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On the approximation for generalized Szász-Durrmeyer type operators in the space $L_p[0,\infty)$

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Abstract
In this paper we give the direct approximation theorem, the inverse theorem, and the equivalence theorem for Szász-Durrmeyer-Bézier operators in the space $L_p[0,\infty)$ $(1 \le p \le \infty)$ with Ditzian-Totik modulus.
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1 Introduction

In 1972 Bézier [1] introduced the Bézier basic function and Bézier-type operators are the generalized types of the original operators. The introduction of these operators should have some background. Some properties of the convergence and approximation for some Bézier-type operators have been studied (*cf.* [2–6]), but there are other aspects that have not yet been considered. For more information as regards the development of the study on this topic or related field, the interested readers can consult the monograph [7] and the paper [8]. In this paper we will consider the direct, inverse and equivalence theorems for the Szász-Durrmeyer-Bézier operator, which is defined by

$$D_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t) f(t)\,dt \big[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\big], \tag{1.1}$$

where $\alpha \ge 1$, $f \in L_p[0,\infty)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} s_{n,j}(x)$, $s_{n,k} = e^{-nx}\frac{(nx)^k}{k!}$. Obviously $D_{n,\alpha}(f,x)$ is bounded and positive in the space $L_p[0,\infty)$. When $\alpha = 1$, $D_{n,\alpha}(f,x)$ is the well-known Durrmeyer operator

$$D_{n,1}(f,x) = \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t) f(t)\,dt s_{n,k}(x).$$

To describe our results, we give the definitions of the first order modulus of smoothness and the K-functional (*cf.* [9]). For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x}$,

$$\omega_{\varphi}(f,t)_p = \sup_{0<h\le t}\left\{\left\| f\left(x + \frac{h\varphi(x)}{2}\right) - f\left(x - \frac{h\varphi(x)}{2}\right)\right\|_p, x - \frac{h\varphi(x)}{2} \ge 0\right\},$$

$$K_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\left\|\varphi g'\right\|_p\right\},$$

$$\overline{K}_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\left\|\varphi g'\right\|_p + t^2\left\|g'\right\|_p\right\},$$

where $W_p = \{f | f \in A.C._{\text{loc}}, \|\varphi f'\|_p < \infty, \|f'\|_p < \infty\}$.

It is well known that (*cf.* [9])

$$\omega_\varphi(f,t)_p \sim K_\varphi(f,t)_p \sim \overline{K}_\varphi(f,t)_p, \tag{1.2}$$

here $a \sim b$ means that there exists $c > 0$ such that $c^{-1}a \le b \le ca$.

Now we state our equivalence theorem as follows.

Theorem *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x}$, $0 < \beta < 1$, *we have*

$$\left\|D_{n,\alpha}(f,x) - f(x)\right\|_p = O\left(\left(\frac{1}{\sqrt{n}}\right)^\beta\right) \quad \Leftrightarrow \quad \omega_\varphi(f,t)_p = O\left(t^\beta\right). \tag{1.3}$$

Throughout this paper, C denotes a constant independent of n and x, but it is not necessarily the same in different cases.

2 Direct theorem

For convenience, we list some basic properties which will be used later and can be found in [9] and [5] or obtained by simple computation:

(1)

$$1 = J_{n,0}(x) > J_{n,1}(x) > \cdots > J_{n,k}(x) > \cdots > 0; \tag{2.1}$$

(2)

$$0 < J_{n,k}^\alpha(x) - J_{n,k+1}^\alpha(x) \le \alpha s_{n,k}(x), \quad \alpha \ge 1; \tag{2.2}$$

(3)

$$s'_{n,k}(x) = \frac{n}{\varphi^2(x)}\left(\frac{k}{n} - x\right)s_{n,k}(x); \tag{2.3}$$

(4)

$$s'_{n,k}(x) = ns_{n,k-1}(x) - ns_{n,k}(x), \qquad s_{n,-1}(x) = 0; \tag{2.4}$$

(5)

$$J'_{n,0}(x) = 0, \qquad J'_{n,k}(x) = ns_{n,k-1}(x) \quad (k = 1,2,\ldots); \tag{2.5}$$

(6)

$$D_{n,1}\left((t-x)^2, x\right) \le n^{-1}\delta_n^2(x), \tag{2.6}$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Now we give the direct theorem.

Theorem 2.1 *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x}$, *we have*

$$\left\|D_{n,\alpha}(f,x) - f(x)\right\|_p \le C\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_p. \tag{2.7}$$

Proof By the definition of $\overline{K}_\varphi(f,t)_p$ and the relation (1.2), for fixed n, we can choose $g = g_n$ such that

$$\|f-g\|_p + \frac{1}{\sqrt{n}}\left\|\varphi g'\right\|_p + \frac{1}{n}\left\|g'\right\|_p \le C\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_p.$$

Since

$$\begin{aligned} \|D_{n,\alpha}f - f\|_p &\le \left\|D_{n,\alpha}(f-g)\right\|_p + \|f-g\|_p + \|D_{n,\alpha}g - g\|_p \\ &\le C\|f-g\|_p + \|D_{n,\alpha}g - g\|_p, \end{aligned} \tag{2.8}$$

we only need to estimate the second term in the above relation. By the Riesz-Thorin theorem (*cf.* [10, Theorem 3.6]), we separate the proof of the assertions for $p = \infty$ and $p = 1$.

