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Generalization of Mizoguchi-Takahashi type contraction and related fixed point theorems

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Abstract

In this paper, we introduce a new notion to generalize a Mizoguchi-Takahashi type contraction. Then, using this notion, we obtain a fixed point theorem for multivalued maps. Our results generalize some results by Minak and Altun, Kamran and those contained therein.

MSC: 47H10; 54H25

Keywords: Mizoguchi-Takahashi contraction; α-admissible maps; α_*-admissible maps

1 Introduction and preliminaries

The notions of α-ψ-contractive and α-admissible mappings were introduced by Samet *et al.* [1]. They proved some fixed point results for such mappings in complete metric spaces. These notions were generalized by Karapınar and Samet [2]. Asl *et al.* [3] extended these notions to multifunctions and introduced the notions of α_*-ψ-contractive and α_*-admissible mappings. Afterwards Ali and Kamran [4] further generalized the notion of α_*-ψ-contractive mappings and obtained some fixed point theorems for multivalued mappings. Some interesting extensions of results by Samet *et al.* [1] are available in [5–13]. Nadler initiated a fixed point theorem for multivalued mappings. Some extensions of Nadler's result can also be found in [14–31]. Mizoguchi and Takahashi [32] extended the Nadler fixed point theorem. Recently, Minak and Altun generalized Mizoguchi and Takahashi's theorem by introducing a function $\alpha : X \times X \to [0,\infty)$. In this paper, we introduce the notion of α_*-Mizoguchi-Takahashi type contraction. By using this notion, we generalize some fixed point theorems presented by Minak and Altun [7], Kamran [26] and those contained therein.

We denote by $CL(X)$ the class of all nonempty closed subsets of X and by $CB(X)$ the class of all nonempty closed and bounded subsets of X. For $A \in CL(X)$ or $CB(X)$ and $x \in X$, $d(x,A) = \inf\{d(x,a) : a \in A\}$, and H is a generalized Hausdorff metric induced by d. Now we recollect some basic definitions and results for the sake of completeness.

If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, then $O(T,x_0) = \{x_0,x_1,x_2,\dots\}$ is said to be an orbit of $T : X \to CL(X)$ at x_0. A mapping $h : X \to \mathbb{R}$ is said to be T-orbitally lower semicontinuous at $\xi \in X$, if $\{x_n\}$ is a sequence in $O(T,x_0)$ and $x_n \to \xi$ implies $h(\xi) \le \liminf h(x_n)$. The following definition is due to Asl *et al.* [3].

Definition 1.1 [3] Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to CL(X)$. Then T is α_*-admissible if for each $x, y \in X$ with $\alpha(x,y) \geq 1 \Rightarrow \alpha_*(Tx,Ty) \geq 1$, where $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$.

Minak and Altun [7] generalized Mizoguchi and Takahashi's theorem in the following way.

Theorem 1.2 [7] *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a mapping satisfying*

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \phi\bigl(d(x,y)\bigr)d(x,y) \quad \text{for each } x,y \in X,$$

where $\phi : [0,\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \phi(r) < 1$ for each $t \in [0,\infty)$. Also assume that

(i) *T is α_*-admissible;*
(ii) *there exists $x_0 \in X$ with $\alpha(x_0,x_1) \geq 1$ for some $x_1 \in Tx_0$;*
(iii) (a) *T is continuous,*
or
(b) *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then we have $\alpha(x_n,x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Kamran in [26] generalized Mizoguchi and Takahashi's theorem in the following way.

Theorem 1.3 [26] *Let (X,d) be a complete metric space and $T : X \to CL(X)$ be a mapping satisfying*

$$d(y,Ty) \leq \phi\bigl(d(x,y)\bigr)d(x,y) \quad \text{for each } x \in X \text{ and } y \in Tx,$$

where $\phi : [0,\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \phi(r) < 1$ for each $t \in [0,\infty)$. Then,

(i) *for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;*
(ii) *ξ is a fixed point of T if and only if the function $h(x) := d(x,Tx)$ is T-orbitally lower semicontinuous at ξ.*

2 Main results

We begin this section with the following definition.

Definition 2.1 Let (X,d) be a metric space, $T : X \to CL(X)$ is said to be an α_*-Mizoguchi-Takahashi type contraction if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\phi : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \phi(r) < 1$ for every $t \in [0,\infty)$ such that

$$\alpha_*(Tx,Ty)d(y,Ty) \leq \phi\bigl(d(x,y)\bigr)d(x,y) \quad \text{for each } x \in X \text{ and } y \in Tx. \tag{2.1}$$

Before moving toward our main results, we prove some lemmas.

