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On the (p,h)-convex function and some integral inequalities

Zhong Bo Fang* and Renjie Shi

*Correspondence:
fangzb7777@hotmail.com
School of Mathematical Sciences,
Ocean University of China, Qingdao,
266100, P.R. China

Abstract

In this paper, we introduce a new class of (p,h)-convex functions which generalize P-functions and convex, h,p,s-convex, Godunova-Levin functions, and we give some properties of the functions. Moreover, we establish the corresponding Schur, Jensen, and Hadamard types of inequalities.

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1 Introduction

Let I and J be intervals in R. To motivate our work, let us recall the definitions of some special classes of functions.

Definition 1 [1] A function $f : I \to R$ is said to be a Godunova-Levin function or belongs to the class $Q(I)$ if f is non-negative and

$$f\bigl(\alpha x + (1-\alpha)y\bigr) \le \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}$$

for all $x, y \in I$ and $\alpha \in (0,1)$.

The class $Q(I)$ was firstly described in [1] by Godunova and Levin. Some further properties of it are given in [2, 3]. It has been known that non-negative convex and monotone functions belong to this class of functions.

Definition 2 [4] Let $s \in (0,1)$ be a fixed real number. A function $f : [0,\infty) \to [0,\infty)$ is said to be an s-convex function (in the second sense) or belongs to the class K_s^2, if

$$f\bigl(\alpha x + (1-\alpha)y\bigr) \le \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

An s-convex function was introduced by Breckner [4] and a number of properties and connections with s-convexity (in the first sense) were discussed in [5]. Of course, s-convexity means just convexity when $s = 1$.

Definition 3 [2] A function $f : I \to R$ is said to be a P-function or belongs to the class $P(I)$, if f is non-negative and

$$f(\alpha x + (1-\alpha)y) \le f(x) + f(y)$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

For some results on the class $P(I)$, see [6, 7].

Definition 4 [8] Let I be a p-convex set. A function $f : I \to R$ is said to be a p-convex function or belongs to the class $PC(I)$, if

$$f\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{\frac{1}{p}}\right) \le \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

Remark 1 [8] An interval I is said to be a p-convex set if $[\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}} \in I$ for all $x, y \in I$ and $\alpha \in [0,1]$, where $p = 2k+1$ or $p = \frac{n}{m}$, $n = 2r+1$, $m = 2t+1$, and $k, r, t \in N$.

Definition 5 [9] Let $h : J \to R$ be a non-negative and non-zero function. We say that $f : I \to R$ is an h-convex function or that f belongs to the class $SX(I)$, if f is non-negative and

$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y)$$

for all $x, y \in I$ and $\alpha \in (0,1)$.

The h- and p-convex functions were introduced by Varšanec, Zhang and Wan, and a number of properties and Jensen's inequalities of the functions were established (*cf.* [8]). As one can see, the definitions of the P-function, convex, h,p,s-convex, Godunova-Levin functions have similar forms. This observation leads us to generalize these varieties of convexity.

2 Definitions and basic results

In this section, we give new definitions and properties of the (p,h)-convex function. Throughout this paper, we assume that $(0,1) \subseteq J$, f and h are real non-negative functions defined on I and J, respectively, and the set I is p-convex when $f \in ghx(p,h,I)$ or $f \in ghv(p,h,I)$. We first give a definition of the new class of convex functions.

Definition 6 Let $h : J \to R$ be a non-negative and non-zero function. We say that $f : I \to R$ is a (p,h)-convex function or that f belongs to the class $ghx(h,p,I)$, if f is non-negative and

$$f\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{\frac{1}{p}}\right) \le h(\alpha)f(x) + h(1-\alpha)f(y) \tag{2.1}$$

for all $x, y \in I$ and $\alpha \in (0,1)$. Similarly, if the inequality sign in (2.1) is reversed, then f is said to be a (p,h)-concave function or belong to the class $ghv(h,p,I)$.

Remark 2 It can be obviously seen that if $h(\alpha) = \alpha$, then all non-negative p-convex and p-concave functions belong to $ghx(h,p,I)$ and $ghv(h,p,I)$, respectively; if $h(\alpha) = \alpha$ and $p = 1$, then all non-negative convex functions belong to $ghx(h,p,I)$; if $h(\alpha) = \frac{1}{\alpha}$ and $p = 1$, then $Q(I) = ghx(h,p,I)$; if $h(\alpha) = \alpha^s$, $s \in (0,1)$, and $p = 1$, then $K_s^2 \subseteq ghx(h,p,I)$; if $h(\alpha) = 1$ and $p = 1$, then $P(I) \subseteq ghx(h,p,I)$, and if $p = 1$, then $SX(I) \subseteq ghx(h,p,I)$.

Example 1 Let $h_k(\alpha) = \alpha^k$, where $k \leq 1$ and $\alpha > 0$. If f is a function defined as $f(x) = x^p$, where p is an odd number and $x \geq 0$, we then have

$$f\bigl(\bigl[\alpha x^p + (1-\alpha)y^p\bigr]^{\frac{1}{p}}\bigr) \leq \alpha f(x) + (1-\alpha)f(y) \leq h_k(\alpha)f(x) + h_k(1-\alpha)f(y),$$

and hence, f belongs to $ghx(h_k,p,I)$.

Next, we discuss some interesting properties of (p,h)-convex (concave) functions, which include linearity, product, composition properties, and an ordered property of h and p. In addition, we give some interesting properties of the (p,h)-convex function, when h is a super(sub)-multiplicative function.

