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The L_p-dual mixed geominimal surface area for multiple star bodies

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Abstract

According to the notion of the L_p-mixed geominimal surface area of multiple convex bodies which were introduced by Ye *et al.*, we define the concept of the L_p-dual mixed geominimal surface area for multiple star bodies, and we establish several inequalities related to this concept.

MSC: 52A20; 52A40

Keywords: L_p-mixed geominimal surface area; multiple convex bodies; L_p-dual mixed geominimal surface area; multiple star bodies

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. $\mathcal{S}_o^n$ and $\mathcal{S}_c^n$, respectively, denote the set of star bodies (about the origin) and the set of star bodies whose centroids lie at the origin in $\mathbb{R}^n$. Let $\mathcal{F}_o^n$ denote the set of $\mathcal{K}_o^n$ that have a positive continuous curvate function. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

The notion of L_p-geominimal surface area was given by Lutwak in [1]. For $K \in \mathcal{K}_o^n$, and $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\bigl\{nV_p(K,L)V\bigl(L^*\bigr)^{\frac{p}{n}} : L \in \mathcal{K}_o^n\bigr\}.$$

Here $V_p(K,L)$ denotes L_p-mixed volume of $K, L \in \mathcal{K}_o^n$ (see [1, 2]) and L^* denotes the polar of L. For the case $p = 1$, $G_p(K)$ is just the classical geominimal surface area which was introduced by Petty [3]. Some affine isoperimetric inequalities related to the classical and L_p-geominimal surface areas can be found in [3–10]. Recently, the L_p-geominimal surface area was successfully extended to any real p ($p \neq -n$) by Ye in [11]. Especially, Ye *et al.* [12] studied the L_p-mixed geominimal surface area for multiple convex bodies. For $p > 0$, they defined the L_p-mixed geominimal surface areas for $K_1, \ldots, K_n \in \mathcal{F}_o^n$ as

$$G_p^{(1)}(K_1,\ldots,K_n) = \inf_{L\in\mathcal{K}_o^n}\bigl\{nV_p(K_1,\ldots,K_n;L,\ldots,L)^{\frac{n}{n+p}}V\bigl(L^*\bigr)^{\frac{p}{n+p}}\bigr\};$$

$$G_p^{(2)}(K_1,\ldots,K_n) = \inf_{L_i\in\mathcal{K}_o^n}\left\{nV_p(K_1,\ldots,K_n;L_1,\ldots,L_n)^{\frac{n}{n+p}}\prod_{i=1}^{n}V\bigl(L_i^*\bigr)^{\frac{p}{(n+p)n}}\right\}.$$

Here Q^* denotes the polar body of Q, and $V_p(K_1,\dots,K_n;L_1,\dots,L_n)$ denotes a type of L_p-mixed volume of $K_1,\dots,K_n \in \mathcal{F}_o^n$, $L_1,\dots,L_n \in \mathcal{K}_o^n$ (see [12]).

Wang and Qi in [13] introduced the L_p-dual geominimal surface area as follows: For $K \in \mathcal{S}_o^n$, and $p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}(K) = \inf\bigl\{n\widetilde{V}_{-p}(K,L)V\bigl(L^*\bigr)^{-\frac{p}{n}} : L \in \mathcal{K}_c^n\bigr\}. \tag{1.1}$$

Here $\widetilde{V}_{-p}(K,L)$ denotes the L_p-dual mixed volume of $K, L \in \mathcal{S}_o^n$ (see Section 2).

Note that we extend L from an origin-symmetric convex body to $L \in \mathcal{K}_c^n$ in definition (1.1). Actually, we can prove that the results of [13] all are correct under this extension.

In this paper, we first define the L_p-dual mixed geominimal surface area for multiple star bodies with the same idea in mind as [12].