I. $p = \infty$. Noting that $g(t) = g(x) + \int_x^t g'(u)\,du$, we write

$$\begin{aligned} \left|D_{n,\alpha}(g,x) - g(x)\right| &= \left|D_{n,\alpha}\left(\int_x^t g'(u)\,du, x\right)\right| \\ &\le \left\|\delta_n g'\right\|_\infty D_{n,\alpha}\left(\left|\int_x^t \delta_n^{-1}(u)\,du\right|, x\right). \end{aligned}$$

Since

$$\left|\int_x^t \varphi^{-1}(u)\,du\right| = \left|\int_x^t \frac{1}{\sqrt{u}}\,du\right| \le \frac{2|t-x|}{\varphi(x)}, \tag{2.9}$$

$$\left|\int_x^t \left(\frac{1}{\sqrt{n}}\right)^{-1} du\right| = \sqrt{n}|t-x|, \tag{2.10}$$

and $\min\{\varphi^{-1}(x), \sqrt{n}\} \sim \delta_n^{-1}(x)$, using the Hölder inequality, we have

$$\begin{aligned} \left|D_{n,\alpha}(g,x) - g(x)\right| &\le C\delta_n^{-1}(x)\left\|\delta_n g'\right\|_\infty D_{n,\alpha}\left(|t-x|, x\right) \\ &\le C\delta_n^{-1}(x)\left\|\delta_n g'\right\|_\infty \left(D_{n,\alpha}\left((t-x)^2, x\right)\right)^{\frac{1}{2}}. \end{aligned}$$

By (2.2) and (2.6) we have

$$\left(D_{n,\alpha}\left((t-x)^2, x\right)\right)^{\frac{1}{2}} \le \left(\alpha D_{n,1}\left((t-x)^2, x\right)\right)^{\frac{1}{2}} \le Cn^{-\frac{1}{2}}\delta_n(x).$$

Then

$$\left|D_{n,\alpha}(g,x) - g(x)\right| \le \frac{C}{\sqrt{n}}\left\|\delta_n g'\right\|_\infty. \tag{2.11}$$

Then, by (2.8) and (2.11), we get

$$\begin{aligned}\|D_{n,\alpha}(f,x)-f(x)\|_\infty &\le C\left(\|f-g\|_\infty+\frac{1}{\sqrt{n}}\|\delta_n g'\|_\infty\right)\\ &\le C\left(\|f-g\|_\infty+\frac{1}{\sqrt{n}}\|\varphi g'\|_\infty+\frac{1}{n}\|g'\|_\infty\right)\\ &\le C\omega_\varphi\left(f,\frac{1}{\sqrt{n}}\right)_\infty.\end{aligned}\tag{2.12}$$

II. $p=1$. By (2.2) and the Fubini theorem, we have

$$\begin{aligned}&\|D_{n,\alpha}(g,x)-g(x)\|_1\\ &\quad\le\alpha\int_0^\infty\sum_{k=0}^\infty s_{n,k}(x)n\int_0^\infty s_{n,k}(t)\,dt\left|\int_x^t g'(u)\,du\right|dx\\ &\quad=n\alpha\int_0^\infty|g'(u)|\left\{\int_u^\infty\int_0^u+\int_0^u\int_u^\infty\right\}\sum_{k=0}^\infty s_{n,k}(t)s_{n,k}(x)\,dt\,dx\,du\\ &\quad=2\alpha\int_0^\infty|g'(u)|\left(\sum_{k=0}^\infty n\int_u^\infty s_{n,k}(t)\,dt\int_0^u s_{n,k}(x)\,dx\right)du\\ &\quad=:2\alpha\int_0^\infty|g'(u)|H_n(u)\,du.\end{aligned}$$

Now we estimate $H_n(u)$, by using $\int_0^\infty s_{n,k}(t)\,dt=\frac{1}{n}$ and $\sum_{k=0}^\infty s_{n,k}(u)=1$:

$$\begin{aligned}H_n(u)&=n\sum_{k=0}^\infty\left(\int_0^\infty s_{n,k}(t)\,dt\int_0^u s_{n,k}(x)\,dx-\int_0^u s_{n,k}(t)\,dt\int_0^u s_{n,k}(x)\,dx\right)\\ &=u-n\sum_{k=0}^\infty\int_0^u s_{n,k}(t)\,dt\int_0^u s_{n,k}(x)\,dx\\ &=u-n\sum_{k=0}^\infty\left(ts_{n,k}(t)|_0^u-\int_0^u ts'_{n,k}(t)\,dt\right)\int_0^u s_{n,k}(x)\,dx\\ &=u-n\sum_{k=0}^\infty us_{n,k}(u)\int_0^u s_{n,k}(x)\,dx+n\sum_{k=0}^\infty\int_0^u ts'_{n,k}(t)\,dt\int_0^u s_{n,k}(x)\,dx\\ &=:u-I_1+I_2.\end{aligned}$$

Since

$$\begin{aligned}\int_0^u s_{n,k}(x)\,dx=\int_0^u e^{-nx}\frac{(nx)^k}{k!}\,dx&=-\frac{1}{n}s_{n,k}(u)+\int_0^u s_{n,k-1}(x)\,dx\\ &=-\frac{1}{n}\big(s_{n,k}(u)+\cdots+s_{n,1}(u)\big)+\int_0^u e^{-nx}\,dx\\ &=-\frac{1}{n}\big(s_{n,k}(u)+\cdots+s_{n,0}(u)\big)+\frac{1}{n},\end{aligned}$$

we have

$$\begin{aligned}
I_1 &= n\sum_{k=0}^{\infty} us_{n,k}(u)\left[-\frac{1}{n}\sum_{j=0}^{k} s_{n,j}(u) + \frac{1}{n}\right] \\
&= \sum_{k=0}^{\infty} us_{n,k}(u) - \sum_{k=0}^{\infty} us_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u) \\
&= u - \sum_{k=0}^{\infty} us_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u).
\end{aligned}$$