Property 1 *If $f,g \in ghx(h,p,I)$ and $\lambda > 0$, then $f+g, \lambda f \in ghx(h,p,I)$. Similarly, if $f,g \in ghv(h,p,I)$ and $\lambda > 0$, then $f+g, \lambda f \in ghv(h,p,I)$.*

Proof The proof immediately follows from the definitions of the classes $ghx(h,p,I)$ and $ghv(h,p,I)$. □

Property 2 *Let h_1 and h_2 be non-negative functions defined on an interval J with $h_2 \leq h_1$ in $(0,1)$. If $f \in ghx(h_2,p,I)$, then $f \in ghx(h_1,p,I)$. Similarly, if $f \in ghv(h_1,p,I)$, then $f \in ghv(h_2,p,I)$.*

Proof If $f \in ghx(h_2,p,I)$, then for any $x,y \in I$ and $\alpha \in (0,1)$ we have

$$\begin{aligned} f\bigl(\bigl[\alpha x^p + (1-\alpha)y^p\bigr]^{\frac{1}{p}}\bigr) &\leq h_2(\alpha)f(x) + h_2(1-\alpha)f(y) \\ &\leq h_1(\alpha)f(x) + h_1(1-\alpha)f(y), \end{aligned}$$

and hence, $f \in ghx(h_1,p,I)$. □

Property 3 *Let $f \in ghx(h,p_1,I)$.*

(a) *For $I \subseteq (0,1]$, if f is monotone increasing (monotone decreasing), and $p_2 \geq p_1 > 0$ or $p_2 \leq p_1 < 0$, and ($p_1 \geq p_2 > 0$ or $p_1 \leq p_2 < 0$), then $f \in ghx(h,p_2,I)$.*

(b) *For $I \subseteq [1,\infty)$, if f is monotone increasing (monotone decreasing), and $p_1 \geq p_2 > 0$ or $p_1 \leq p_2 < 0$, and ($p_2 \geq p_1 > 0$ or $p_2 \leq p_1 < 0$), then $f \in ghx(h,p_2,I)$.*

Let $f \in ghv(h,p_1,I)$.

(c) *For $I \subseteq (0,1]$, if f is monotone increasing (monotone decreasing), and $p_1 \geq p_2 > 0$ or $p_1 \leq p_2 < 0$, and ($p_2 \geq p_1 > 0$ or $p_2 \leq p_1 < 0$), then $f \in ghv(h,p_2,I)$.*

(d) *For $I \subseteq [1,\infty)$, if f is monotone increasing (monotone decreasing), and $p_2 \geq p_1 > 0$ or $p_2 \leq p_1 < 0$, and ($p_1 \geq p_2 > 0$ or $p_1 \leq p_2 < 0$), then $f \in ghv(h,p_2,I)$.*

Proof (a) Setting $g(p) = (\alpha x^p + (1-\alpha)y^p)^{\frac{1}{p}}$, we have

$$g'(p) = \frac{1}{p}\bigl(\alpha x^p + (1-\alpha)y^p\bigr)^{\frac{1}{p}-1}\bigl(\alpha x^p \ln(x) + (1-\alpha)y^p \ln(y)\bigr).$$

When $p > 0$ and $x, y \in (0,1]$, we have $g'(p) < 0$, and so $g(p_2) \leq g(p_1)$. We then obtain

$$f\bigl(g(p_2)\bigr) \leq f\bigl(g(p_1)\bigr) \leq h(\alpha)f(x) + (1-\alpha)f(y),$$

since f is monotone increasing and $f \in ghx(h, p_1, I)$. Therefore, we get $f \in ghx(h, p_2, I)$.

The results of (b), (c), and (d) follow by similar arguments as above. □

Property 4 *Let f and g be similarly ordered functions on I, i.e.,*

$$\bigl(f(x) - f(y)\bigr)\bigl(g(x) - g(y)\bigr) \geq 0 \tag{2.2}$$

for all $x, y \in I$. If $f \in ghx(h_1, p, I)$, $g \in ghx(h_2, p, I)$, and $h(\alpha) + h(1-\alpha) \leq c$ for all $\alpha \in (0,1)$, where $h(t) = \max(h_1(t), h_2(t))$ and c is a fixed positive number, then the product fg belongs to $ghx(ch, p, I)$. Similarly, let f and g be oppositely ordered, i.e.,

$$\bigl(f(x) - f(y)\bigr)\bigl(g(x) - g(y)\bigr) \leq 0$$

for all $x, y \in I$. If $f \in ghv(h_1, p, I)$, $g \in ghv(h_2, p, I)$, and $h(\alpha) + h(1-\alpha) \geq c$ for all $\alpha \in (0,1)$, where $h(t) = \min(h_1(t), h_2(t))$ and c is a fixed positive number, then the product fg belongs to $ghv(ch, p, I)$.

Proof We only give a proof for the first part, since the result of the second part of this theorem follows by a similar argument. By (2.2), we have

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x).$$

Let α and β be positive numbers such that $\alpha + \beta = 1$. We then obtain

$$\begin{aligned} fg\bigl(\bigl[\alpha x^p + \beta y^p\bigr]^{\frac{1}{p}}\bigr) &\leq \bigl(h_1(\alpha)f(x) + h_1(\beta)f(y)\bigr)\bigl(h_2(\alpha)g(x) + h_2(\beta)g(y)\bigr) \\ &\leq h^2(\alpha)fg(x) + h(\alpha)h(\beta)f(x)g(y) + h(\alpha)h(\beta)f(y)g(x) + h^2(\beta)fg(y) \\ &\leq h^2(\alpha)fg(x) + h(\alpha)h(\beta)f(x)g(x) + h(\alpha)h(\beta)f(y)g(y) + h^2(\beta)fg(y) \\ &= \bigl(h(\alpha) + h(\beta)\bigr)\bigl(h(\alpha)fg(x) + h(\beta)fg(y)\bigr) \\ &\leq ch(\alpha)fg(x) + ch(\beta)fg(y), \end{aligned}$$

which completes the proof. □

Definition 7 [9] A function $h : I \to R$ is called a super-multiplicative function if

$$h(xy) \geq h(x)h(y) \tag{2.3}$$

for all $x, y \in J$.

If the inequality sign in (2.3) is reversed, then h is said to be a sub-multiplicative function, and if the equality holds in (2.3), then h is called a multiplicative function.