Lemma 2.2 *Let (X,d) be a metric space, $\{A_k\}$ be a sequence in $CL(X)$, $\{x_k\}$ be a sequence in X such that $x_k \in A_{k-1}$. Let $\phi : [0,\infty) \to [0,1)$ be a function satisfying $\limsup_{r\to t^+} \phi(r) < 1$*

for every $t \in [0, \infty)$. Suppose that $\{d(x_{k-1}, x_k)\}$ is a nonincreasing sequence such that

$$d(x_k, A_k) \le \phi\big(d(x_{k-1}, x_k)\big) d(x_{k-1}, x_k), \tag{2.2}$$

$$d(x_k, x_{k+1}) \le d(x_k, A_k) + \phi^{n_k}\big(d(x_{k-1}, x_k)\big), \tag{2.3}$$

where $n_1 < n_2 < \cdots$, $k, n_k \in \mathbb{N}$. Then $\{x_k\}$ is a Cauchy sequence in X.

Proof The proof runs on the same lines as the proof of [18, Lemma 3.2]. We include its details for completeness. Let $d_k := d(x_{k-1}, x_k)$. Since d_k is a nonincreasing sequence of nonnegative real numbers, therefore $\lim_{k\to\infty} d_k = c \ge 0$. By hypothesis, for $t = c$, we get $\limsup_{t\to c^+} \phi(t) < 1$. Therefore, there exists k_0 such that $k \ge k_0$ implies that $\phi(d_k) < h$, where $\limsup_{t\to c^+} \phi(t) \le h < 1$. From (2.2) and (2.3), we have

$$\begin{aligned}
d_{k+1} &\le \phi(d_k) d_k + \phi^{n_k}(d_k) \\
&\le \phi(d_k)\phi(d_{k-1}) d_{k-1} + \phi(d_k)\phi^{n_{k-1}}(d_{k-1}) + \phi^{n_k}(d_k) \\
&\quad \cdots \\
&\le \prod_{i=1}^{k} \phi(d_i) d_1 + \sum_{m=1}^{k-1} \prod_{i=m+1}^{k} \phi(d_i)\phi^{n_m}(d_m) + \phi^{n_k}(d_k) \\
&\le \prod_{i=1}^{k} \phi(d_i) d_1 + \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \phi(d_i)\phi^{n_m}(d_m) + \phi^{n_k}(d_k).
\end{aligned} \tag{2.4}$$

We have deleted some factors of ϕ from the product in (2.4) using the fact that $\phi < 1$. Let S denote the second term on the right-hand side of (2.4),

$$\begin{aligned}
S &\le (k_0 - 1)h^{k-k_0+1} \sum_{m=1}^{k_0-1} \phi^{n_m}(d_m) + \sum_{m=k_0}^{k-1} h^{k-m}\phi^{n_m}(d_m) \\
&\le (k_0 - 1)h^{k-k_0+1} \sum_{m=1}^{k_0-1} \phi^{n_m}(d_m) + \sum_{m=k_0}^{k-1} h^{k-m+n_m} \\
&\le Ch^k + \sum_{m=k_0}^{k-1} h^{k-m+n_m} \\
&\le Ch^k + h^{k+n_{k_0}-k_0} + h^{k+n_{k_0-1}-(k_0-1)} + \cdots + h^{k+n_{k-1}-(k-1)} \\
&\le Ch^k + \sum_{m=k+n_{k_0}-k_0}^{k+n_{k-1}-(k-1)} h^m \\
&= Ch^k + \frac{h^{k+n_{k_0}-k_0+1} - h^{k+n_{k-1}-k+2}}{1-h} \\
&< Ch^k + h^k \frac{h^{n_{k_0}-k_0+1}}{1-h} \\
&= Ch^k,
\end{aligned}$$

where C is a generic positive constant. Now, it follows from (2.4) that

$$\begin{aligned} d_{k+1} &\leq \prod_{i=1}^{k} \phi(d_i)d_1 + Ch^k + \phi^{n_k}(d_k) \\ &< h^{k-k_0+1}\prod_{i=1}^{k_0-1} \phi(d_i)d_1 + Ch^k + h^{n_k} \\ &< Ch^k + Ch^k + k \\ &= Ch^k, \end{aligned}$$