Using the equation $\frac{k+1}{n}s_{n,k+1}(u) = us_{n,k}(u)$ and (2.4), we have

$$\begin{aligned}
I_2 &= n\sum_{k=0}^{\infty}\int_0^u tn\bigl(s_{n,k-1}(t) - s_{n,k}(t)\bigr)\,dt\int_0^u s_{n,k}(x)\,dx \\
&= n\sum_{k=0}^{\infty}\int_0^u ts_{n,k}(t)\,dt\int_0^u n\bigl(s_{n,k+1}(x) - s_{n,k}(x)\bigr)\,dx \\
&= n\sum_{k=0}^{\infty}\int_0^u ts_{n,k}(t)\,dt\int_0^u \bigl(-s'_{n,k+1}(x)\bigr)\,dx \\
&= -n\sum_{k=0}^{\infty}\int_0^u \frac{k+1}{n}s_{n,k+1}(t)\,dts_{n,k+1}(u) \\
&= -n\sum_{k=0}^{\infty}\int_0^u s_{n,k+1}(t)\,dtus_{n,k}(u) \\
&= -n\sum_{k=0}^{\infty}\left[-\frac{1}{n}\bigl(s_{n,k+1}(u) + \cdots + s_{n,0}(u)\bigr) + \frac{1}{n}\right]us_{n,k}(u) \\
&= \sum_{k=0}^{\infty} us_{n,k}(u)\sum_{j=0}^{k+1} s_{n,j}(u) - \sum_{k=0}^{\infty} us_{n,k}(u) \\
&= \sum_{k=0}^{\infty} us_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u) + u\sum_{k=0}^{\infty} s_{n,k}(u)s_{n,k+1}(u) - u.
\end{aligned}$$

So

$$H_n(u) = u - u + 2u\sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u) + u\sum_{k=0}^{\infty} s_{n,k}(u)s_{n,k+1}(u) - u.$$

Since

$$\begin{aligned}
\sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u) &= \sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=k}^{\infty} s_{n,j}(u) \\
&= \sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=k+1}^{\infty} s_{n,j}(u) + \sum_{k=0}^{\infty} s_{n,k}(u)s_{n,k}(u),
\end{aligned}$$

we can write

$$\begin{aligned}H_n(u) &= u\left(\sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=0}^{k} s_{n,j}(u) + \sum_{k=0}^{\infty} s_{n,k}(u)\sum_{j=k+1}^{\infty} s_{n,j}(u)\right)\\ &\quad + u\sum_{k=0}^{\infty} s_{n,k}(u)\big(s_{n,k}(u)+s_{n,k+1}(u)\big) - u\\ &= u\sum_{k=0}^{\infty} s_{n,k}(u)\big(s_{n,k}(u)+s_{n,k+1}(u)\big).\end{aligned}$$

Using the result of [5, Lemma 3]

$$s_{n,k}(u) \le \frac{1}{\sqrt{nu}},$$

we get

$$H_n(u) \le 2\frac{\sqrt{u}}{\sqrt{n}}.$$

Consequently

$$\begin{aligned}\big\|D_{n,\alpha}(g,x)-g(x)\big\|_1 &\le 4\alpha\int_0^{\infty}\big|g'(u)\big|\frac{\varphi(u)}{\sqrt{n}}\,du\\ &= \frac{4\alpha}{\sqrt{n}}\big\|\varphi' g\big\|_1.\end{aligned} \tag{2.13}$$

By (2.8) and (2.13) we have

$$\big\|D_{n,\alpha}(f,x)-f(x)\big\|_1 \le C\omega_\varphi\left(f,\frac{1}{\sqrt{n}}\right)_1. \tag{2.14}$$

From (2.12) and (2.14), (2.7) is obtained. □

Remark 1 In [11] we show that the second order modulus cannot be used for the Baskakov-Bézier operators. Similarly in (2.7) $\omega_\varphi^2(f,x)_p$ cannot be used instead of $\omega_\varphi(f,x)_p$.

3 Inverse theorem

To prove the inverse theorem, we need the following lemmas.

Lemma 3.1 *For $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, we have*

$$\big\|\delta_n D'_{n,\alpha}(f)\big\|_p \le C\sqrt{n}\|f\|_p. \tag{3.1}$$

Proof We will show (3.1) for the two cases of $p = \infty$ and $p = 1$. Since

$$\begin{aligned}D'_{n,\alpha}(f,x) &= \sum_{k=0}^{\infty}\alpha\big[J_{n,k}^{\alpha-1}(x)J'_{n,k}(x) - J_{n,k+1}^{\alpha-1}(x)J'_{n,k+1}(x)\big]n\int_0^{\infty} s_{n,k}(t)f(t)\,dt\\ &= \alpha\sum_{k=0}^{\infty}\big[\big(J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x)\big)J'_{n,k+1}(x) + J_{n,k}^{\alpha-1}(x)s'_{n,k}(x)\big]n\int_0^{\infty} s_{n,k}(t)f(t)\,dt\end{aligned}$$

using $\int_0^\infty s_{n,k}(t)\,dt = \frac{1}{n}$, we have

$$\begin{aligned}\left|D'_{n,\alpha}(f,x)\right| &\le \alpha\|f\|_\infty\left(\sum_{k=0}^\infty\left[J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x)\right]J'_{n,k+1}(x) + \sum_{k=0}^\infty J_{n,k}^{\alpha-1}(x)\left|s'_{n,k}(x)\right|\right)\\ &=: \alpha\|f\|_\infty(I_1 + I_2).\end{aligned} \tag{3.2}$$