Example 2 Let $h(x) = ce^x$. If $c = 1$, then h is a multiplicative function. If $c > 1$, then h is a sub-multiplicative function, and if $0 < c < 1$, then h is a super-multiplicative function.

Property 5 *Let I be an interval such that $0 \in I$. We then have the following.*

(a) *If $f \in ghx(h,p,I)$, $f(0) = 0$, and h is super-multiplicative, then the inequality*

$$f\left(\left[\alpha x^p + \beta y^p\right]^{\frac{1}{p}}\right) \leq h(\alpha)f(x) + h(\beta)f(y) \tag{2.4}$$

holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$.

(b) *Let h be a non-negative function with $h(\alpha) < \frac{1}{2}$ for some $\alpha \in (0, \frac{1}{2})$. If f is a non-negative function satisfying* (2.4) *for all $x, y \in I$ and all $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, then $f(0) = 0$.*

(c) *If $f \in ghv(h,p,I)$, $f(0) = 0$, and h is sub-multiplicative, then the inequality*

$$f\left(\left[\alpha x^p + \beta y^p\right]^{\frac{1}{p}}\right) \geq h(\alpha)f(x) + h(\beta)f(y) \tag{2.5}$$

holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$.

(d) *Let h be a non-negative function with $h(\alpha) > \frac{1}{2}$ for some $\alpha \in (0, \frac{1}{2})$. If f is a non-negative function satisfying* (2.5) *for all $x, y \in I$ and all $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, then $f(0) = 0$.*

Proof (a) Let $\alpha, \beta > 0$, $\alpha + \beta = \gamma < 1$, and let a and b be numbers such that $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. We then have $a + b = 1$ and

$$\begin{aligned}
f\left(\left[\alpha x^p + \beta y^p\right]^{\frac{1}{p}}\right) &= f\left(\left[a\gamma x^p + b\gamma y^p\right]^{\frac{1}{p}}\right) \\
&\leq h(a)f\left(\gamma^{\frac{1}{p}}x\right) + h(b)f\left(\gamma^{\frac{1}{p}}y\right) \\
&= h(a)f\left(\left[\gamma x^p + (1-\gamma)0^p\right]^{\frac{1}{p}}\right) + h(b)f\left(\left[\gamma y^p + (1-\gamma)0^p\right]^{\frac{1}{p}}\right) \\
&\leq h(a)h(\gamma)f(x) + h(a)h(1-\gamma)f(0) \\
&\quad + h(b)h(\gamma)f(y) + h(b)h(1-\gamma)f(0) \\
&= h(a)h(\gamma)f(x) + h(b)h(\gamma)f(y) \\
&\leq h(a\gamma)f(x) + h(b\gamma)f(y) = h(\alpha)f(x) + h(\beta)f(y).
\end{aligned}$$

(b) If $f(0) \neq 0$, then $f(0) > 0$. Setting $x = y = 0$ in (2.4), we get

$$f(0) \leq h(\alpha)f(0) + h(\beta)f(0).$$

By setting $\alpha = \beta$, where $\alpha \in (0, \frac{1}{2})$, and dividing both sides of the inequality above by $f(0)$, we obtain $2h(\alpha) \geq 1$ for all $\alpha \in (0, \frac{1}{2})$, which is a contradiction to the assumption $h(\alpha) < \frac{1}{2}$ for some $\alpha \in (0, \frac{1}{2})$, and so $f(0) = 0$.

The results of (c) and (d) follow by using similar arguments as above, and so we omit the proofs here. □

Corollary 1 *Let $h_s(x) = x^s$, where $s, x > 0$, and let $0 \in I$. For all $f \in ghx(h_s, p, I)$, inequality* (2.4) *holds for all $\alpha, \beta > 0$ with $\alpha + \beta \le 1$ if and only if $f(0) = 0$. For all $f \in ghv(h_s, p, I)$, inequality* (2.5) *holds for all $\alpha, \beta > 0$ with $\alpha + \beta \le 1$ if and only if $f(0) = 0$.*

Proof Let $\alpha, \beta > 0$, $\alpha + \beta = \gamma < 1$, and let a and b be positive numbers such that $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. We then have $a + b = 1$ and

$$\begin{aligned} f\left(\left[\alpha x^p + \beta y^p\right]^{\frac{1}{p}}\right) &= f\left(\left[a\gamma x^p + b\gamma y^p\right]^{\frac{1}{p}}\right) \\ &\le a^s f\left(\gamma^{\frac{1}{p}} x\right) + b^s f\left(\gamma^{\frac{1}{p}} y\right) \\ &= a^s f\left(\left[\gamma x^p + (1-\gamma)0^p\right]^{\frac{1}{p}}\right) + b^s f\left(\left[\gamma y^p + (1-\gamma)0^p\right]^{\frac{1}{p}}\right) \\ &\le a^s \gamma^s f(x) + a^s (1-\gamma)^s f(0) + b^s \gamma^s f(y) + b^s (1-\gamma)^s f(0) \\ &= a^s \gamma^s f(x) + b^s \gamma^s f(y) \\ &= \alpha^s f(x) + \beta^s f(y). \end{aligned}$$

Setting $x = y = \alpha = \beta = 0$ in (2.4), we get $f(0) \le 0$, while $f(0) \ge 0$ by the definition of the (p, h)-convex function, and hence $f(0) = 0$. □

Property 6 *Suppose that $h_i : J_i \to (0, \infty)$, $i = 1, 2$, are functions such that $h_2(J_2) \subseteq J_1$ and $h_2(\alpha) + h_2(1-\alpha) \le 1$ for all $\alpha \in (0,1)$, and that $f : I_1 \to [0, \infty)$ and $g : I_2 \to [0, \infty)$ are functions with $g(I_2) \subseteq I_1$, $0 \in I_1$, and $f(0) = 0$.*

If h_1 is a super-multiplicative function, $f \in SX(h_1, I_1)$, and f is increasing (decreasing) and $g \in ghx(h_2, p, I_2)$ ($g \in ghv(h_2, p, I_2)$), then the composite function $f \circ g$ belongs to $ghx(h_1 \circ h_2, p, I_2)$. If h_1 is a sub-multiplicative function, $f \in SV(h_1, I_1)$, and f is increasing (decreasing) and $g \in ghv(h_2, p, I_2)$ ($g \in ghx(h_2, p, I_2)$), then the composite function $f \circ g$ belongs to $ghv(h_1 \circ h_2, p, I_2)$.

