambda N_1(x_n, G(y_n)), \\ w_n = A_2(y_n) - \rho N_2(F(x_n), y_n), \end{cases}$$

where $\lambda, \rho > 0$ *are constants, and for all* $n \geq 0$, *the sequences* $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *with* $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\{d_n\}, \{e_n\} \subset \mathbb{B}_1$, *and* $\{f_n\}, \{h_n\} \subset \mathbb{B}_2$ *are the sequences of errors and satisfy the following conditions*:

(1) $d_n = d'_n + d''_n$ *and* $f_n = f'_n + f''_n$, *where* $\{d'_n\}, \{d''_n\} \subset \mathbb{B}_1$ *and* $\{f'_n\}, \{f''_n\} \subset \mathbb{B}_2$;

(2) $\lim_{n\to\infty}\|d'_n\| = 0$ *and* $\lim_{n\to\infty}\|f'_n\| = 0$;

(3) $\sum_{n=0}^{\infty}\|d''_n\| < \infty$, $\sum_{n=0}^{\infty}\|e_n\| < \infty$, $\sum_{n=0}^{\infty}\|f''_n\| < \infty$ *and* $\sum_{n=0}^{\infty}\|h_n\| < \infty$.

If conditions (3.4) *and* (3.5) *in Theorem* 3.1 *hold, then the iterative sequence* $\{(x_n, y_n)\}$ *converges strongly to the unique solution* (x^*, y^*) *of the problem* (1.2). *Further, if, in addition, there exists* $\alpha > 0$ *such that* $\alpha_n \geq \alpha$ *for all* $n \geq 0$, *then*

$$\lim_{n\to\infty}(\varphi_n, \psi_n) = (x^*, y^*) \quad \Longleftrightarrow \quad \lim_{n\to\infty}(\epsilon_n, \varepsilon_n) = (0, 0),$$

where $(\epsilon_n, \varepsilon_n) \in \mathbb{R} \times \mathbb{R}$ *is defined by*

$$\begin{cases} \epsilon_n = \|\varphi_{n+1} - \{(1-\alpha_n)\varphi_n + \alpha_n R^{\lambda,A_1}_{\eta_1,M_1(\cdot,\varphi_n)}(s_n) + \alpha_n d_n + e_n\}\|, \\ \varepsilon_n = \|\psi_{n+1} - \{(1-\alpha_n)\psi_n + \alpha_n R^{\rho,A_2}_{\eta_2,M_2(\cdot,\psi_n)}(t_n) + \alpha_n f_n + h_n\}\|, \\ s_n = A_1(\varphi_n) - \lambda N_1(\varphi_n, G(\psi_n)), \\ t_n = A_2(\psi_n) - \rho N_2(F(\varphi_n), \psi_n) \end{cases}$$

for any sequence $\{(\varphi_n, \psi_n)\} \subset \mathbb{B}_1 \times \mathbb{B}_2$.

Proof For any $\lambda > 0$ and $\rho > 0$, define $P_\lambda : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1$ and $Q_\rho : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_2$ by

$$\begin{cases} P_\lambda(x, y) = R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x)}[A_1(x) - \lambda N_1(x, G(y))], \\ Q_\rho(x, y) = R^{\rho,A_2}_{\eta_2,M_2(\cdot,y)}[A_2(y) - \rho N_2(F(x), y)] \end{cases} \tag{3.21}$$

for all $(x,y) \in \mathbb{B}_1 \times \mathbb{B}_2$. Now, define the norm $\|\cdot\|_*$ on $\mathbb{B}_1 \times \mathbb{B}_2$ by

$$\|(x,y)\|_* = \|x\| + \|y\|$$

for all $(x,y) \in \mathbb{B}_1 \times \mathbb{B}_2$. It is easy to see that $(\mathbb{B}_1 \times \mathbb{B}_2, \|\cdot\|_*)$ is a Banach space (see [4]). By (3.21), for any $\lambda > 0$ and $\rho > 0$, define $T_{\lambda,\rho} : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1 \times \mathbb{B}_2$ by

$$T_{\lambda,\rho}(x,y) = \bigl(P_\lambda(x,y), Q_\rho(x,y)\bigr)$$

for all $(x,y) \in \mathbb{B}_1 \times \mathbb{B}_2$.