For $x \in E_n = (\frac{1}{n}, \infty)$, $\delta_n(x) \sim \varphi(x)$, by (2.1) and (2.3) we get

$$\begin{aligned}\delta_n(x)I_2 &\le \delta_n(x)\sum_{k=0}^\infty\left|s'_{n,k}(x)\right| \le \frac{n\delta_n(x)}{\varphi^2(x)}\sum_{k=0}^\infty\left|\frac{k}{n} - x\right|s_{n,k}(x)\\ &\le \frac{n\delta_n(x)}{\varphi^2(x)}\left(\sum_{k=0}^\infty\left(\frac{k}{n} - x\right)^2 s_{n,k}(x)\right)^{\frac{1}{2}}\\ &= \frac{n\delta_n(x)}{\varphi^2(x)}\frac{\varphi(x)}{\sqrt{n}} \le 2\sqrt{n},\end{aligned} \tag{3.3}$$

here we used (*cf.* [9, p.128, Lemma 9.4.3])

$$\sum_{k=0}^\infty\left(\frac{k}{n} - x\right)^2 s_{n,k}(x) = \frac{\varphi^2(x)}{n}.$$

For $x \in E_n^c = [0, \frac{1}{n}]$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, by (2.4) we have

$$\delta_n(x)I_2 \le \delta_n(x)\sum_{k=0}^\infty n\big(s_{n,k-1}(x) + s_{n,k}(x)\big) \le 4\sqrt{n}. \tag{3.4}$$

By (3.3) and (3.4), we get

$$\delta_n(x)I_2 \le 6\sqrt{n}. \tag{3.5}$$

Noting $J'_{n,0}(x) = 0$, we have

$$\begin{aligned}I_1 &= \sum_{k=0}^\infty\big(J_{n,k}^{\alpha-1}(x)\big(J'_{n,k}(x) - s'_{n,k}(x)\big) - J_{n,k+1}^{\alpha-1}(x)J'_{n,k+1}(x)\big)\\ &\le \sum_{k=0}^\infty J_{n,k}^{\alpha-1}(x)J'_{n,k}(x) - \sum_{k=0}^\infty J_{n,k+1}^{\alpha-1}(x)J'_{n,k+1}(x) + \sum_{k=0}^\infty J_{n,k}^{\alpha-1}(x)\left|s'_{n,k}(x)\right|\\ &= I_2,\end{aligned}$$

then

$$\delta_n(x)I_1 \le 6\sqrt{n}. \tag{3.6}$$

Combining (3.2), (3.5), and (3.6) we get for $p = \infty$

$$\left\|\delta_n D'_{n,\alpha}(f)\right\|_\infty \le C\sqrt{n}\|f\|_\infty. \tag{3.7}$$

For $p = 1$, we have

$$\begin{aligned} \left|D'_{n,\alpha}(f)\right| &\le \alpha \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt \left[\left(J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x)\right)J'_{n,k+1}(x) + J_{n,k}^{\alpha-1}(x)\left|s'_{n,k}(x)\right|\right] \\ &=: \alpha(\widetilde{J}_1 + \widetilde{J}_2). \end{aligned} \tag{3.8}$$

Hence we can write

$$\int_0^{\infty} \left|\delta_n(x)D'_{n,\alpha}(f,x)\right| dx \le \alpha \int_0^{\infty} \delta_n(x)(\widetilde{J}_1 + \widetilde{J}_2)\, dx = \alpha\left(\int_{E_n^c} + \int_{E_n}\right)\delta_n(x)(\widetilde{J}_1 + \widetilde{J}_2)\, dx. \tag{3.9}$$

Now we estimate the last part of (3.9) in four phases:

$$\begin{aligned} \int_{E_n^c} \delta_n(x)\widetilde{J}_2\, dx &\le \int_{E_n^c} \delta_n(x) \sum_{k=1}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt n\left(s_{n,k-1}(x) + s_{n,k}(x)\right) dx \\ &\quad + \int_{E_n^c} \delta_n(x) n \int_0^{\infty} s_{n,0}(t)\left|f(t)\right| dt n s_{n,0}(x)\, dx. \end{aligned}$$

For $x \in E_n^c$, $\delta_n(x) \le \frac{2}{\sqrt{n}}$, $s_{n,0}(t) \le 1$, noting $\int_0^{\infty} s_{n,k}(x)\, dx = \frac{1}{n}$, we have

$$\int_{E_n^c} \delta_n(x)\widetilde{J}_2\, dx \le \frac{4n}{\sqrt{n}}\|f\|_1 + \frac{2n}{\sqrt{n}}\|f\|_1 \le 6\sqrt{n}\|f\|_1. \tag{3.10}$$

Since $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le 1$ and $J'_{n,k+1}(x) = n s_{n,k}(x)$, we have

$$\int_{E_n^c} \delta_n(x)\widetilde{J}_1\, dx \le \int_{E_n^c} \delta_n(x) \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt n s_{n,k}(x)\, dx \le 2\sqrt{n}\|f\|_1. \tag{3.11}$$