C again being a generic constant. Now, for $k \geq k_0$, $m \in \mathbb{N}$,

$$\begin{aligned} d(x_k, x_{k+m}) &\leq \sum_{i=k+1}^{k+m} d_i \\ &< \sum_{i=k+1}^{k+m} Ch^{i-1} \\ &= C\frac{h^{k+1} - h^{k+m}}{1-h} \\ &\leq h^k, \end{aligned}$$

which shows that $\{x_k\}$ is a Cauchy sequence in X. □

Lemma 2.3 *Let (X,d) be a metric space, $T : X \to CL(X)$ be an α_*-Mizoguchi-Takahashi type contraction. Let $\{x_k\}$ be an orbit of T at x_0 such that $\alpha_*(Tx_{k-1}, Tx_k) \geq 1$ and*

$$d(x_k, x_{k+1}) \leq d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr), \tag{2.5}$$

where $x_k \in Tx_{k-1}$, $n_1 < n_2 < \cdots$ and $k, n_k \in \mathbb{N}$ and $\{d(x_{k-1}, x_k)\}$ is a nonincreasing sequence. Then $\{x_k\}$ is a Cauchy sequence in X.

Proof Given that $\{x_k\}$ is an orbit of T at x_0, *i.e.*, $x_k \in Tx_{k-1}$ for each $k \in \mathbb{N}$, with $\alpha_*(Tx_{k-1}, Tx_k) \geq 1$ for each $k \in \mathbb{N}$, as T is an α_*-Mizoguchi-Takahashi type contraction. From (2.1), we have

$$\begin{aligned} d(x_k, Tx_k) &\leq \alpha_*(Tx_{k-1}, Tx_k)d(x_k, Tx_k) \\ &\leq \phi\bigl(d(x_{k-1}, x_k)\bigr)d(x_{k-1}, x_k). \end{aligned}$$

From (2.5), we have

$$d(x_k, x_{k+1}) \leq d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr).$$

Since all the conditions of Lemma 2.2 are satisfied, $\{x_k\}$ is a Cauchy sequence in X. □

Theorem 2.4 *Let (X,d) be a complete metric space, $T : X \to CL(X)$ be an α_*-Mizoguchi-Takahashi type contraction and α_*-admissible. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then,*

(i) *there exists an orbit* $\{x_n\}$ *of* T *and* $x^* \in X$ *such that* $\lim x_n = x^*$;
(ii) x^* *is a fixed point of* T *if and only if* $h(x) = d(x, Tx)$ *is* T*-orbitally lower semicontinuous at* x^*.

Proof By hypothesis, we have $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. Thus, for $x_1 \in Tx_0$, we can choose a positive integer n_1 such that

$$\phi^{n_1}\big(d(x_0, x_1)\big) \leq \big[1 - \phi\big(d(x_0, x_1)\big)\big] d(x_0, x_1). \tag{2.6}$$

There exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d(x_1, Tx_1) + \phi^{n_1}\big(d(x_0, x_1)\big). \tag{2.7}$$

As T is α_*-admissible, we have $\alpha_*(Tx_0, Tx_1) \geq 1$. From (2.6) and (2.7) it follows that

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, Tx_1) + \phi^{n_1}\big(d(x_0, x_1)\big) \\ &\leq \alpha_*(Tx_0, Tx_1) d(x_1, Tx_1) + \phi^{n_1}\big(d(x_0, x_1)\big) \\ &\leq \phi\big(d(x_0, x_1)\big) d(x_0, x_1) + \big[1 - \phi\big(d(x_0, x_1)\big)\big] d(x_0, x_1) \\ &= d(x_0, x_1). \end{aligned}$$

Now we can choose a positive integer $n_2 > n_1$ such that

$$\phi^{n_2}\big(d(x_1, x_2)\big) \leq \big[1 - \phi\big(d(x_1, x_2)\big)\big] d(x_1, x_2). \tag{2.8}$$

There exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq d(x_2, Tx_2) + \phi^{n_2}\big(d(x_1, x_2)\big). \tag{2.9}$$

As T is α_*-admissible, then $\alpha(x_1, x_2) \geq \alpha_*(Tx_0, Tx_1) \geq 1$ implies $\alpha_*(Tx_1, Tx_2) \geq 1$. Using (2.8) and (2.9) we have that

$$\begin{aligned} d(x_2, x_3) &\leq d(x_2, Tx_2) + \phi^{n_2}\big(d(x_1, x_2)\big) \\ &\leq \alpha_*(Tx_1, Tx_2) d(x_2, Tx_2) + \phi^{n_2}\big(d(x_1, x_2)\big) \\ &\leq \phi\big(d(x_1, x_2)\big) d(x_1, x_2) + \big[1 - \phi\big(d(x_1, x_2)\big)\big] d(x_1, x_2) \\ &= d(x_1, x_2). \end{aligned}$$