Proof If $g \in ghx(h_2, p, I_2)$ and f is an increasing function, then we have

$$(f \circ g)\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{\frac{1}{p}}\right) \le f\left(h_2(\alpha)g(x) + h_2(1-\alpha)g(y)\right)$$

for all $x, y \in I_2$ and $\alpha \in (0,1)$. Using Property 5(a) with $p = 1$, we obtain

$$f\left(h_2(\alpha)g(x) + h_2(1-\alpha)g(y)\right) \le h_1\left(h_2(\alpha)\right)f\left(g(x)\right) + h_1\left(h_2(1-\alpha)\right)f\left(g(y)\right),$$

which implies that $f \circ g$ belongs to $ghx(h_1 \circ h_2, p, I_2)$. □

If f is a convex or concave function, then we may give a similar statement on the composite function of f and g.

Property 7 *Let $f : I_1 \to [0, \infty)$ and $g : I_2 \to [0, \infty)$ be functions with $g(I_2) \subseteq I_1$. If the function f is convex and increasing (decreasing), and $g \in ghx(h, p, I_2)$ ($g \in ghv(h, p, I_2)$) with $h(\alpha) + h(1-\alpha) = 1$ for $\alpha \in (0,1)$, then $f \circ g$ belongs to $ghx(h, p, I_2)$. If the function f is concave and increasing (decreasing), and $g \in ghv(h, p, I_2)$ ($g \in ghx(h, p, I_2)$) with $h(\alpha) + h(1-\alpha) = 1$ for $\alpha \in (0,1)$, then $f \circ g$ belongs to $ghv(h, p, I_2)$.*

Proof If $g \in ghx(h,p,I_2)$ and f is an increasing function, we then have

$$(f \circ g)\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{\frac{1}{p}}\right) \le f\left(h(\alpha)g(x) + h(1-\alpha)g(y)\right)$$

for all $x, y \in I_2$ and $\alpha \in (0,1)$. Since $h(\alpha) + h(1-\alpha) = 1$ and f is convex, we obtain

$$f\left(h(\alpha)g(x) + h(1-\alpha)g(y)\right) \le h(\alpha)f\left(g(x)\right) + h(1-\alpha)f\left(g(y)\right),$$

which implies that $f \circ g$ belongs to $ghx(h,p,I_2)$. □

3 Schur-type inequalities

In this section, we establish Schur-type inequalities of (p,h)-convex functions.

Theorem 1 *Let $h : J \to R$ be a non-negative super-multiplicative function and let $f : I \to R$ be a function such that $f \in ghx(h,p,I)$. Then for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $x_3^p - x_1^p, x_3^p - x_2^p, x_2^p - x_1^p \in J$, the following inequality holds*:

$$h\left(x_3^p - x_2^p\right)f(x_1) - h\left(x_3^p - x_1^p\right)f(x_2) + h\left(x_2^p - x_1^p\right)f(x_3) \ge 0. \tag{3.1}$$

If the function h is sub-multiplicative and $f \in ghv(h,p,I)$, then the inequality sign in (3.1) *is reversed.*

Proof Let $f \in ghx(h,p,I)$ and let $x_1, x_2, x_3 \in I$ be the numbers stated in this theorem. Then one can easily see that

$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p}, \frac{x_2^p - x_1^p}{x_3^p - x_1^p} \in (0,1) \subseteq J \quad \text{and} \quad \frac{x_3^p - x_2^p}{x_3^p - x_1^p} + \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1.$$

We also have

$$h\left(x_3^p - x_2^p\right) = h\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\left(x_3^p - x_1^p\right)\right) \ge h\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)h\left(x_3^p - x_1^p\right)$$

and

$$h\left(x_2^p - x_1^p\right) \ge h\left(\frac{x_2^p - x_1^p}{x_3^p - x_1^p}\right)h\left(x_3^p - x_1^p\right).$$

Setting $\alpha = \frac{x_3^p - x_2^p}{x_3^p - x_1^p}$, $x = x_1$, and $y = x_3$ in (2.1), we have $x_2^p = \alpha x^p + (1-\alpha)y^p$ and

$$\begin{aligned} f(x_2) &\le h\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)f(x_1) + h\left(\frac{x_2^p - x_1^p}{x_3^p - x_1^p}\right)f(x_3) \\ &\le \frac{h(x_3^p - x_2^p)}{h(x_3^p - x_1^p)}f(x_1) + \frac{h(x_2^p - x_1^p)}{h(x_3^p - x_1^p)}f(x_3). \end{aligned} \tag{3.2}$$

Assuming $h(x_3^p - x_1^p) > 0$ and multiplying both sides of the inequality above by $h(x_3^p - x_1^p)$, we obtain inequality (3.1). □

Remark 3 In fact, if $f(x) = x^\lambda$, $\lambda \in R$, $h(x) = h_{-1}(x) = \frac{1}{x}$, $p = 1$, and $x_1, x_2, x_3 \in I = (0,1)$, then inequality (3.1) gives the Schur inequality, see [10, p.177].

The following corollary gives a Schur-type inequality for the (p,h)-convex function.