Definition 1.1 For $K_1,\dots,K_n \in \mathcal{S}_o^n$, $p \geq 1$, the L_p-dual mixed geominimal surface areas, $\widetilde{G}_{-p}^{(j)}(K_1,\dots,K_n)$ $(j = 1,2)$, of $K_1,\dots,K_n$, are defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}^{(1)}(K_1,\dots,K_n) = \inf_{L\in\mathcal{K}_c^n}\bigl\{n\widetilde{V}_{-p}(K_1,\dots,K_n;L,\dots,L)V\bigl(L^*\bigr)^{-\frac{p}{n}}\bigr\}; \tag{1.2}$$

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}^{(2)}(K_1,\dots,K_n) = \inf_{L_i\in\mathcal{K}_c^n}\left\{n\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)\prod_{i=1}^{n} V\bigl(L_i^*\bigr)^{-\frac{p}{n^2}}\right\}. \tag{1.3}$$

Here $\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)$ denotes a type of L_p-dual mixed volume of the star bodies $K_1,\dots,K_n$, $L_1,\dots,L_n$ (see (2.5)).

Comparing the definitions (1.2) and (1.3), we easily obtain

$$\widetilde{G}_{-p}^{(2)}(K_1,\dots,K_n) \leq \widetilde{G}_{-p}^{(1)}(K_1,\dots,K_n). \tag{1.4}$$

When $K_1 = \cdots = K_n = K$ in (1.2), then

$$\widetilde{G}_{-p}^{(1)}(K,\dots,K) = \widetilde{G}_{-p}(K). \tag{1.5}$$

Further, we establish some inequalities for the L_p-dual mixed geominimal surface area. Our results can be stated as follows.

Theorem 1.1 *If $K_1,\dots,K_n \in \mathcal{S}_o^n$, $p \geq 1$, $1 \leq m \leq n$, then*

$$\bigl[\widetilde{G}_{-p}^{(1)}(K_1,\dots,K_n)\bigr]^m \leq \prod_{i=0}^{m-1} \widetilde{G}_{-p}^{(1)}(K_1,\dots,K_{n-m},\underbrace{K_{n-i},\dots,K_{n-i}}_{m}); \tag{1.6}$$

$$\bigl[\widetilde{G}_{-p}^{(2)}(K_1,\dots,K_n)\bigr]^m \leq \prod_{i=0}^{m-1} \widetilde{G}_{-p}^{(2)}(K_1,\dots,K_{n-m},\underbrace{K_{n-i},\dots,K_{n-i}}_{m}). \tag{1.7}$$

Equality holds in inequality (1.6) *if and only if K_i $(i = n-m+1,\dots,n)$ all are dilates of each other. Equality holds in inequality* (1.7) *if and only if there exist constants $c_1, c_2,\dots,c_m$ (not all zero) such that, for all $u \in S^{n-1}$,*

$$c_1\rho_{K_n}^{n+p}(u)\rho_{L_n}^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_{L_{n-1}}^{-p}(u) = \cdots = c_m\rho_{K_{n-m+1}}^{n+p}(u)\rho_{L_{n-m+1}}^{-p}(u).$$

In particular, if $m = n$, then we have the following.

Corollary 1.1 *If $K_1, \dots, K_n \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$\big[\widetilde{G}_{-p}^{(2)}(K_1, \dots, K_n)\big]^n \leq \big[\widetilde{G}_{-p}^{(1)}(K_1, \dots, K_n)\big]^n \leq \widetilde{G}_{-p}(K_1) \cdots \widetilde{G}_{-p}(K_n). \tag{1.8}$$

Equality holds in the second inequality of (1.8) *if and only if K_i ($i = 1, 2, \dots, n$) all are dilates of each other.*

Using Corollary 1.1, we may get the following Blaschke-Santalö type inequality.