By repeating this process for all $k \in \mathbb{N}$, we can choose a positive integer n_k such that

$$\phi^{n_k}\big(d(x_{k-1}, x_k)\big) \leq \big[1 - \phi\big(d(x_{k-1}, x_k)\big)\big] d(x_{k-1}, x_k). \tag{2.10}$$

There exists $x_k \in Tx_{k-1}$ such that

$$d(x_k, x_{k+1}) \leq d(x_k, Tx_k) + \phi^{n_k}\big(d(x_{k-1}, x_k)\big). \tag{2.11}$$

Also, by α_*-admissibility of T, we have $\alpha_*(Tx_{k-1}, Tx_k) \geq 1$ for each $k \in \mathbb{N}$. From (2.10) and (2.11) it follows that

$$\begin{aligned} d(x_k, x_{k+1}) &\leq d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr) \\ &\leq \alpha_*(Tx_{k-1}, Tx_k) d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr) \\ &\leq \phi\bigl(d(x_{k-1}, x_k)\bigr) d(x_{k-1}, x_k) + \bigl[1 - \phi\bigl(d(x_{k-1}, x_k)\bigr)\bigr] d(x_{k-1}, x_k) \\ &= d(x_{k-1}, x_k), \end{aligned}$$

which implies that $\{d(x_k, x_{k+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Thus, by Lemma 2.3, $\{x_k\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_k \to x^*$ as $k \to \infty$. Since $x_k \in Tx_{k-1}$, it follows from (2.1) that

$$\begin{aligned} d(x_k, Tx_k) &\leq \alpha_*(Tx_{k-1}, Tx_k) d(x_k, Tx_k) \\ &\leq \phi\bigl(d(x_{k-1}, x_k)\bigr) d(x_{k-1}, x_k) \\ &< d(x_{k-1}, x_k). \end{aligned}$$

Letting $k \to \infty$, in the above inequality, we have

$$\lim_{k\to\infty} d(x_k, Tx_k) = 0. \tag{2.12}$$

Suppose that $h(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x^*, then

$$d\bigl(x^*, Tx^*\bigr) = h\bigl(x^*\bigr) \leq \liminf_k h(x_k) = \liminf_k d(x_k, Tx_k) = 0.$$

By the closedness of T it follows that $x^* \in Tx^*$. Conversely, suppose that x^* is a fixed point of T, then $h(x^*) = 0 \leq \liminf_k h(x_k)$. □

Example 2.5 Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup (1, \infty)$ be endowed with the usual metric d. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x = 0, \\ \{\frac{1}{n+2}, \frac{1}{n+3}\} & \text{if } x = \frac{1}{n} : 1 \leq n \leq 6, \\ \{\frac{1}{n}, 0\} & \text{if } x = \frac{1}{n} : n > 6, \\ [2x, \infty) & \text{if } x > 1, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi : [0, \infty) \to [0, 1)$ by

$$\phi(t) = \begin{cases} \frac{4}{5} & \text{if } 0 \leq t \leq \frac{1}{6}, \\ \frac{1}{2} & \text{if } t > \frac{1}{6}. \end{cases}$$

One can check that for each $x \in X$ and $y \in Tx$, we have

$$\alpha_*(Tx, Ty)d(y, Ty) \le \phi\big(d(x,y)\big)d(x,y).$$

Also, T is α_*-admissible and for $x_0 = 1$ we have $x_1 = \frac{1}{3} \in Tx_0$ with $\alpha(x_0, x_1) = 1$. Moreover, all the other conditions of Theorem 2.4 are satisfied. Therefore T has a fixed point. Note that Theorem 5 of Minak and Altun [7] is not applicable here; see, for example, $x = \frac{1}{7}$ and $y = \frac{1}{8}$. Further Theorem 2.1 of Kamran [26] is also not applicable; see, for example, $x = 2$ and $y = 4 \in Tx$.

The proofs of the following theorems run on the same lines as the proof of Theorem 2.4.