Corollary 2 *If $f : I = (0,1) \to I$ belongs to the class $ghx(h_{-k}, p, I)$ and $h_{-k} = \frac{1}{x^k}$, then we have the inequality*

$$\begin{aligned} &f(x_1)\left(x_3^p - x_1^p\right)^k\left(x_2^p - x_1^p\right)^k - f(x_2)\left(x_3^p - x_2^p\right)^k\left(x_2^p - x_1^p\right)^k \\ &\quad + f(x_3)\left(x_3^p - x_1^p\right)^k\left(x_3^p - x_2^p\right)^k \ge 0 \end{aligned} \tag{3.3}$$

for all $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. If $f \in ghv(h_{-k}, p, I)$, then the inequality sign in (3.3) *is reversed. If $k = 1$, $p = 1$, and $f(x) = x^\lambda$, $\lambda \in R$, then $f \in ghx(h_{-1}, 1, I)$ and inequality* (3.3) *gives the Schur inequality.*

4 Jensen-type inequalities

In this section, we introduce some Jensen-type inequalities of (p,h)-convex functions.

Theorem 2 *Let $w_1, \ldots, w_n$ be positive real numbers with $n \ge 2$. If h is a non-negative super-multiplicative function and if $f \in ghx(h, p, I)$ and $x_1, \ldots, x_n \in I$, then we have the inequality*

$$f\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_i x_i^p\right]^{\frac{1}{p}}\right) \le \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i), \quad \text{where } W_n = \sum_{i=1}^n w_i. \tag{4.1}$$

If h is sub-multiplicative and $f \in ghv(h, p, I)$, then the inequality sign in (4.1) *is reversed.*

Proof When $n = 2$, inequality (4.1) holds by (2.1) with $\alpha = \frac{w_1}{W_2}$. Assuming inequality (4.1) holds for $n-1$, we obtain

$$\begin{aligned} f\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_i x_i^p\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{w_n}{W_n}x_n^p + \sum_{i=1}^{n-1}\frac{w_i}{W_n}x_i^p\right]^{\frac{1}{p}}\right) \\ &= f\left(\left[\frac{w_n}{W_n}x_n^p + \frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}x_i^p\right]^{\frac{1}{p}}\right) \\ &\le h\left(\frac{w_n}{W_n}\right)f(x_n) + h\left(\frac{W_{n-1}}{W_n}\right)f\left(\left[\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}x_i^p\right]^{\frac{1}{p}}\right) \\ &\le h\left(\frac{w_n}{W_n}\right)f(x_n) + h\left(\frac{W_{n-1}}{W_n}\right)\sum_{i=1}^{n-1}h\left(\frac{w_i}{W_{n-1}}\right)f(x_i) \\ &\le \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right)f(x_i), \end{aligned}$$

and, hence, the result follows by mathematical induction. □

Remark 4 For $h(\alpha) = \alpha$ and $p = 1$, inequality (4.1) becomes the classical Jensen inequality.

Theorem 3 *Let $w_1, \ldots, w_n$ be positive real numbers and let (m, M) be an interval in I. If $h : (0, \infty) \to R$ is a non-negative super-multiplicative function and $f \in ghx(h, p, I)$, then for all $x_1, \ldots, x_n \in (m, M)$ we have the inequality*

$$\sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) f(x_i) \le f(m) \sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) h\left(\frac{M^p - x_i^p}{M^p - m^p}\right) + f(M) \sum_{i=1}^{n} h\left(\frac{w_i}{W_n}\right) h\left(\frac{x_i^p - m^p}{M^p - m^p}\right). \tag{4.2}$$

If h is a non-negative sub-multiplicative function and $f \in ghv(h, p, I)$, then the inequality sign in (4.2) *is reversed.*

Proof Setting $x_1 = m$, $x_2 = x_i$, and $x_3 = M$ in (3.2), we get the inequalities

$$f(x_i) \le h\left(\frac{M^p - x_i^p}{M^p - m^p}\right) f(m) + h\left(\frac{x_i^p - m^p}{M^p - m^p}\right) f(M), \quad i = 1, \ldots, n.$$

Multiplying both sides of the above inequality with $h(\frac{w_i}{W_n})$ and adding all inequalities side by side for $i = 1, \ldots, n$, we obtain (4.2). □

Let K be a finite nonempty set of positive integers and let F be an index set function defined by

$$F(K) = h(W_K) f\left(\left[\frac{1}{W_K} \sum_{i \in K} w_i x_i^p\right]^{\frac{1}{p}}\right) - \sum_{i \in K} h(w_i) f(x_i), \quad \text{where } W_K = \sum_{i \in K} w_i.$$

Theorem 4 *Let $h : (0, \infty) \to R$ be a non-negative function, and let M and K be finite nonempty sets of positive integers such that $M \cap K = \emptyset$. If h is super-multiplicative and $f : I \to R$ belongs to the class $ghx(h, p, I)$, then for $w_i > 0$, $x_i \in I$, $i \in M \cup K$ we have the inequality*

$$F(M \cup K) \le F(M) + F(K). \tag{4.3}$$

If h is sub-multiplicative and $f \in ghv(h, p, I)$, then the inequality sign in (4.3) *is reversed.*

Proof Setting $x = [\frac{1}{W_M} \sum_{i \in M} w_i x_i^p]^{\frac{1}{p}}$, $y = [\frac{1}{W_K} \sum_{i \in K} w_i x_i^p]^{\frac{1}{p}}$, and $\alpha = \frac{W_M}{W_{M \cup K}}$ in (2.1), we obtain the inequality

$$f\left(\left[\frac{1}{W_{M \cup K}} \sum_{i \in M \cup K} w_i x_i^p\right]^{\frac{1}{p}}\right) \le h\left(\frac{W_M}{W_{M \cup K}}\right) f\left(\left[\frac{1}{W_M} \sum_{i \in M} w_i x_i^p\right]^{\frac{1}{p}}\right) + h\left(\frac{W_K}{W_{M \cup K}}\right) f\left(\left[\frac{1}{W_K} \sum_{i \in K} w_i x_i^p\right]^{\frac{1}{p}}\right).$$

Multiplying both sides of the above inequality with $h(W_{M\cup K})$, we get the inequality

$$\begin{aligned} &h(W_{M\cup K})f\left(\left[\frac{1}{W_{M\cup K}}\sum_{i\in M\cup K} w_i x_i^p\right]^{\frac{1}{p}}\right) \\ &\quad \le h(W_M)f\left(\left[\frac{1}{W_M}\sum_{i\in M} w_i x_i^p\right]^{\frac{1}{p}}\right) + h(W_K)f\left(\left[\frac{1}{W_K}\sum_{i\in K} w_i x_i^p\right]^{\frac{1}{p}}\right). \end{aligned}$$

Subtracting $\sum_{i\in M\cup K} h(w_i)f(x_i)$ from both sides of the inequality above and using the identity $\sum_{i\in M\cup K} h(w_i)f(x_i) = \sum_{i\in M} h(w_i)f(x_i) + \sum_{i\in K} h(w_i)f(x_i)$, we obtain (4.3). □

A simple consequence of Theorem 4 is stated in the following corollary without proof.