Corollary 1.2 *If $K_1, \dots, K_n \in \mathcal{K}_c^n$, $n \geq p \geq 1$, then*

$$\widetilde{G}_{-p}^{(2)}(K_1, \dots, K_n)\widetilde{G}_{-p}^{(2)}\big(K_1^*, \dots, K_n^*\big) \leq \widetilde{G}_{-p}^{(1)}(K_1, \dots, K_n)\widetilde{G}_{-p}^{(1)}\big(K_1^*, \dots, K_n^*\big) \leq n^2\omega_n^2. \tag{1.9}$$

Equality holds in the second inequality of (1.9) *if and only if K_i ($i = 1, 2, \dots, n$) all are balls centered at the origin.*

Theorem 1.2 *If $K_1, \dots, K_n \in \mathcal{K}_c^n$, $p \geq 1$, then*

$$\widetilde{G}_{-p}^{(2)}(K_1, \dots, K_n) \leq n\omega_n^{\frac{2n+p}{n}} \prod_{i=1}^{n} V\big(K_i^*\big)^{\frac{-(n+p)}{n^2}}.$$

Theorem 1.3 *If $K_1, \dots, K_n \in \mathcal{S}_o^n$, $1 \leq p < q$, then*

$$\left(\frac{\widetilde{G}_{-p}^{(1)}(K_1, \dots, K_n)^n}{n^n \widetilde{V}(K_1, \dots, K_n)^{n+p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{-q}^{(1)}(K_1, \dots, K_n)^n}{n^n \widetilde{V}(K_1, \dots, K_n)^{n+q}}\right)^{\frac{1}{q}}; \tag{1.10}$$

$$\left(\frac{\widetilde{G}_{-p}^{(2)}(K_1, \dots, K_n)^n}{n^n \widetilde{V}(K_1, \dots, K_n)^{n+p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{-q}^{(2)}(K_1, \dots, K_n)^n}{n^n \widetilde{V}(K_1, \dots, K_n)^{n+q}}\right)^{\frac{1}{q}}. \tag{1.11}$$

Equality holds in (1.10) *and* (1.11) *if and only if each $K_i \in \mathcal{K}_c^n$ ($i = 1, 2, \dots, n$).*

2 Notations and background materials

2.1 Radial function and polar set

If K is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [6, 14])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a nonempty subset in $\mathbb{R}^n$, the polar set, E^*, of E is defined by (see [6, 14])

$$E^* = \big\{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\big\}.$$

For $K \in \mathcal{K}_o^n$ and its polar body, the well-known Blaschke-Stantalö inequality can be stated (see [14]): *If* $K \in \mathcal{K}_c^n$, *then*

$$V(K)V\big(K^*\big) \leq \omega_n^2, \tag{2.1}$$

with equality if and only if K *is an ellipsoid centered at the origin.*

2.2 Dual mixed volume

The dual mixed volume of star bodies was introduced by Lutwak (see [15]). For $K_1, \ldots, K_n \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n$ is given by

$$\widetilde{V}(K_1, K_2, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\rho_{K_2}(u)\cdots\rho_{K_n}(u)\,du. \tag{2.2}$$

The classical Alexander-Fenchel inequality for the dual mixed volume (see [6, 14]) asserts that the integer m satisfies $1 \leq m \leq n$ such that

$$\widetilde{V}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m-1} \widetilde{V}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m}),$$

with equality if and only if $K_{n-m+1}, \ldots, K_n$ *are all dilations of each other.*

In particular, if $m = n$, one has the Minkowski inequality

$$\widetilde{V}(K_1, K_2, \ldots, K_n)^n \leq V(K_1)V(K_2)\cdots V(K_n), \tag{2.3}$$

with equality if and only if $K_1, \ldots, K_n$ *are all dilations of each other.*

2.3 L_p-Dual mixed volume

Lutwak in [1] introduced the L_p-dual mixed volume. For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)\,du. \tag{2.4}$$

Associated with (2.4), for all $K_1, \ldots, K_n \in \mathcal{S}_o^n$, $L_1, \ldots, L_n \in \mathcal{S}_o^n$, and $p \geq 1$, we define

$$\widetilde{V}_{-p}(K_1, \ldots, K_n; L_1, \ldots, L_n) = \frac{1}{n}\int_{S^{n-1}} \prod_{i=1}^{n}\big[\rho_{K_i}^{n+p}(u)\rho_{L_i}^{-p}(u)\big]^{\frac{1}{n}}\,du. \tag{2.5}$$

From (2.2) and (2.5), we easily get

$$\widetilde{V}_{-p}(K_1, \ldots, K_n; K_1, \ldots, K_n) = \widetilde{V}(K_1, \ldots, K_n). \tag{2.6}$$

If $K_1 = \cdots = K_n = K$ and $L_1 = \cdots = L_n = L$ in (2.5), then (2.4) and (2.5) yield

$$\widetilde{V}_{-p}(K, \ldots, K; L, \ldots, L) = \widetilde{V}_{-p}(K, L).$$

3 Results and proofs

In this section, we will prove Theorems 1.1-1.3 and Corollaries 1.1-1.2.