To estimate $\int_{E_n} \delta_n(x)\widetilde{J}_2\, dx$, we will need the relation [9, p.129, (9.4.15)]

$$\int_{E_n} \frac{(\frac{k}{n} - x)^2}{\varphi^2(x)} s_{n,k}(x)\, dx \le Cn^{-2}.$$

By the Hölder inequality and (2.3), we get

$$\begin{aligned} &\int_{E_n} \delta_n(x)\widetilde{J}_2\, dx \\ &\quad \le 2\sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt \int_{E_n} \varphi(x)\frac{n}{\varphi^2(x)}\left|\frac{k}{n} - x\right| s_{n,k}(x)\, dx \\ &\quad \le 2\sum_{k=0}^{\infty} n^2 \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt \left(\int_{E_n} \frac{(\frac{k}{n} - x)^2}{\varphi^2(x)} s_{n,k}(x)\, dx\right)^{\frac{1}{2}} \left(\int_{E_n} s_{n,k}(x)\, dx\right)^{\frac{1}{2}} \\ &\quad \le C\sqrt{n}\sum_{k=0}^{\infty} \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt \\ &\quad = C\sqrt{n}\|f\|_1. \end{aligned} \tag{3.12}$$

In order to estimate $\int_{E_n}\delta_n(x)\widetilde{J}_1\,dx$, we consider the two cases of $\alpha\geq 2$ and $1<\alpha<2$ (when $\alpha=1$, $\widetilde{J}_1=0$).

For $\alpha\geq 2$, $J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)\leq(\alpha-1)s_{n,k}(x)$. Using integration by parts, we can deduce

$$\begin{aligned}
\int_{E_n}\delta_n(x)\widetilde{J}_1\,dx &\leq C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\int_{E_n}\varphi(x)s_{n,k}(x)J'_{n,k+1}(x)\,dx\\
&= C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\\
&\quad\times\Big(\varphi(x)s_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty}-\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\,d\big(\varphi(x)s_{n,k}(x)\big)\Big)\\
&= C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\varphi(x)s_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty}\\
&\quad- C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\\
&\quad\times\Big(\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\frac{1}{2\sqrt{x}}s_{n,k}(x)\,dx+\int_{\frac{1}{n}}^{\infty}\varphi(x)s'_{n,k}(x)J_{n,k+1}(x)\,dx\Big).
\end{aligned}$$

Noting that $\varphi(x)s_{n,k}(x)J_{n,k+1}(x)|_{\frac{1}{n}}^{\infty}<0$ and $\int_{\frac{1}{n}}^{\infty}J_{n,k+1}(x)\frac{1}{2\sqrt{x}}s_{n,k}(x)\,dx>0$, and from (2.3), we have

$$\begin{aligned}
&\int_{E_n}\delta_n(x)\widetilde{J}_1\,dx\\
&\quad\leq C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\int_{\frac{1}{n}}^{\infty}\big|\varphi(x)s'_{n,k}(x)J_{n,k+1}(x)\big|\,dx\\
&\quad\leq C\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dtn\bigg(\int_{E_n}\frac{(\frac{k}{n}-x)^2}{\varphi^2(x)}s_{n,k}(x)\,dx\bigg)^{\frac{1}{2}}\bigg(\int_{E_n}s_{n,k}(x)\,dx\bigg)^{\frac{1}{2}}\\
&\quad\leq C\sqrt{n}\|f\|_1.
\end{aligned}\tag{3.13}$$

For $1<\alpha<2$, using the mean value theorem, we know

$$J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)=(\alpha-1)\big(\xi_k(x)\big)^{\alpha-2}s_{n,k}(x),$$

where $J_{n,k+1}(x)<\xi_k(x)<J_{n,k}(x)$ and $\alpha-2<0$, then

$$J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)\leq(\alpha-1)J_{n,k+1}^{\alpha-2}(x)s_{n,k}(x).$$

For $1<\alpha<2$, we get from the procedure of (3.13)

$$\begin{aligned}
\int_{E_n}\delta_n(x)\widetilde{J}_1\,dx &\leq\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\int_{E_n}(\alpha-1)\varphi(x)s_{n,k}(x)J_{n,k+1}^{\alpha-2}(x)J'_{n,k+1}(x)\,dx\\
&=\sum_{k=0}^{\infty}n\int_0^{\infty}s_{n,k}(t)\big|f(t)\big|\,dt\int_{E_n}\varphi(x)s_{n,k}(x)\,dJ_{n,k+1}^{\alpha-1}(x)
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|f(t)\right| dt \int_{E_n} \varphi(x)\left|s'_{n,k}(x)\right| dx \\ &\leq C\sqrt{n}\|f\|_1.\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we get for $\alpha \geq 1$

$$\int_{E_n} \delta_n(x)\widetilde{J}_1\, dx \leq C\sqrt{n}\|f\|_1. \tag{3.15}$$

From (3.8)-(3.12) and (3.15), we obtain

$$\left\|\delta_n D'_{n,\alpha}(f)\right\|_1 \leq C\sqrt{n}\|f\|_1. \tag{3.16}$$

By (3.7) and (3.16), Lemma 3.1 holds. □

Lemma 3.2 *For $f \in W_p$, $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, we have*

$$\left\|\delta_n D'_{n,\alpha}(f)\right\|_p \leq C\left\|\delta_n f'\right\|_p. \tag{3.17}$$