Corollary 3 *Let $h:(0,\infty)\to R$ be a non-negative super-multiplicative function. If $w_i > 0$, $i = 1,\dots,n$, and $M_k = \{1,\dots,K\}$, then for $f \in ghx(h,p,I)$ we have*

$$F(M_n) \le F(M_{n-1}) \le \cdots \le F(M_2) \le 0 \tag{4.4}$$

and

$$F(M_n) \le \min_{1\le i<j\le n}\left\{h(w_i+w_j)f\left(\left[\frac{w_i x_i^p + w_j x_j^p}{w_i + w_j}\right]^{\frac{1}{p}}\right) - h(w_i)f(x_i) - h(w_j)f(x_j)\right\}. \tag{4.5}$$

If h is sub-multiplicative and $f \in ghv(h,p,I)$, then the inequality signs in (4.4) *and* (4.5) *are reversed, and* min *is replaced with* max.

Remark 5 Some results obtained from Theorem 4 and Corollary 3 are given in [11, p.7], when $h(\alpha) = \alpha$, $p = 1$, and h is a convex or concave function.

5 Hadamard-type inequalities

In this section, we give some Hadamard-type inequalities of (p,h)-convex functions.

Theorem 5 *If $f \in ghx(h,p,I) \cap L_1([a,b])$ for $a, b \in I$ with $a < b$, then we have*

$$\frac{1}{2h(\frac{1}{2})}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\,dx \le \big(f(a)+f(b)\big)\int_0^1 h(t)\,dt. \tag{5.1}$$

Proof Setting $x^p = \frac{y-a}{b-a}b^p + \frac{b-y}{b-a}a^p$, we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\,dx = \frac{1}{b-a}\int_a^b f\left(\left[\frac{y-a}{b-a}b^p + \frac{b-y}{b-a}a^p\right]^{\frac{1}{p}}\right)dy.$$

By using inequality (2.1) we obtain

$$f\left(\left[\frac{y-a}{b-a}b^p + \frac{b-y}{b-a}a^p\right]^{\frac{1}{p}}\right) \le h\left(\frac{y-a}{b-a}\right)f(b) + h\left(\frac{b-y}{b-a}\right)f(a),$$

and hence, by integrating the above inequality over $[a,b]$, we have

$$\begin{aligned}\int_a^b f\left(\left[\frac{y-a}{b-a}b^p + \frac{b-y}{b-a}a^p\right]^{\frac{1}{p}}\right) dy &\le f(b)\int_a^b h\left(\frac{y-a}{b-a}\right) dy + f(a)\int_a^b h\left(\frac{b-y}{b-a}\right) dy \\ &= (b-a)\bigl(f(a)+f(b)\bigr)\int_0^1 h(t)\,dt,\end{aligned}$$

which gives the second inequality.

Setting $y = \frac{1}{2}(a+b)+t$, we obtain

$$\begin{aligned}&\int_{-\frac{1}{2}(b-a)}^{\frac{1}{2}(b-a)} f\left(\left[\frac{1}{2}(a^p+b^p) + \frac{b^p-a^p}{b-a}t\right]^{\frac{1}{p}}\right) dt \\ &\quad= \int_0^{\frac{1}{2}(b-a)} f\left(\left[\frac{1}{2}(a^p+b^p) + \frac{b^p-a^p}{b-a}t\right]^{\frac{1}{p}}\right) dt \\ &\quad\quad+ \int_0^{\frac{1}{2}(b-a)} f\left(\left[\frac{1}{2}(a^p+b^p) - \frac{b^p-a^p}{b-a}t\right]^{\frac{1}{p}}\right) dt \\ &\quad\ge \frac{1}{h(1/2)}\int_0^{\frac{1}{2}(b-a)} f\left(\left[\frac{1}{2}(a^p+b^p)\right]^{\frac{1}{p}}\right) dt = \frac{b-a}{2h(1/2)} f\left(\left[\frac{1}{2}(a^p+b^p)\right]^{\frac{1}{p}}\right),\end{aligned}$$

and, hence, the first inequality follows. □

Remark 6 If $h(\alpha)=\alpha$ and $p=1$, then inequality (5.1) gives the classical Hadamard inequality.

Theorem 6 *Suppose that f and g are functions such that $f \in ghx(h_1,p,I)$, $g \in ghx(h_2,p,I)$, $fg \in L_1([a,b])$, and $h_1h_2 \in L_1([0,1])$ with $a,b \in I$ and $a<b$. We then have*

$$\begin{aligned}\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)g(x)\,dx &\le M(a,b)\int_0^1 h_1(t)h_2(t)\,dt \\ &\quad+ N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt, \qquad (5.2)\end{aligned}$$

where $M(a,b) = f(a)g(a)+f(b)g(b)$ and $N(a,b) = f(a)g(b)+f(b)g(a)$.