Proof of Theorem 1.1 We first prove inequality (1.7) is true.

Let $\boldsymbol{\rho}_0, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m$ be nonnegative bounded Borel functions on S^{n-1}. By the Hölder inequality (see [16]), we have (see [14])

$$\left(\frac{1}{n}\int_{S^{n-1}} \boldsymbol{\rho}_0(u)\boldsymbol{\rho}_1(u)\cdots\boldsymbol{\rho}_m(u)\,du\right)^m \leq \prod_{i=0}^{m-1}\left(\frac{1}{n}\int_{S^{n-1}} \boldsymbol{\rho}_0(u)\left[\boldsymbol{\rho}_{i+1}(u)\right]^m du\right), \tag{3.1}$$

with equality if and only if there exist constants $b_1, \ldots, b_m \geq 0$ (not all zero) such that $b_1\boldsymbol{\rho}_1^m(u) = \cdots = b_m\boldsymbol{\rho}_m^m(u)$ for all $u \in S^{n-1}$.

For $i = 0, \ldots, m-1$, we let

$$\begin{aligned}
&\boldsymbol{\rho}_0(u) = \left[\rho_{K_1}^{n+p}(u)\rho_{L_1}^{-p}(u)\cdots\rho_{K_{n-m}}^{n+p}(u)\rho_{L_{n-m}}^{-p}(u)\right]^{\frac{1}{n}},\\
&\boldsymbol{\rho}_{i+1}(u) = \left[\rho_{K_{n-i}}^{n+p}(u)\rho_{L_{n-i}}^{-p}(u)\right]^{\frac{1}{n}}.
\end{aligned}$$

In association with (2.5), we get

$$\begin{aligned}
&\widetilde{V}_{-p}(K_1, \ldots, K_n; L_1, \ldots, L_n)^m\\
&\quad= \left(\frac{1}{n}\int_{S^{n-1}} \boldsymbol{\rho}_0(u)\boldsymbol{\rho}_1(u)\cdots\boldsymbol{\rho}_m(u)\,du\right)^m\\
&\quad\leq \prod_{i=0}^{m-1}\left(\frac{1}{n}\int_{S^{n-1}} \boldsymbol{\rho}_0(u)\left[\boldsymbol{\rho}_{i+1}(u)\right]^m du\right)\\
&\quad= \prod_{i=0}^{m-1} \widetilde{V}_{-p}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m}; L_1, \ldots, L_{n-m}, \underbrace{L_{n-i}, \ldots, L_{n-i}}_{m}).
\end{aligned} \tag{3.2}$$

Combining with (1.3), (3.2), we get

$$\begin{aligned}
&\left[\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(2)}(K_1, \ldots, K_n)\right]^m\\
&\quad= \inf_{L_i \in \mathcal{K}_c^n}\left\{n\widetilde{V}_{-p}(K_1, \ldots, K_n; L_1, \ldots, L_n)\prod_{i=1}^{n} V\left(L_i^*\right)^{-\frac{p}{n^2}}\right\}^m\\
&\quad\leq \prod_{i=0}^{m-1}\inf_{L_i \in \mathcal{K}_c^n}\left[nV_{-p}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m}; L_1, \ldots, L_{n-m}, \underbrace{L_{n-i}, \ldots, L_{n-i}}_{m})\right.\\
&\qquad\left.\times V\left(L_{n-i}^*\right)^{-\frac{mp}{n^2}}\prod_{i=1}^{n-m} V\left(L_i^*\right)^{-\frac{p}{n^2}}\right]\\
&\quad= \prod_{i=0}^{m-1}\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(2)}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m})\\
&\quad= \omega_n^{-\frac{mp}{n}}\prod_{i=0}^{m-1}\widetilde{G}_{-p}^{(2)}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}}_{m}),
\end{aligned}$$

i.e.,

$$\big[\widetilde{G}_{-p}^{(2)}(K_1,\ldots,K_n)\big]^m \le \prod_{i=0}^{m-1}\widetilde{G}_{-p}^{(2)}(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m}).$$

This gives (1.7).

