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# Some identities of special $q$-polynomials

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**Abstract**

In this paper, we investigate some identities of $q$-extensions of special polynomials which are derived from the fermonic $q$-integral on $\mathbb{Z}_p$ and the bosonic $q$-integral on $\mathbb{Z}_p$.

## 1 Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ and $\mathrm{UD}(\mathbb{Z}_p)$ be the space of all uniformly differentiable functions on $\mathbb{Z}_p$. The $q$-analog of $x$ is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x \quad \text{(see [1, 2])} \tag{1.1}$$

and the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is also defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [1–3])}. \tag{1.2}$$

From (1.1) and (1.2), we have

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0) \tag{1.3}$$

and

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0) \quad \text{(see [1–3])}. \tag{1.4}$$

As is well known, the $q$-analog of the Bernoulli polynomials is given by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} \quad \text{(see [1, 2, 4–20])}, \tag{1.5}$$

and the $q$-analog of the Euler polynomials is given by

$$\frac{[2]_q}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}\quad\text{(see [1, 2, 4–21])}.\tag{1.6}$$

The higher-order $q$-Daehee polynomials are given by

$$\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}(1+t)^x=\sum_{n=0}^{\infty}D_{n,q}(x)\frac{t^n}{n!},\tag{1.7}$$

where $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$.

Now, we define the $q$-analog of the Changhee polynomials, which are given by the generating function to be

$$\left(\frac{[2]_q}{qt+[2]_q}\right)(1+t)^x=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,q}(x)\frac{t^n}{n!}.\tag{1.8}$$

In this paper, we investigate some properties for the $q$-analog of several special polynomials which are derived from the bosonic or fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$.

## 2 Some special $q$-polynomials

In this section, we assume that $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$. Now, we define the higher-order $q$-Bernoulli numbers,

$$\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_{n,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.1}$$

When $x=0$, $B_{n,q}^{(r)}=B_{n,q}^{(r)}(0)$ are called the higher-order $q$-Bernoulli numbers.

We also consider the higher-order $q$-Daehee polynomials as follows:

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_{n,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.2}$$

When $x=0$, $D_{n,q}^{(r)}=D_{n,q}^{(r)}(0)$ are called the higher-order $q$-Daehee numbers.

From (1.3), we can derive the following equation:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}\,d\mu_q(x_1)\cdots d\mu_q(x_r)\\&\quad=\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r(1+t)^x\\&\quad=\sum_{n=0}^{\infty}D_{n,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

Thus, by (2.3), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_r+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_r)=\frac{D_{n,q}^{(r)}(x)}{n!}\quad(n\geq 0).\tag{2.4}$$

By replacing $t$ by $e^t - 1$ in (2.2), we get

$$\sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\frac{(e^t-1)^n}{n!} = \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{t^n}{n!} \tag{2.5}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\frac{1}{n!}\left(e^t-1\right)^n &= \sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\frac{1}{n!}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,q}^{(r)}(x)S_2(m,n)\right)\frac{t^m}{m!}. \end{aligned} \tag{2.6}$$

Thus, by (2.5) and (2.6), we get

$$B_{n,q}^{(r)}(x) = \sum_{m=0}^{n} D_{m,q}^{(r)}(x)S_2(n,m). \tag{2.7}$$

Therefore, by (2.4) and (2.7), we obtain the following theorem.

**Theorem 1** *For $n \geq 0$, we have*

$$B_{n,q}^{(r)}(x) = \sum_{m=0}^{n} D_{m,q}^{(r)}(x)S_2(n,m)$$

*and*

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_r+x}{n}\,d\mu_q(x_1)\cdots d\mu_q(x_r) \\ &\quad = \frac{D_{n,q}^{(r)}(x)}{n!},\end{aligned}$$

*where $S_2(n,m)$ is the Stirling number of the second kind.*

From (2.1), by replacing $t$ by $\log(1+t)$, we obtain

$$\begin{aligned}&\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r(1+t)^x \\ &\quad = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{1}{n!}\left(\log(1+t)\right)^n \\ &\quad = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{1}{n!}n!\sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!} \\ &\quad = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} S_1(m,n)B_{n,q}^{(r)}(x)\right)\frac{t^m}{m!},\end{aligned} \tag{2.8}$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Therefore, by (2.2) and (2.8), we obtain the following theorem.