Proof Since $f \in ghx(h_1,p,I)$ and $g \in ghx(h_2,p,I)$, we have

$$\begin{aligned}f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr) &\le h_1(t)f(a)+h_1(1-t)f(b), \\ g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr) &\le h_2(t)g(a)+h_2(1-t)g(b)\end{aligned}$$

for all $t \in [0,1]$. Because f and g are non-negative, we get the inequality

$$\begin{aligned}&f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr) \\ &\quad\le h_1(t)h_2(t)f(a)g(a)+h_1(1-t)h_2(t)f(b)g(a)+h_1(t)h_2(1-t)f(a)g(b) \\ &\quad\quad+ h_1(1-t)h_2(1-t)f(b)g(b).\end{aligned}$$

Integrating both sides of the above inequality over $(0,1)$, we obtain the inequality

$$\begin{aligned}
&\int_0^1 f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)\,dt \\
&\quad\le f(a)g(a)\int_0^1 h_1(t)h_2(t)\,dt+f(b)g(a)\int_0^1 h_1(1-t)h_2(t)\,dt \\
&\qquad+f(a)g(b)\int_0^1 h_1(t)h_2(1-t)\,dt+f(b)g(b)\int_0^1 h_1(1-t)h_2(1-t)\,dt.
\end{aligned}$$

Setting $x=[ta^p+(1-t)b^p]^{\frac{1}{p}}$, we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)g(x)\,dx\le M(a,b)\int_0^1 h_1(t)h_2(t)\,dt+N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt.$$

□

Theorem 7 *Let $f\in ghx(h_1,p,I)$, $g\in ghx(h_2,p,I)$ be functions such that $fg\in L_1([a,b])$ and $h_1h_2\in L_1([0,1])$, and let $a,b\in I$ with $a<b$. We then have*

$$\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\biggl(\biggl[\frac{a^p+b^p}{2}\biggr]^{\frac{1}{p}}\biggr)g\biggl(\biggl[\frac{a^p+b^p}{2}\biggr]^{\frac{1}{p}}\biggr)-\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)g(x)\,dx \\
&\quad\le M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt+N(a,b)\int_0^1 h_1(t)h_2(t)\,dt.
\end{aligned}\tag{5.3}$$

Proof Since $\frac{a^p+b^p}{2}=\frac{ta^p+(1-t)b^p}{2}+\frac{(1-t)a^p+tb^p}{2}$, we have

$$\begin{aligned}
&f\biggl(\biggl[\frac{a^p+b^p}{2}\biggr]^{\frac{1}{p}}\biggr)g\biggl(\biggl[\frac{a^p+b^p}{2}\biggr]^{\frac{1}{p}}\biggr) \\
&\quad=f\biggl(\biggl[\frac{ta^p+(1-t)b^p}{2}+\frac{(1-t)a^p+tb^p}{2}\biggr]^{\frac{1}{p}}\biggr)g\biggl(\biggl[\frac{ta^p+(1-t)b^p}{2}+\frac{(1-t)a^p+tb^p}{2}\biggr]^{\frac{1}{p}}\biggr) \\
&\quad=h_1\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)+f\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)\bigr] \\
&\qquad\times h_2\biggl(\frac{1}{2}\biggr)\bigl[g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)+g\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)\bigr] \\
&\quad\le h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)\bigr] \\
&\qquad+h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)\bigr] \\
&\qquad+h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[h_1(t)f(a)+h_1(1-t)f(b)\bigr]\bigl[h_2(1-t)g(a)+h_2(t)g(b)\bigr] \\
&\qquad+h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[h_1(1-t)f(a)+h_1(t)f(b)\bigr]\bigl[h_2(t)g(a)+h_2(1-t)g(b)\bigr] \\
&\quad=h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[ta^p+(1-t)b^p\bigr]^{\frac{1}{p}}\bigr)\bigr] \\
&\qquad+h_1\biggl(\frac{1}{2}\biggr)h_2\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)g\bigl(\bigl[(1-t)a^p+tb^p\bigr]^{\frac{1}{p}}\bigr)\bigr]
\end{aligned}$$

$$+ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\left(h_1(t)h_2(1-t) + h_1(1-t)h_2(t)\right)M(a,b)\right]$$
$$+ h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\left(h_1(t)h_2(t) + h_1(1-t)h_2(1-t)\right)N(a,b)\right].$$

Integrating the above inequality over $[0,1]$, we obtain

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) - \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)g(x)\,dx$$
$$\le M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt + N(a,b)\int_0^1 h_1(t)h_2(t)\,dt. \qquad \square$$

Theorem 8 *Let $f \in ghx(h_1,p,I)$ and $g \in ghx(h_2,p,I)$ be functions such that $fg \in L_1([a,b])$, $h_1h_2 \in L_1([0,1])$, and let $a,b \in I$ with $a < b$. We then have the inequality*

$$\frac{p^2}{2(b^p-a^p)^2}\int_a^b\int_a^b\int_0^1 x^{p-1}y^{p-1}f\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)dx\,dy\,dt$$
$$\le \frac{p}{b^p-a^p}\int_0^1 h_1(t)h_2(t)\,dt\int_a^b x^{p-1}f(x)g(x)\,dx$$
$$+ \int_0^1 h_1(t)\,dt\int_0^1 h_2(t)\,dt\int_0^1 h_1(t)h_2(1-t)\,dt\left[M(a,b)+N(a,b)\right]. \tag{5.4}$$

Proof Since $f \in ghx(h_1,p,I)$ and $g \in ghx(h_2,p,I)$, we have

$$f\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right) \le h_1(t)f(x) + h_1(1-t)f(y),$$
$$g\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right) \le h_2(t)g(x) + h_2(1-t)g(y)$$

for all $t \in [0,1]$. Because f and g are non-negative, we get the inequality

$$f\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)$$
$$\le h_1(t)h_2(t)f(x)g(x) + h_1(1-t)h_2(t)f(y)g(x) + h_1(t)h_2(1-t)f(x)g(y)$$
$$+ h_1(1-t)h_2(1-t)f(y)g(y).$$