According to the equality condition of inequality (3.1), we see that equality holds in inequality (1.7) if and only if

$$c_1\rho_{K_n}^{n+p}(u)\rho_{L_n}^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_{L_{n-1}}^{-p}(u) = \cdots = c_m\rho_{K_{n-m+1}}^{n+p}(u)\rho_{L_{n-m+1}}^{-p}(u)$$

for all $u \in S^{n-1}$, where $c_i = b_i^{n/m}$ ($i = 1,2,\ldots,m$).

Now we complete the proof of (1.6). For $i = 0,\ldots,m-1$, we let

$$\begin{aligned}
\boldsymbol{\rho}_0(u) &= \big[\rho_{K_1}^{n+p}(u)\rho_L^{-p}(u)\cdots\rho_{K_{n-m}}^{n+p}(u)\rho_L^{-p}(u)\big]^{\frac{1}{n}},\\
\boldsymbol{\rho}_{i+1}(u) &= \big[\rho_{K_{n-i}}^{n+p}(u)\rho_L^{-p}(u)\big]^{\frac{1}{n}}.
\end{aligned}$$

In association with (2.5) and (3.1), we get

$$\begin{aligned}
&\widetilde{V}_{-p}(K_1,\ldots,K_n;L,\ldots,L)^m\\
&\quad\le \prod_{i=0}^{m-1}\widetilde{V}_{-p}(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m};L,\ldots,L,\underbrace{L,\ldots,L}_{m}).
\end{aligned}\tag{3.3}$$

Similar to the proof of (1.7), combining with (1.2) and (3.3), we obtain

$$\begin{aligned}
&\big[\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(1)}(K_1,\ldots,K_n)\big]^m\\
&\quad= \inf_{L\in\mathcal{K}_c^n}\big\{n\widetilde{V}_{-p}(K_1,\ldots,K_n;L,\ldots,L)V\big(L^*\big)^{-\frac{p}{n}}\big\}^m\\
&\quad\le \prod_{i=0}^{m-1}\inf_{L\in\mathcal{K}_c^n}\big\{n\widetilde{V}_{-p}(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m};L,\ldots,L,\underbrace{L,\ldots,L}_{m})V\big(L^*\big)^{-\frac{p}{n}}\big\}\\
&\quad\le \prod_{i=0}^{m-1}\omega_n^{-\frac{p}{n}}\widetilde{G}_{-p}^{(1)}(K_1,\ldots,K_{n-m},\underbrace{K_{n-i},\ldots,K_{n-i}}_{m}).
\end{aligned}$$

According to the equality condition of inequality (3.1), we see that equality holds in inequality (1.6) if and only if

$$c_1\rho_{K_n}^{n+p}(u)\rho_L^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_L^{-p}(u) = \cdots = c_m\rho_{K_{n-m+1}}^{n+p}(u)\rho_L^{-p}(u)$$

for all $u \in S^{n-1}$, where $c_i = b_i^{n/m}$ ($i = 1,2,\ldots,m$). This means that

$$c_1\rho_{K_n}^{n+p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u) = \cdots = c_m\rho_{K_{n-m+1}}^{n+p}(u)$$

for all $u \in S^{n-1}$, *i.e.*, K_i ($i = n-m+1,\ldots,n$) all are dilates of each other. $\square$

Proof of Corollary 1.1 Let $m = n$ in (1.6), and together with (1.4) and (1.5), we easily obtain

$$\begin{aligned}\bigl[\widetilde{G}^{(2)}_{-p}(K_1,\dots,K_n)\bigr]^n &\le \bigl[\widetilde{G}^{(1)}_{-p}(K_1,\dots,K_n)\bigr]^n \\ &\le \prod_{i=0}^{n-1}\widetilde{G}^{(1)}_{-p}(K_{n-i},\dots,K_{n-i}) \\ &\le \widetilde{G}_{-p}(K_1)\widetilde{G}_{-p}(K_2)\cdots\widetilde{G}_{-p}(K_n).\end{aligned}$$

This gives (1.8).