**Theorem 2** *For $n \geq 0$, we have*

$$D_{n,q}^{(r)}(x) = \sum_{m=0}^{n} S_1(n,m) B_{m,q}^{(r)}(x).$$

Now, we define the higher-order $q$-Changhee polynomials as follows:

$$\left(\frac{[2]_q}{qt + [2]_q}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.9}$$

When $x = 0$, $\mathrm{Ch}_{n,q}^{(r)} = \mathrm{Ch}_{n,q}^{(r)}(0)$ are called the higher-order $q$-Changhee numbers.

From (1.4), we note that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1+\cdots+x_r+x} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \left(\frac{[2]_q}{qt + [2]_q}\right)^r (1+t)^x. \tag{2.10}$$

Thus, by (2.10), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_r + x}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \frac{\mathrm{Ch}_{n,q}^{(r)}(x)}{n!}. \tag{2.11}$$

In view of (1.6), we define the higher-order $q$-Euler polynomials which are given by the generating function to be

$$\left(\frac{[2]_q}{qe^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.12}$$

From (2.10), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1+\cdots+x_r+x} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&= \left(\frac{[2]_q}{qe^{\log(1+t)} + 1}\right)^r e^{x\log(1+t)} \\
&= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{1}{n!} \bigl(\log(1+t)\bigr)^n \\
&= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \sum_{m=n}^{\infty} S_1(m,n) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} E_{n,q}^{(r)}(x) S_1(m,n)\right) \frac{t^m}{m!}.
\end{aligned} \tag{2.13}$$

Therefore, by (2.11) and (2.13), we obtain the following theorem.

**Theorem 3** *For $n \geq 0$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_r + x}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
&= \frac{\mathrm{Ch}_{n,q}^{(r)}(x)}{n!} = \frac{1}{n!} \sum_{m=0}^{n} E_{m,q}^{(r)}(x) S_1(n,m).
\end{aligned}$$

By replacing $t$ by $e^t - 1$ in (2.9), we get

$$\sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(r)}(x) \frac{(e^t-1)^n}{n!} = \left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt} \tag{2.14}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(r)}(x) \frac{1}{n!}\left(e^t-1\right)^n &= \sum_{n=0}^{\infty} \mathrm{Ch}_{n,q}^{(r)}(x) \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \mathrm{Ch}_{n,q}^{(r)}(x) S_2(m,n)\right)\frac{t^m}{n!}. \end{aligned} \tag{2.15}$$

Therefore, by (2.12), (2.14), and (2.15), we obtain the following theorem.

**Theorem 4** *For $m \geq 0$, we have*

$$E_{m,q}^{(r)}(x) = \sum_{n=0}^{m} \mathrm{Ch}_{n,q}^{(r)}(x) S_2(m,n).$$

Now, we consider the $q$-analog of the higher-order Cauchy polynomials, which are defined by the generating function to be

$$\left(\frac{q(1+t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} C_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.16}$$

When $x = 0$, $C_{n,q}^{(r)} = C_{n,q}^{(r)}(0)$ are called the higher-order $q$-Cauchy numbers. Indeed,

$$\begin{aligned} &\lim_{q\to 1}\left(\frac{q(1+t)-1}{q-1+\frac{q-1}{\log q}\log(1+t)}\right)^r (1+t)^x \\ &\quad = \left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.17}$$

where $C_n^{(r)}(x)$ are called the higher-order Cauchy polynomials.

We observe that

$$\begin{aligned} (1+t)^x &= \left(\frac{q(1+t)-1}{q-1+\frac{q-1}{\log q}\log(1+t)}\right)^r (1+t)^x \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r \\ &= \left(\sum_{l=0}^{\infty} C_{l,q}^{(r)}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_{m,q}^{(r)}\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} C_{l,q}^{(r)}(x) D_{n-l,q}^{(r)}\right)\frac{t^n}{n!} \end{aligned} \tag{2.18}$$

and

$$(1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}. \tag{2.19}$$

By (2.18) and (2.19), we get

$$(x)_n = \sum_{l=0}^{n} \binom{n}{l} C_{l,q}^{(r)}(x) D_{n-l,q}^{(r)}. \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

**Theorem 5** *For $n \geq 0$, we have*

$$\binom{x}{n} = \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} C_{l,q}^{(r)}(x) D_{n-l,q}^{(r)}.$$