Proof By the Riesz-Thorin theorem, we shall prove Lemma 3.2 for $p = \infty$ and $p = 1$. For $f \in W_p$ and noting that $D'_{n,\alpha}(1,x) = 0$, we have

$$\sum_{k=0}^{\infty} n \int_0^{\infty} f(x)s_{n,k}(t)\, dt\big(J^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x)\big)' = 0.$$

Then

$$\begin{aligned}\left|D'_{n,\alpha}(f,x)\right| &= \left|\sum_{k=0}^{\infty} n \int_0^{\infty} \big(f(t) - f(x)\big)s_{n,k}(t)\, dt\big(J^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x)\big)'\right| \\ &= \left|\sum_{k=0}^{\infty} n \int_0^{\infty} \int_x^t f'(u)\, du s_{n,k}(t)\, dt\big(J^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x)\big)'\right| \\ &\leq \alpha \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)\left|\int_x^t f'(u)\, du\right| dt \\ &\quad \times \left\{\left[J^{\alpha-1}_{n,k}(x) - J^{\alpha-1}_{n,k+1}(x)\right]J'_{n,k+1}(x) + J^{\alpha-1}_{n,k}(x)\left|s'_{n,k}(x)\right|\right\}.\end{aligned} \tag{3.18}$$

By (2.9) and (2.10) we have

$$\left|\int_x^t \delta_n(u)\, du\right| \leq C\delta_n^{-1}(x)|t-x|,$$

hence

$$\begin{aligned}\left|\delta_n(x)D'_{n,\alpha}(f,x)\right| &\leq C\left\|\delta_n f'\right\|_{\infty} \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\, dt \\ &\quad \times \left\{\left[J^{\alpha-1}_{n,k}(x) - J^{\alpha-1}_{n,k+1}(x)\right]J'_{n,k+1}(x) + J^{\alpha-1}_{n,k}(x)\left|s'_{n,k}(x)\right|\right\} \\ &=: C\left\|\delta_n f'\right\|_{\infty}(J_1 + J_2).\end{aligned} \tag{3.19}$$

For $x \in E_n^c$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$ and by (2.1) and (2.4) we have

$$\begin{aligned} J_2 &= \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt J_{n,k}^{\alpha-1}(x)\left|s'_{n,k}(x)\right| \\ &\le n \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt\bigl(s_{n,k-1}(x)+s_{n,k}(x)\bigr) \\ &= n \sum_{k=1}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt s_{n,k-1}(x) + n \sum_{k=0}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt s_{n,k}(x) \\ &=: K_1 + K_2. \end{aligned}$$

By (2.6) we get

$$K_2 = nD_{n,1}\bigl(|t-x|,x\bigr) \le n\bigl(D_{n,1}\bigl((t-x)^2,x\bigr)\bigr)^{\frac{1}{2}} \le \sqrt{n}\delta_n(x) \le 2.$$

For K_1, we write

$$K_1 = n \sum_{k=2}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt s_{n,k-1}(x) + n^2 \int_0^{\infty} s_{n,1}(t)|t-x|\,dt s_{n,0}(x).$$

First, using (2.6), $\int_0^{\infty} s_{n,k}(t)\,dt = \frac{1}{n}$, and the Hölder inequality, we have

$$\begin{aligned} & n \sum_{k=2}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt s_{n,k-1}(x) \\ &\quad = n \sum_{k=2}^{\infty} n \int_0^{\infty} s_{n,k}(t)|t-x|\,dt s_{n,k}(x)\frac{k}{nx} \\ &\quad \le n\left(\sum_{k=2}^{\infty} n^2\left(\int_0^{\infty} s_{n,k}(t)|t-x|\,dt\right)^2 s_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty} s_{n,k}(x)\frac{k^2}{(nx)^2}\right)^{\frac{1}{2}} \\ &\quad \le n\left(\sum_{k=2}^{\infty} n^2 \int_0^{\infty} s_{n,k}(t)\,dt \cdot \int_0^{\infty} s_{n,k}(t)(t-x)^2\,dt s_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty} s_{n,k}(x)\frac{k^2}{(nx)^2}\right)^{\frac{1}{2}} \\ &\quad \le Cn\left(\sum_{k=2}^{\infty} n \int_0^{\infty} s_{n,k}(t)(t-x)^2\,dt s_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty} s_{n,k-2}(x)\right)^{\frac{1}{2}} \\ &\quad \le Cn \cdot \frac{\delta_n(x)}{\sqrt{n}} \\ &\quad \le C. \end{aligned}$$

Next, for $x \in E_n^c$, $e^{-nx} \le 1$, and $\int_0^{\infty} s_{n,k}(t)\,dt = \frac{1}{n}$, we have

$$\begin{aligned} n^2 \int_0^{\infty} s_{n,1}(t)|t-x|\,dt s_{n,0}(x) &\le n^2 \int_0^{\infty} s_{n,1}(t)(t+x)\,dt e^{-nx} \\ &\le n^2\left(\frac{2}{n}\int_0^{\infty} s_{n,2}(t)\,dt + x\int_0^{\infty} s_{n,1}(t)\,dt\right) \end{aligned}$$

$$\begin{aligned}
&= n^2\left(\frac{2}{n^2}+\frac{x}{n}\right)\\
&\le 3.
\end{aligned}$$