From the equality condition of (1.6), we easily find that equality holds in the second inequality of (1.8) if and only if there exist constants $c_1, c_2, \dots, c_n$ (not all zero) such that, for all $u \in S^{n-1}$, $c_1\rho_{K_n}^{n+p}(u)\rho_L^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_L^{-p}(u) = \cdots = c_n\rho_{K_1}^{n+p}(u)\rho_L^{-p}(u)$. This means all K_i ($i = 1, 2, \dots, n$) are dilates of each other. □

In order to prove Corollary 1.2, we give the following lemma.

Lemma 3.1 ([13]) *If $K \in \mathcal{K}_c^n$, $n \ge p \ge 1$, then*

$$\widetilde{G}_{-p}(K)\widetilde{G}_{-p}(K^*) \le n^2\omega_n^2, \tag{3.4}$$

with equality if and only if K is a ball centered at the origin.

Proof of Corollary 1.2 Corollary 1.1, for K and K^*, immediately yields

$$\bigl[\widetilde{G}^{(2)}_{-p}(K_1,\dots,K_n)\bigr]^n \le \bigl[\widetilde{G}^{(1)}_{-p}(K_1,\dots,K_n)\bigr]^n \le \widetilde{G}_{-p}(K_1)\cdots\widetilde{G}_{-p}(K_n), \tag{3.5}$$

$$\bigl[\widetilde{G}^{(2)}_{-p}(K_1^*,\dots,K_n^*)\bigr]^n \le \bigl[\widetilde{G}^{(1)}_{-p}(K_1^*,\dots,K_n^*)\bigr]^n \le \widetilde{G}_{-p}(K_1^*)\cdots\widetilde{G}_{-p}(K_n^*). \tag{3.6}$$

Combining with (3.5), (3.6), and (3.4), we obtain

$$\begin{aligned}&\bigl[\widetilde{G}^{(2)}_{-p}(K_1,\dots,K_n)\bigr]^n\bigl[\widetilde{G}^{(2)}_{-p}(K_1^*,\dots,K_n^*)\bigr]^n \\ &\quad\le \bigl[\widetilde{G}^{(1)}_{-p}(K_1,\dots,K_n)\bigr]^n\bigl[\widetilde{G}^{(1)}_{-p}(K_1^*,\dots,K_n^*)\bigr]^n \\ &\quad\le \widetilde{G}_{-p}(K_1)\widetilde{G}_{-p}(K_1^*)\cdots\widetilde{G}_{-p}(K_n)\widetilde{G}_{-p}(K_n^*) \le \bigl[n^2\omega_n^2\bigr]^n,\end{aligned}$$

i.e.,

$$\widetilde{G}^{(2)}_{-p}(K_1,\dots,K_n)\widetilde{G}^{(2)}_{-p}(K_1^*,\dots,K_n^*) \le \widetilde{G}^{(1)}_{-p}(K_1,\dots,K_n)\widetilde{G}^{(1)}_{-p}(K_1^*,\dots,K_n^*) \le n^2\omega_n^2.$$

This yields (1.9).