For $n \in \mathbb{N} \cup \{0\}$, we define the $q$-analog of the Bernoulli-Euler mixed-type polynomials of order $(r,s)$ as follows:

$$BE_{n,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_{n,q}^{(s)}(x + y_1 + \cdots + y_r)\, d\mu_q(y_1) \cdots d\mu_q(y_r). \tag{2.21}$$

Then, by (2.21), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x) \frac{t^n}{n!} \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} E_{n,q}^{(s)}(x + y_1 + \cdots + y_r) \frac{t^n}{n!}\, d\mu_q(y_1) \cdots d\mu_q(y_r) \\
&\quad = \left( \frac{[2]_q}{qe^t + 1} \right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x+y_1+\cdots+y_r)t}\, d\mu_q(y_1) \cdots d\mu_q(y_r) \\
&\quad = \left( \frac{[2]_q}{qe^t + 1} \right)^s \left( \frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt}.
\end{aligned} \tag{2.22}$$

It is easy to show that

$$\left( \frac{[2]_q}{qe^t + 1} \right)^s \left( \frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(s)} B_{n-l,q}^{(r)}(x) \right) \frac{t^n}{n!}. \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

**Theorem 6** *For $n \geq 0$, we have*

$$BE_{n,q}^{(r,s)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(s)} B_{n-l,q}^{(r)}(x).$$

By replacing $t$ by $\log(1+t)$ in (2.22), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x) \frac{(\log(1+t))^n}{n!} &= \left( \frac{[2]_q}{qt + [2]_q} \right)^s \left( \frac{q - 1 + \frac{q-1}{\log q} \log(1+t)}{q(1+t) - 1} \right)^r (1+t)^x \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x)\, \mathrm{Ch}_{n-m,q}^{(s)} \right\} \frac{t^n}{n!}
\end{aligned} \tag{2.24}$$

and

$$\sum_{m=0}^{\infty} BE_{m,q}^{(r,s)}(x)\frac{(\log(1+t))^m}{m!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} BE_{m,q}^{(r,s)}(x)S_1(n,m)\right\}\frac{t^n}{n!}. \tag{2.25}$$

Therefore, by (2.24) and (2.25), we obtain the following theorem.

**Theorem 7** *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}D_{m,q}^{(r)}(x)\,\mathrm{Ch}_{n-m,q}^{(s)} = \sum_{m=0}^{n} BE_{m,q}^{(r,s)}(x)S_1(n,m).$$

Let us consider the $q$-analog of the Daehee-Changhee mixed-type polynomials of order $(r,s)$ as follows: for $n \geq 0$,

$$DC_{n,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} D_{n,q}^{(r)}(x+y_1+\cdots+y_s)\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s). \tag{2.26}$$

Thus, by (2.26), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} DC_{n,q}^{(r,s)}(x)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty} D_{n,q}^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n!}\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s) \\
&= \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+y_1+\cdots+y_s}\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s) \\
&= \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^r\left(\frac{[2]_q}{qt+[2]_q}\right)^s(1+t)^x \\
&= \left(\sum_{m=0}^{\infty} D_{m,q}^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\mathrm{Ch}_{l,q}^{(s)}(x)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}D_{m,q}^{(r)}\,\mathrm{Ch}_{n-m,q}^{(s)}(x)\right\}\frac{t^n}{n!}
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} DC_{n,q}^{(r,s)}(x)\frac{(e^t-1)^n}{n!} \\
&= \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r\left(\frac{[2]_q}{qe^t+1}\right)^s e^{xt} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}B_{m,q}^{(r)}E_{n-m,q}^{(s)}(x)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{2.28}$$

Now, we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} DC_{n,q}^{(r,s)}(x)\frac{(e^t-1)^n}{n!}\\
&\quad=\sum_{n=0}^{\infty} DC_{n,q}^{(r,s)}(x)\frac{1}{n!}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}\\
&\quad=\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} DC_{n,q}^{(r,s)}(x)S_2(m,n)\right\}\frac{t^m}{m!}.
\end{aligned}\tag{2.29}$$

Therefore, by (2.27), (2.28), and (2.29), we obtain the following theorem.