Then we get

$$J_2 \le C, \quad x \in E_n^c. \tag{3.20}$$

Noting that $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le 1$, by (2.5) we have

$$J_1 \le K_2 \le 2. \tag{3.21}$$

For $x \in E_n$, $\delta_n(x) \sim \varphi(x)$, using (2.3), $\int_0^\infty s_{n,k}(t)\,dt = \frac{1}{n}$ and the Hölder inequality, we have

$$\begin{aligned}
J_2 &\le \sum_{k=0}^{\infty} n\int_0^\infty s_{n,k}(t)|t-x|\,dt\frac{n}{\varphi^2(x)}\left|\frac{k}{n}-x\right|s_{n,k}(x)\\
&\le \left(\sum_{k=0}^{\infty} n\int_0^\infty s_{n,k}(t)(t-x)^2\,dts_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2 s_{n,k}(x)\right)^{\frac{1}{2}}\cdot\frac{n}{\varphi^2(x)}\\
&\le \frac{\delta_n(x)}{\sqrt{n}}\cdot\frac{\varphi(x)}{\sqrt{n}}\cdot\frac{n}{\varphi^2(x)}\\
&\le 2.
\end{aligned}$$

Noting that $J'_{n,0}(x) = 0$, one has

$$\begin{aligned}
J_1 &= \sum_{k=0}^{\infty} n\int_0^\infty s_{n,k}(t)|t-x|\,dt\big[J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x)\big]J'_{n,k+1}(x)\\
&\le \sum_{k=1}^{\infty} n\int_0^\infty s_{n,k}(t)|t-x|\,dtJ_{n,k}^{\alpha-1}(x)J'_{n,k}(x) - \sum_{k=0}^{\infty} n\int_0^\infty s_{n,k}(t)|t-x|\,dtJ_{n,k+1}^{\alpha-1}(x)J'_{n,k+1}(x)\\
&\quad+\sum_{k=0}^{\infty} n\int_0^\infty s_{n,k}(t)|t-x|\,dtJ_{n,k}^{\alpha-1}(x)\big|s'_{n,k}(x)\big|.
\end{aligned}$$

The third term of the above is J_2, J_3 denotes the difference of the front two terms, and we need only to consider J_3. By (2.1), (2.4), (2.5), and integration by parts, we have

$$\begin{aligned}
J_3 &\le \sum_{k=1}^{\infty} n\int_0^\infty \big(s_{n,k}(t) - s_{n,k-1}(t)\big)|t-x|\,dtJ_{n,k}^{\alpha-1}(x)J'_{n,k}(x)\\
&\le \sum_{k=1}^{\infty} n\left|\int_0^\infty \frac{1}{n}s'_{n,k}(t)|t-x|\,dt\right|ns_{n,k-1}(x)\\
&\le \sum_{k=1}^{\infty}\left(\left|\int_0^x s'_{n,k}(t)(x-t)\,dt\right| + \left|\int_x^\infty s'_{n,k}(t)(t-x)\,dt\right|\right)ns_{n,k-1}(x)\\
&= \sum_{k=1}^{\infty}\left(\left|\int_0^x (x-t)\,ds_{n,k}(t)\right| + \left|\int_x^\infty (t-x)\,ds_{n,k}(t)\right|\right)ns_{n,k-1}(x)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \int_0^{\infty} s_{n,k}(t)\,dtns_{n,k-1}(x) \\
&= 1.
\end{aligned}$$

Thus

$$J_1 \leq J_2 + J_3 \leq 3. \tag{3.22}$$

So we get

$$\left\| \delta_n D'_{n,\alpha}(f) \right\|_{\infty} \leq C \left\| \delta_n f' \right\|_{\infty}. \tag{3.23}$$

For $p = 1$, using (2.1), (2.4), (2.5), and integration by parts, we have

$$\begin{aligned}
D'_{n,\alpha}(f,x) &= \alpha \sum_{k=1}^{\infty} n \int_0^{\infty} f(t) s_{n,k}(t)\,dt J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&\quad - \alpha \sum_{k=0}^{\infty} n \int_0^{\infty} f(t) s_{n,k}(t)\,dt J_{n,k+1}^{\alpha-1}(x) J'_{n,k+1}(x) \\
&= \alpha \sum_{k=1}^{\infty} n \int_0^{\infty} f(t) \big(s_{n,k}(t) - s_{n,k-1}(t) \big)\,dt J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&= -\alpha \sum_{k=1}^{\infty} \int_0^{\infty} f(t) s'_{n,k}(t)\,dt J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&= \alpha \sum_{k=1}^{\infty} \int_0^{\infty} f'(t) s_{n,k}(t)\,dt J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&\leq \alpha \sum_{k=1}^{\infty} \int_0^{\infty} f'(t) s_{n,k}(t)\,dtns_{n,k-1}(x).
\end{aligned}$$

Let

$$\begin{aligned}
\left\| \delta_n D'_{n,\alpha}(f) \right\|_1 &= \left\| \delta_n D'_{n,\alpha}(f) \right\|_1^{E_n^c} + \left\| \delta_n D'_{n,\alpha}(f) \right\|_1^{E_n} \\
&=: \widetilde{K}_1 + \widetilde{K}_2.
\end{aligned} \tag{3.24}$$