By the equality conditions of inequality (3.4) and the second inequality of (1.8), we know that equality holds in the second inequality of (1.9) if and only if $K_1, \dots, K_n$ all are balls centered at the origin. □

Proof of Theorem 1.2 From (1.3), it follows that, for any $L_i \in \mathcal{K}_c^n$,

$$\widetilde{G}^{(2)}_{-p}(K_1,\dots,K_n) \le n\omega_n^{\frac{p}{n}}\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)\prod_{i=1}^n V(L_i^*)^{\frac{-p}{n^2}}.$$

Since $K_1,\dots,K_n \in \mathcal{K}_c^n$, taking $K_1,\dots,K_n$ for $L_1,\dots,L_n$, and using (2.6), (2.3), and (2.1), we get

$$\begin{aligned}
\widetilde{G}_{-p}^{(2)}(K_1,\dots,K_n) &\le n\omega_n^{\frac{p}{n}}\widetilde{V}_{-p}(K_1,\dots,K_n;K_1,\dots,K_n)\prod_{i=1}^{n}V\bigl(K_i^*\bigr)^{-\frac{p}{n^2}} \\
&= n\omega_n^{\frac{p}{n}}\widetilde{V}(K_1,\dots,K_n)\prod_{i=1}^{n}V\bigl(K_i^*\bigr)^{-\frac{p}{n^2}} \\
&\le n\omega_n^{\frac{p}{n}}\bigl[V(K_1)\cdots V(K_n)\bigr]^{\frac{1}{n}}\prod_{i=1}^{n}V\bigl(K_i^*\bigr)^{-\frac{p}{n^2}} \\
&= n\omega_n^{\frac{p}{n}}\bigl[V(K_1)\cdots V(K_n)V\bigl(K_1^*\bigr)\cdots V\bigl(K_n^*\bigr)\bigr]^{\frac{1}{n}}\prod_{i=1}^{n}V\bigl(K_i^*\bigr)^{-\frac{(n+p)}{n^2}} \\
&\le n\omega_n^{\frac{2n+p}{n}}\prod_{i=1}^{n}V\bigl(K_i^*\bigr)^{-\frac{(n+p)}{n^2}}.
\end{aligned}$$

This gives the proof of Theorem 1.2. □

Proof of Theorem 1.3 Using the Hölder inequality, (2.5), and (2.2), we get

$$\begin{aligned}
\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n) &= \frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\bigl[\rho_{K_i}^{n+p}(u)\rho_{L_i}^{-p}(u)\bigr]^{\frac{1}{n}}\,du \\
&= \frac{1}{n}\int_{S^{n-1}}\left(\prod_{i=1}^{n}\bigl[\rho_{K_i}^{n+q}(u)\rho_{L_i}^{-q}(u)\bigr]^{\frac{1}{n}}\right)^{\frac{p}{q}}\left(\prod_{i=1}^{n}\bigl[\rho_{K_i}^{n}(u)\bigr]^{\frac{1}{n}}\right)^{\frac{q-p}{q}}du \\
&\le \widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)^{\frac{p}{q}}\widetilde{V}(K_1,\dots,K_n)^{\frac{q-p}{q}},
\end{aligned}$$

that is,

$$\left(\frac{\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)}{\widetilde{V}(K_1,\dots,K_n)}\right)^{\frac{1}{p}} \le \left(\frac{\widetilde{V}_{-q}(K_1,\dots,K_n;L_1,\dots,L_n)}{\widetilde{V}(K_1,\dots,K_n)}\right)^{\frac{1}{q}}. \tag{3.7}$$

According to the equality condition in the Hölder inequality, we know that equality holds in (3.7) if and only if there exist constants $c_i > 0$ ($i = 1,2,\dots,n$) such that $\rho(K_i,u) = c_i\rho(L_i,u)$ for any $u \in \mathcal{S}^{n-1}$, *i.e.*, for each $i = 1,2,\dots,n$, K_i and L_i both are dilates.