Thus, $T_{\lambda,\rho}$ is a contractive mapping. In fact, for any $(x_i,y_i) \in \mathbb{B}_1 \times \mathbb{B}_2$ and $i = 1,2$, it follows from (3.4) and (3.21) that

$$\begin{aligned}
P_\lambda(x_1,y_1) &= R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_1)}\bigl[A_1(x_1) - \lambda N_1\bigl(x_1, G(y_1)\bigr)\bigr],\\
Q_\rho(x_1,y_1) &= R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_1)}\bigl[A_2(y_1) - \rho N_2\bigl(F(x_1), y_1\bigr)\bigr],\\
P_\lambda(x_2,y_2) &= R^{\lambda,A_1}_{\eta_1,M_1(\cdot,x_2)}\bigl[A_1(x_2) - \lambda N_1\bigl(x_2, G(y_2)\bigr)\bigr],\\
Q_\rho(x_2,y_2) &= R^{\rho,A_2}_{\eta_2,M_2(\cdot,y_2)}\bigl[A_2(y_2) - \rho N_2\bigl(F(x_2), y_2\bigr)\bigr],
\end{aligned}$$

and

$$\bigl\|T_{\lambda,\rho}(x_1,y_1) - T_{\lambda,\rho}(x_2,y_2)\bigr\|_* \le \vartheta \bigl\|(x_1,y_1) - (x_2,y_2)\bigr\|_*,$$

where

$$\begin{aligned}
\vartheta = \max\Biggl\{ &\frac{\tau_1^{q_1-1}}{r_1 - \lambda m_1}\sqrt[q_1]{\sigma_1^{q_1} - q_1\lambda\iota_1 + q_1\lambda\pi_1\delta_1^{q_1} + c_{q_1}\lambda^{q_1}\delta_1^{q_1}} + \frac{\rho k\beta_1\tau_2^{q_2-1}}{r_2 - \rho m_2} + \nu_1,\\
&\frac{\tau_2^{q_2-1}}{r_2 - \rho m_2}\sqrt[q_2]{\sigma_2^{q_2} - q_2\rho\iota_2 + q_2\rho\pi_2\delta_2^{q_2} + c_{q_2}\rho^{q_2}\delta_2^{q_2}} + \frac{\lambda\kappa\beta_2\tau_1^{q_1-1}}{r_1 - \lambda m_1} + \nu_2\Biggr\}.
\end{aligned}$$

From (3.5), now we know that $T_{\lambda,\rho} : \mathbb{B}_1 \times \mathbb{B}_2 \to \mathbb{B}_1 \times \mathbb{B}_2$ is a Banach contraction mapping. Hence, (x^*, y^*) is unique solution of the problem (1.1). The rest of proof is similar to that of Theorem 3.1 and we omit the details. This completes the proof. □

Remark 3.2 If $d_n = 0$ or $e_n = 0$ or $f_n = 0$ or $h_n = 0$ for all $n \ge 0$ in Algorithm 3.1 and Corollary 3.2, then the conclusions of Theorem 3.1 also hold. The results of Theorem 3.1 improve and generalize the corresponding results of [3, 9, 10]. For other related works, we refer to [1–8, 11–14].

4 Conclusions

In this paper, we first introduced a system of generalized nonlinear mixed quasi-variational inclusions with (A,η)-accretive mappings in Banach spaces, which includes some systems of quasi-variational inclusions and variational inequality problems as special cases. Then, by using the new resolvent operator technique associated with (A,η)-accretive mappings, Nadler's fixed point theorem, and Liu's inequality, we constructed some new Mann iterative algorithms with mixed errors for the existence of solutions for generalized nonlinear variational inclusion systems in q-uniformly smooth Banach spaces. Furthermore, we proved the convergence and stability of the iterative sequences generated by the perturbed iterative algorithm. The results presented in this paper improve and generalize the corresponding results in the literature.

By similar methods to the ones this paper, we can study the existence of solutions and the convergence and stability to the following system of general nonlinear mixed quasi-variational inclusions:

Find $(x_1, x_2, \ldots, x_m) \in \mathbb{B}_1 \times \mathbb{B}_2 \times \cdots \times \mathbb{B}_m$ and $u_i \in F_i(x_i)$ $(i = 1, 2, \ldots, m)$ such that

$$\begin{cases} 0 \in N_1(x_1, u_2, x_3, \ldots, x_m) + M_1(x_1, x_1), \\ 0 \in N_2(x_1, x_2, u_3, \ldots, x_m) + M_2(x_2, x_2), \\ \cdots \\ 0 \in N_m(u_1, x_2, x_3, \ldots, x_m) + M_m(x_m, x_m), \end{cases}$$

where $N_i : \mathbb{B}_1 \times \mathbb{B}_2 \times \cdots \times \mathbb{B}_m \to \mathbb{B}_i$, $A_i : \mathbb{B}_i \to \mathbb{B}_i$, and $\eta_i : \mathbb{B}_i \times \mathbb{B}_i \to \mathbb{B}_i$ are single-valued mappings, $F_i : \mathbb{B}_i \to 2^{\mathbb{B}_i}$ is multi-valued mapping, $M_i : \mathbb{B}_i \times \mathbb{B}_i \to 2^{\mathbb{B}_i}$ is an any nonlinear mapping such that $M_i(\cdot, t) : \mathbb{B}_i \to 2^{\mathbb{B}_i}$ is an (A_i, η_i)-accretive mapping for all $t \in \mathbb{B}_i$ and $i = 1, 2$, which are still worthy of being studied in further research.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper and they read and approved the final manuscript.