Multiplying both sides of the above inequality with $\frac{p^2x^{p-1}y^{p-1}}{(b^p-a^p)^2}$ and integrating the result over $[a,b]$ and $[0,1]$, we obtain the inequality

$$\frac{p^2}{(b^p-a^p)^2}\int_a^b\int_a^b\int_0^1 x^{p-1}y^{p-1}f\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)y^p\right]^{\frac{1}{p}}\right)dx\,dy\,dt$$
$$\le \int_0^1 h_1(t)h_2(t)\,dt\left[\frac{p^2}{(b^p-a^p)^2}\left(\int_a^b x^{p-1}f(x)g(x)\,dx\int_a^b y^{p-1}\,dy\right.\right.$$
$$\left.\left. + \int_a^b y^{p-1}f(y)g(y)\,dy\int_a^b x^{p-1}\,dx\right)\right]$$
$$+ 2\int_0^1 h_1(t)h_2(1-t)\,dt\left[\frac{p^2}{(b^p-a^p)^2}\int_a^b x^{p-1}f(x)\,dx\int_a^b y^{p-1}f(y)\,dy\right].$$

By (5.1), we have the inequality

$$\begin{aligned}
&\frac{p^2}{2(b^p-a^p)^2}\int_a^b\int_a^b\int_0^1 x^{p-1}y^{p-1}f\big(\big[tx^p+(1-t)y^p\big]^{\frac{1}{p}}\big)g\big(\big[tx^p+(1-t)y^p\big]^{\frac{1}{p}}\big)\,dx\,dy\,dt\\
&\quad\le\frac{p}{b^p-a^p}\int_0^1 h_1(t)h_2(t)\,dt\int_a^b x^{p-1}f(x)g(x)\,dx\\
&\qquad+\int_0^1 h_1(t)\,dt\int_0^1 h_2(t)\,dt\int_0^1 h_1(t)h_2(1-t)\,dt\big[M(a,b)+N(a,b)\big]. \qquad\square
\end{aligned}$$

Theorem 9 *Let $f\in ghx(h_1,p,I)$, $g\in ghx(h_2,p,I)$ be functions such that $fg\in L_1([a,b])$, $h_1h_2\in L_1([0,1])$, and let $a,b\in I$ with $a<b$. We then have the inequality*

$$\begin{aligned}
&\int_a^b\int_0^1 x^{p-1}f\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)dt\,dx\\
&\quad\le\int_0^1 h_1(t)h_2(t)\,dt\int_a^b x^{p-1}f(x)g(x)\,dx+\frac{b^p-a^p}{p}\big[M(a,b)+N(a,b)\big]\\
&\qquad\times\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)h_2(t)\,dt\right.\\
&\qquad\left.+\left[h_1\left(\frac{1}{2}\right)\int_0^1 h_2(t)\,dt+h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)\,dt\right]\int_0^1 h_1(t)h_2(1-t)\,dt\right]. \qquad (5.5)
\end{aligned}$$

Proof Since $f\in ghx(h_1,p,I)$ and $g\in ghx(h_2,p,I)$, we have the inequalities

$$\begin{aligned}
&f\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\le h_1(t)f(x)+h_1(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right),\\
&g\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\le h_2(t)g(x)+h_2(1-t)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)
\end{aligned}$$

for all $t\in[0,1]$. Because f and g are non-negative, we get the inequality

$$\begin{aligned}
&f\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\\
&\quad\le h_1(t)h_2(t)f(x)g(x)+h_1(1-t)h_2(t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g(x)\\
&\qquad+h_1(t)h_2(1-t)f(x)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\\
&\qquad+h_1(1-t)h_2(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right).
\end{aligned}$$

Multiplying both sides of the inequality above with x^{p-1} and integrating the result over $[a,b]$ and $[0,1]$, we obtain

$$\begin{aligned}
&\int_a^b\int_0^1 x^{p-1}f\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)dt\,dx\\
&\quad\le\int_0^1 h_1(t)h_2(t)\,dt\left[\int_a^b x^{p-1}f(x)g(x)\,dx+\frac{b^p-a^p}{p}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right]
\end{aligned}$$

$$+\int_0^1 h_1(t)h_2(1-t)\,dt\left[g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_a^b x^{p-1}f(x)\,dx\right.$$
$$\left.+f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_a^b x^{p-1}g(x)\,dx\right].$$

By inequality (5.1), we have

$$\begin{aligned}
&\int_a^b\int_0^1 x^{p-1}f\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[tx^p+(1-t)\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)dt\,dx\\
&\quad\le\int_0^1 h_1(t)h_2(t)\,dt\left[\int_a^b x^{p-1}f(x)g(x)\,dx\right.\\
&\qquad\left.+\frac{b^p-a^p}{p}h_1\left(\frac{1}{2}\right)\bigl(f(a)+f(b)\bigr)h_2\left(\frac{1}{2}\right)\bigl(g(a)+g(b)\bigr)\right]\\
&\qquad+\int_0^1 h_1(t)h_2(1-t)\,dt\frac{b^p-a^p}{p}h_2\left(\frac{1}{2}\right)\bigl(f(a)+f(b)\bigr)\bigl(g(a)+g(b)\bigr)\int_0^1 h_1(t)\,dt\\
&\qquad+\int_0^1 h_1(t)h_2(1-t)\,dt\frac{b^p-a^p}{p}h_1\left(\frac{1}{2}\right)\bigl(f(a)+f(b)\bigr)\bigl(g(a)+g(b)\bigr)\int_0^1 h_2(t)\,dt\\
&\quad=\int_0^1 h_1(t)h_2(t)\,dt\int_a^b x^{p-1}f(x)g(x)\,dx+\frac{b^p-a^p}{p}\bigl[M(a,b)+N(a,b)\bigr]\\
&\qquad\times\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)h_2(t)\,dt\right.\\
&\qquad\left.+\left[h_1\left(\frac{1}{2}\right)\int_0^1 h_2(t)\,dt+h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)\,dt\right]\int_0^1 h_1(t)h_2(1-t)\,dt\right].\qquad\square
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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