For $x \in E_n^c$, noting that $\sqrt{n}\delta_n(t) \geq 1$ and $\sqrt{n}\delta_n(x) \leq 2$, we have

$$\begin{aligned}
\widetilde{K}_1 &\leq \alpha \int_{E_n^c} \delta_n(x) \sum_{k=1}^{\infty} \int_0^{\infty} \left| f'(t) \right| s_{n,k}(t)\,dtns_{n,k-1}(x)\,dx \\
&\leq \alpha \int_{E_n^c} \sqrt{n}\delta_n(x) \sum_{k=1}^{\infty} \int_0^{\infty} \left| \delta_n(t) f'(t) \right| s_{n,k}(t)\,dtns_{n,k-1}(x)\,dx \\
&\leq 2\alpha \sum_{k=1}^{\infty} \int_0^{\infty} \left| \delta_n(t) f'(t) \right| s_{n,k}(t)\,dtn \int_{E_n^c} s_{n,k-1}(x)\,dx \\
&\leq 2\alpha \left\| \delta_n f' \right\|_1.
\end{aligned} \tag{3.25}$$

For $x \in E_n$, we estimate $\widetilde{K}_2$

$$\begin{aligned}\widetilde{K}_2 &\le 2\alpha \int_{E_n} \varphi(x) \sum_{k=1}^{\infty} \int_0^{\infty} \frac{|\delta_n(t) f'(t)|}{\varphi(t)} s_{n,k}(t)\, dt n s_{n,k-1}(x)\, dx \\ &= 2n\alpha \sum_{k=1}^{\infty} \int_0^{\infty} \frac{|\delta_n(t) f'(t)|}{\varphi(t)} s_{n,k}(t)\, dt \int_{E_n} \varphi(x) s_{n,k-1}(x)\, dx.\end{aligned}$$

Using the Hölder inequality and $2\sqrt{ab} \le a + b$ $(a, b > 0)$, we have

$$\begin{aligned}&\int_0^{\infty} \frac{|\delta_n(t) f'(t)|}{\varphi(t)} s_{n,k}(t)\, dt \\ &\quad\le \left(\int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k}(t)\, dt \int_0^{\infty} \frac{|\delta_n(t) f'(t)|}{\varphi^2(t)} s_{n,k}(t)\, dt \right)^{\frac{1}{2}} \\ &\quad= \left(\int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k}(t)\, dt \int_0^{\infty} \left|\delta_n(t) f'(t)\right| \frac{n}{k} s_{n,k-1}(t)\, dt \right)^{\frac{1}{2}} \\ &\quad\le \left(\frac{n}{k}\right)^{\frac{1}{2}} \left(\int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k}(t)\, dt + \int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k-1}(t)\, dt \right)\end{aligned}$$

and

$$\begin{aligned}\int_{E_n} \varphi(x) s_{n,k-1}(x)\, dx &\le \frac{1}{\sqrt{n}} \left(\int_{E_n} \varphi^2(x) s_{n,k-1}(x)\, dx \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{n}} \left(\frac{k}{n}\right)^{\frac{1}{2}} \left(\int_{E_n} s_{n,k}(x)\, dx \right)^{\frac{1}{2}} \\ &\le \frac{1}{n} \left(\frac{k}{n}\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\widetilde{K}_2 &\le 2\alpha \sum_{k=1}^{\infty} \left(\int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k}(t)\, dt + \int_0^{\infty} \left|\delta_n(t) f'(t)\right| s_{n,k-1}(t)\, dt \right) \\ &\le 2\alpha \left\| \delta_n f' \right\|_1. \end{aligned} \tag{3.26}$$

By (3.24)-(3.26) we have

$$\left\| \delta_n D'_{n,\alpha}(f) \right\|_1 \le C \left\| \delta_n f' \right\|_1. \tag{3.27}$$

By (3.23) and (3.27), Lemma 3.2 holds. □

Using Lemmas 3.1 and 3.2, we can prove the inverse theorem.

Theorem 3.3 *For* $f \in L_p[0,\infty)$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x}$, $0 < \beta < 1$,

$$\left\| D_{n,\alpha}(f,x) - f(x) \right\|_p = O\left(n^{-\frac{\beta}{2}}\right)$$

implies $\omega_\varphi(f,t)_p = O(t^\beta)$.

Proof Using Lemmas 3.1 and 3.2, for a suitable function g, we have

$$\begin{aligned}
K_\varphi(f,t)_p &\le \|f - D_{n,\alpha}(f)\|_p + t\|\delta_n D'_{n,\alpha}(f)\|_p \\
&\le Cn^{-\frac{\beta}{2}} + t\left(\|\delta_n D'_{n,\alpha}(f-g)\|_p + \|\delta_n D'_{n,\alpha}(g)\|_p\right) \\
&\le Cn^{-\frac{\beta}{2}} + Ct\left(\sqrt{n}\|f-g\|_p + \|\delta_n g'\|_p\right) \\
&\le Cn^{-\frac{\beta}{2}} + Ct\sqrt{n}\left(\|f-g\|_p + \frac{1}{\sqrt{n}}\|\varphi g'\|_p + \frac{1}{n}\|g'\|_p\right) \\
&\le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}\overline{K}_\varphi\left(f, n^{-\frac{1}{2}}\right)_p\right) \\
&\le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}K_\varphi\left(f, n^{-\frac{1}{2}}\right)_p\right)
\end{aligned}$$

which by the Berens-Lorentz lemma (*cf.* [9, Lemma 9.3.4]) implies that

$$K_\varphi(f,t)_p = O\left(t^\beta\right). \tag{3.28}$$

From (1.2) and (3.28), we see that the proof of Theorem 3.3 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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