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References

1. Cho, YJ, Fang, YP, Huang, NJ, Hwang, HJ: Algorithms for systems of nonlinear variational inequalities. J. Korean Math. Soc. **41**(3), 489-499 (2004)
2. Fang, YP, Huang, NJ: *H*-Monotone operator and resolvent operator technique for variational inclusions. Appl. Math. Comput. **145**, 795-803 (2003)
3. Fang, YP, Huang, NJ, Thompson, HB: A new system of variational inclusions with (H, η)-monotone operators in Hilbert spaces. Comput. Math. Appl. **49**(2-3), 365-374 (2005)
4. Fang, YP, Huang, NJ: Iterative algorithm for a system of variational inclusions involving *H*-accretive operators in Banach spaces. Acta Math. Hung. **108**(3), 183-195 (2005)
5. Huang, NJ, Fang, YP: A new class of general variational inclusions involving maximal η-monotone mappings. Publ. Math. (Debr.) **62**(1-2), 83-98 (2003)
6. Huang, NJ, Fang, YP: Fixed point theorems and a new system of multivalued generalized order complementarity problems. Positivity **7**(3), 257-265 (2003)
7. Huang, NJ: Nonlinear implicit quasi-variational inclusions involving generalized *m*-accretive mappings. Arch. Inequal. Appl. **2**(4), 413-425 (2004)
8. Kassay, G, Kolumbán, J: System of multi-valued variational inequalities. Publ. Math. (Debr.) **56**(1-2), 185-195 (2000)
9. Lan, HY, Cho, YJ, Verma, RU: On nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings in Banach spaces. Comput. Math. Appl. **51**(9-10), 1529-1538 (2006)
10. Verma, RU: *A*-Monotononicity and applications to nonlinear variational inclusion problems. J. Appl. Math. Stoch. Anal. **17**(2), 193-195 (2004)
11. Verma, RU: Approximation-solvability of a class of *A*-monotone variational inclusion problems. J. Korea Soc. Ind. Appl. Math. **8**(1), 55-66 (2004)
12. Verma, RU: Generalized system for relaxed cocoercive variational inequalities and projection methods. J. Optim. Theory Appl. **121**(1), 203-210 (2004)
13. Verma, RU: Nonlinear *H*-monotone variational inclusions and resolvent operator technique. Int. J. Pure Appl. Math. Sci. **2**(1), 53-57 (2005)

14. Verma, RU: Nonlinear *A*-monotone mixed variational inclusion problems based on resolvent operator techniques. Math. Sci. Res. J. **9**(10), 255-267 (2005)
15. Lan, HY: Stability of iterative processes with errors for a system of nonlinear (A,η)-accretive variational inclusions in Banach spaces. Comput. Math. Appl. **56**(1), 290-303 (2008)
16. Kazmi, KR, Ahmad, N, Shahzad, M: Convergence and stability of an iterative algorithm for a system of generalized implicit variational-like inclusions in Banach spaces. Appl. Math. Comput. **218**(18), 9208-9219 (2012)
17. Suwannawit, J, Petrot, N: Existence and stability of iterative algorithm for a system of random set-valued variational inclusion problems involving (A,m,η)-generalized monotone operators. J. Appl. Math. **2012**, Article ID 590676 (2012)
18. Petrot, N: A resolvent operator technique for approximate solving of generalized system mixed variational inequality and fixed point problems. Appl. Math. Lett. **23**(4), 440-445 (2010)
19. Ceng, LC, Latif, A, Al-Mazrooei, AE: Mann-type viscosity approximation methods for multivalued variational inclusions with finitely many variational inequality constraints in Banach spaces. Abstr. Appl. Anal. **2013**, Article ID 328740 (2013)
20. Adly, S, Outrata, JV: Qualitative stability of a class of non-monotone variational inclusions. Application in electronics. J. Convex Anal. **20**(1), 43-66 (2013)
21. Lan, HY, Kim, JK, Liu, ZS: Stable perturbed iteration procedures for solving new strongly nonlinear operator inclusions in Banach spaces. Nonlinear Funct. Anal. Appl. **18**(3), 433-444 (2013)
22. Guan, JL, Hu, CS: A system of generalized variational inclusions involving a new monotone mapping in Banach spaces. Abstr. Appl. Anal. **2013**, Article ID 654537 (2013)
23. Zeidler, E: Nonlinear Functional Analysis and Its Applications II: Monotone Operators. Springer, Berlin (1985)
24. Lan, HY: Generalized Yosida approximations based on relatively *A*-maximal *m*-relaxed monotonicity frameworks. Abstr. Appl. Anal. **2013**, Article ID 157190 (2013)
25. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127-1138 (1991)
26. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**(1), 114-125 (1995)
27. Nadler, SB: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)

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