From definition (1.3) and inequality (3.7), we have

$$\begin{aligned}
&\omega_n^{-1}\left(\frac{\widetilde{G}_{-p}^{(2)}(K_1,\dots,K_n)^n}{n^n\widetilde{V}(K_1,\dots,K_n)^{n+p}}\right)^{\frac{1}{p}} \\
&\quad= \inf_{L_i\in\mathcal{K}_c^n}\left\{\left(\frac{\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)}{\widetilde{V}(K_1,\dots,K_n)}\right)^{\frac{n}{p}}\frac{1}{\widetilde{V}(K_1,\dots,K_n)}\left(\prod_{i=1}^{n}V\bigl(L_i^*\bigr)^{-\frac{p}{n^2}}\right)^{\frac{n}{p}}\right\} \\
&\quad= \inf_{L_i\in\mathcal{K}_c^n}\left\{\left(\frac{\widetilde{V}_{-p}(K_1,\dots,K_n;L_1,\dots,L_n)}{\widetilde{V}(K_1,\dots,K_n)}\right)^{\frac{n}{p}}\frac{1}{\widetilde{V}(K_1,\dots,K_n)}\prod_{i=1}^{n}V\bigl(L_i^*\bigr)^{-\frac{1}{n}}\right\} \\
&\quad\le \inf_{L_i\in\mathcal{K}_c^n}\left\{\left(\frac{\widetilde{V}_{-q}(K_1,\dots,K_n;L_1,\dots,L_n)}{\widetilde{V}(K_1,\dots,K_n)}\right)^{\frac{n}{q}}\frac{1}{\widetilde{V}(K_1,\dots,K_n)}\prod_{i=1}^{n}V\bigl(L_i^*\bigr)^{-\frac{1}{n}}\right\}
\end{aligned}$$

$$= \inf_{L_i \in \mathcal{K}_c^n} \left\{ \left(\frac{\widetilde{V}_{-q}(K_1, \ldots, K_n; L_1, \ldots, L_n)}{\widetilde{V}(K_1, \ldots, K_n)} \right)^{\frac{n}{q}} \frac{1}{\widetilde{V}(K_1, \ldots, K_n)} \left(\prod_{i=1}^{n} V(L_i)^{-\frac{q}{n^2}} \right)^{\frac{n}{q}} \right\}$$

$$= \omega_n^{-1} \left(\frac{\widetilde{G}_{-q}^{(2)}(K_1, \ldots, K_n)^n}{n^n \widetilde{V}(K_1, \ldots, K_n)^{n+q}} \right)^{\frac{1}{q}},$$

i.e.,

$$\left(\frac{\widetilde{G}_{-p}^{(2)}(K_1, \ldots, K_n)^n}{n^n \widetilde{V}(K_1, \ldots, K_n)^{n+p}} \right)^{\frac{1}{p}} \le \left(\frac{\widetilde{G}_{-q}^{(2)}(K_1, \ldots, K_n)^n}{n^n \widetilde{V}(K_1, \ldots, K_n)^{n+q}} \right)^{\frac{1}{q}}.$$

This is just (1.11). Because of each $L_i \in \mathcal{K}_c^n$ in inequality (3.7), together with the equality condition of (3.7), we see that equality holds in (1.11) if and only if each $K_i \in \mathcal{K}_c^n$.

In order to prove (1.10), (3.7) can be written

$$\left(\frac{\widetilde{V}_{-p}(K_1, \ldots, K_n; L, \ldots, L)}{\widetilde{V}(K_1, \ldots, K_n)} \right)^{\frac{1}{p}} \le \left(\frac{\widetilde{V}_{-q}(K_1, \ldots, K_n; L, \ldots, L)}{\widetilde{V}(K_1, \ldots, K_n)} \right)^{\frac{1}{q}}. \tag{3.8}$$

In the same way as (1.11), from definition (1.2) and (3.8), we get

$$\left(\frac{\widetilde{G}_{-p}^{(1)}(K_1, \ldots, K_n)^n}{n^n \widetilde{V}(K_1, \ldots, K_n)^{n+p}} \right)^{\frac{1}{p}} \le \left(\frac{\widetilde{G}_{-q}^{(1)}(K_1, \ldots, K_n)^n}{n^n \widetilde{V}(K_1, \ldots, K_n)^{n+q}} \right)^{\frac{1}{q}}.$$

From the equality condition in (1.11), we see that equality holds in (1.10) if and only if each $K_i \in \mathcal{K}_c^n$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed in designing a new algorithm and obtaining complexity results. All authors read and approved the final manuscript.

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