Theorem 2.6 *Let (X,d) be a complete metric space, $T : X \to CL(X)$ be an α_*-admissible mapping such that*

$$\alpha_*(y, Ty)d(y, Ty) \le \phi\big(d(x,y)\big)d(x,y) \quad \text{for each } x \in X \text{ and } y \in Tx, \tag{2.13}$$

where $\phi : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \phi(r) < 1$ for every $t \in [0,\infty)$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then,

(i) *there exists an orbit $\{x_n\}$ of T and $x^* \in X$ such that $\lim x_n = x^*$;*

(ii) *x^* is a fixed point of T if and only if $h(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x^*.*

Theorem 2.7 *Let (X,d) be a complete metric space, $T : X \to CL(X)$ be an α_*-admissible mapping such that*

$$\alpha(x,y)d(y, Ty) \le \phi\big(d(x,y)\big)d(x,y) \quad \text{for each } x \in X \text{ and } y \in Tx, \tag{2.14}$$

where $\phi : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \phi(r) < 1$ for every $t \in [0,\infty)$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then,

(i) *there exists an orbit $\{x_n\}$ of T and $x^* \in X$ such that $\lim x_n = x^*$;*

(ii) *x^* is a fixed point of T if and only if $h(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x^*.*

Corollary 2.8 [26] *Let (X,d) be a complete metric space and $T : X \to CL(X)$ be a mapping satisfying*

$$d(y, Ty) \le \phi\big(d(x,y)\big)d(x,y) \quad \text{for each } x \in X \text{ and } y \in Tx,$$

where $\phi : [0,\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \phi(r) < 1$ for each $t \in [0,\infty)$. Then,

(i) *for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;*

(ii) *ξ is a fixed point of T if and only if the function $h(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at ξ.*

Proof Define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = 1$ for each $x, y \in X$. Then the proof follows from Theorem 2.4 as well as from Theorem 2.6, and from Theorem 2.7. □

3 Application

From Definition 2.1, we get the following definition by considering only those $x \in X$ and $y \in Tx$ for which we have $\alpha_*(Tx, Ty) \geq 1$.

Definition 3.1 Let (X, d) be a metric space, $T : X \to CL(X)$ is said to be a modified α_*-Mizoguchi-Takahashi type contraction if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\phi : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \phi(r) < 1$ for every $t \in [0, \infty)$ such that for each $x \in X$ and $y \in Tx$,

$$\alpha_*(Tx, Ty) \geq 1 \quad \Rightarrow \quad d(y, Ty) \leq \phi\bigl(d(x, y)\bigr)d(x, y). \tag{3.1}$$

Lemma 3.2 *Let (X, d) be a metric space, $T : X \to CL(X)$ be a modified α_*-Mizoguchi-Takahashi contraction. Let $\{x_k\}$ be an orbit of T at x_0 such that $\alpha_*(Tx_{k-1}, Tx_k) \geq 1$ and*

$$d(x_k, x_{k+1}) \leq d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr), \tag{3.2}$$

where $x_k \in Tx_{k-1}$, $n_1 < n_2 < \cdots$ and $k, n_k \in \mathbb{N}$ and $\{d(x_{k-1}, x_k)\}$ is a nonincreasing sequence. Then $\{x_k\}$ is a Cauchy sequence in X.

Proof Given that $\{x_k\}$ is an orbit of T at x_0, *i.e.*, $x_k \in Tx_{k-1}$ for each $k \in \mathbb{N}$, with $\alpha_*(Tx_{k-1}, Tx_k) \geq 1$ for each $k \in \mathbb{N}$, as T is a modified α_*-Mizoguchi-Takahashi contraction. From (3.1), we have

$$d(x_k, Tx_k) \leq \phi\bigl(d(x_{k-1}, x_k)\bigr)d(x_{k-1}, x_k).$$

From (3.2), we have

$$d(x_k, x_{k+1}) \leq d(x_k, Tx_k) + \phi^{n_k}\bigl(d(x_{k-1}, x_k)\bigr).$$

Since all the conditions of Lemma 2.2 are satisfied, $\{x_k\}$ is a Cauchy sequence in X. □

Working on the same lines as the proof of Theorem 2.4 is done, one may obtain the proof of the following result.

Theorem 3.3 *Let (X, d) be a complete metric space, $T : X \to CL(X)$ be a modified α_*-Mizoguchi-Takahashi contraction and α_*-admissible. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then,*

(i) *there exists an orbit $\{x_n\}$ of T and $x^* \in X$ such that $\lim x_n = x^*$;*
(ii) *x^* is a fixed point of T if and only if $h(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x^*.*

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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