**Theorem 8** *For $n \geq 0$, we have*

$$DC_{n,q}^{(r,s)}(x)=\sum_{m=0}^{n}\binom{n}{m}D_{m,q}^{(r)}\operatorname{Ch}_{n-m,q}^{(s)}(x)$$

*and*

$$\sum_{m=0}^{n}\binom{n}{m}B_{m,q}^{(r)}E_{n-m,q}^{(s)}(x)=\sum_{m=0}^{n}DC_{m,q}^{(r,s)}(x)S_2(n,m).$$

Now, we consider the $q$-extension of the Cauchy-Changhee mixed-type polynomials of order $(r,s)$ as follows: for $n \geq 0$,

$$CC_{n,q}^{(r,s)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} C_{n,q}^{(r)}(x+y_1+\cdots+y_s)\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r).\tag{2.30}$$

Thus, by (2.30), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} CC_{n,q}^{(r,s)}(x)\frac{t^n}{n!}\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty} C_{n,q}^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n}\,d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s)\\
&\quad=\left(\frac{q(1+t)-1}{q-1+\frac{q-1}{\log q}\log(1+t)}\right)^r\left(\frac{[2]_q}{qt+[2]_q}\right)^s(1+t)^x\\
&\quad=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}C_{m,q}^{(r)}\operatorname{Ch}_{n-m,q}^{(s)}(x)\right\}\frac{t^n}{n!},
\end{aligned}\tag{2.31}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} CC_{n,q}^{(r,s)}(x)\frac{(e^t-1)^n}{n!}\\
&\quad=\left(\frac{qe^t-1}{q-1+\frac{q-1}{\log q}t}\right)^r\left(\frac{[2]_q}{qe^t+1}\right)^s e^{tx}\\
&\quad=\left(\sum_{m=0}^{\infty}B_{m,q}^{(-r)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}E_{l,q}^{(s)}(x)\frac{t^l}{l!}\right)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,q}^{(-r)}E_{n-m,q}^{(s)}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.32}$$

Note that

$$\sum_{n=0}^{\infty} CC_{n,q}^{(r,s)}(x)\frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} CC_{m,q}^{(r,s)}(x)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.33}$$

Therefore, by (2.31), (2.32), and (2.33), we obtain the following theorem.

**Theorem 9** *For $n \geq 0$, we have*

$$CC_{n,q}^{(r,s)}(x) = \sum_{m=0}^{n}\binom{n}{m} C_{m,q}^{(r)} \operatorname{Ch}_{n-m,q}^{(s)}(x)$$

*and*

$$\sum_{m=0}^{n}\binom{n}{m} B_{m,q}^{(-r)} E_{n-m,q}^{(s)}(x) = \sum_{m=0}^{n} CC_{m,q}^{(r,s)}(x)S_2(n,m).$$

Finally, we define the $q$-extension of the Cauchy-Daehee mixed-type polynomials of order $(r,s)$ as follows:

$$CD_{n,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} C_{n,q}^{(r)}(x+y_1+\cdots+y_r)\,d\mu_q(x_1)\cdots d\mu_q(x_r). \tag{2.34}$$

Thus, by (2.34), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} CD_{n,q}^{(r,s)}(x)\frac{t^n}{n!} \\
&\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty} C_{n,q}^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n!}\,d\mu_q(y_1)\cdots d\mu_q(y_s) \\
&\quad = \left(\frac{q(1+t)-1}{q-1+\frac{q-1}{\log q}\log(1+t)}\right)^r \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+y_1+\cdots+y_s}\,d\mu_q(y_1)\cdots d\mu_q(y_s) \\
&\quad = \left(\frac{q(1+t)-1}{q-1+\frac{q-1}{\log q}\log(1+t)}\right)^r \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{q(1+t)-1}\right)^s (1+t)^x \\
&\quad = \begin{cases} \sum_{n=0}^{\infty} C_{n,q}^{(r-s)}(x)\frac{t^n}{n!} & \text{if } r>s, \\ \sum_{n=0}^{\infty} D_{n,q}^{(s-r)}(x)\frac{t^n}{n!} & \text{if } r<s, \\ \sum_{n=0}^{\infty} (x)_n\frac{t^n}{n!} & \text{if } r=s. \end{cases}
\end{aligned} \tag{2.35}$$

Therefore, by (2.35), we obtain the following equation:

$$CD_{n,q}^{(r,s)}(x) = \begin{cases} C_{n,q}^{(r-s)}(x) & \text{if } r>s, \\ D_{n,q}^{(s-r)}(x) & \text{if } r<s, \\ (x)_n & \text{if } r=s. \end{cases}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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