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Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions

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Abstract

In this paper, we obtain some fixed point results for generalized weakly contractive mappings with some auxiliary functions in the framework of b-metric spaces. The proved results generalize and extend the corresponding well-known results of the literature. Some examples are also provided in order to show that these results are more general than the well-known results existing in literature.

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1 Introduction

The Banach contraction principle [1] is a basic result on fixed points for contractive-type mappings. So far, there have been a lot of fixed point results dealing with mappings satisfying diverse types of contractive inequalities. Various researchers have worked on different types of inequalities having continuity on mapping or not on different abstract spaces *viz.* metric spaces [2–4], convex metric spaces [5], ordered metric spaces [6], cone metric spaces [7, 8], generalized metric spaces [9, 10], b-metric spaces [11–15] and many more (see [16–20] and references cited therein).

In 1993, Czerwik [12] introduced the b-metric spaces. These form a nontrivial generalization of metric spaces and several fixed point results for single and multivalued mappings in such spaces have been obtained since then (see [11, 14, 15, 21] and references cited therein).

Let (X,d) be a metric space and $T\colon X\to X$. A mapping T is said to be a K-contraction [4] if there exists $\alpha\in(0,\frac{1}{2})$ such that for all $x,y\in X$ the following inequality holds:

$$d(Tx,Ty)\leq\alpha\bigl(d(x,Tx)+d(y,Ty)\bigr).$$

In 1968, Kannan [4] proved that if (X,d) is a complete metric space, then every K-contraction on X has a unique fixed point.

In 1972, Chatterjea [2] established a fixed point theorem for C-contractions mappings, that is, a mapping T is said to be a C-contraction if there exists $\alpha\in(0,\frac{1}{2})$ such that for all $x,y\in X$ the following inequality holds:

$$d(Tx,Ty)\leq\alpha\bigl(d(x,Ty)+d(y,Tx)\bigr).$$

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Various researchers generalize and/or extend Kannan and Chatterjea type contraction mappings to obtain fixed point results in abstract spaces (see [3, 5, 7, 8, 12, 13, 22–31] and references cited therein). In this paper, we generalize and extend the Kannan and Chatterjea type contractions with some auxiliary functions to obtain some new fixed point results in the framework of b-metric spaces. The proved results generalize and extend the corresponding well-known results of Chandok [22–25], Choudhury [27], Filipović *et al.* [7], Harjani *et al.* [28], Moradi [29], Morales and Rojas [8], Razani and Parvaneh [30] and of Shatanawi [31].

2 Preliminaries

To begin with, we give some basic definitions and notations which will be used in the sequel.

Definition 2.1 ([12]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d\colon X \times X \to \mathbb{R}^+$ is a *b-metric* if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a *b-metric space*.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. We present an easy example to show that in general a b-metric need not be a metric.

Example 2.1 Let (X,ρ) be a metric space, and $d(x,y) = (\rho(x,y))^p$, where $p \geq 1$ is a real number. Then d is a b-metric with $s = 2^{p-1}$. However, (X,d) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $\rho(x,y) = |x-y|$ is the usual Euclidean metric, then $d(x,y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but it is not a metric on $\mathbb{R}$.

It should also be noted that a b-metric might not be a continuous function (see Example 3 of [21]). Thus, while working in b-metric spaces, the following lemma is useful.

Lemma 2.1 ([11]) *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x,z).$$

Definition 2.2 Let (X,d) be a metric space. A mapping $T\colon X \to X$ is said to be *sequentially convergent* [32] (respectively, *subsequentially convergent*) if, for every sequence $\{x_n\}$ in X for which $\{Tx_n\}$ is convergent, $\{x_n\}$ is also convergent (respectively, $\{x_n\}$ has a convergent subsequence).

3 Main results

We denote by Ψ the family of functions $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous, strictly increasing and $\psi^{-1}(\{0\}) = 0$.

Also we denote by Φ the family of functions $\varphi : [0,\infty)^2 \to [0,\infty)$ such that $\varphi(0,0) \geq 0$, $\varphi(x,y) > 0$ if $(x,y) \neq (0,0)$, and $\varphi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n) \leq \liminf_{n\to\infty} \varphi(a_n, b_n)$.

Theorem 3.1 *Let (X,d) be a complete b-metric space with parameter $s \geq 1$, $T, f : X \to X$ be such that, for some $\psi \in \Psi$, $\varphi \in \Phi$, and all $x, y \in X$,*

$$\psi\big(sd(Tfx, Tfy)\big) \leq \frac{\psi(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1})}{1+\varphi(d(Tx,Tfy), d(Ty,Tfx))}, \tag{3.1}$$

and let T be one-to-one and continuous. Then:

(1) *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
(2) *If T is subsequentially convergent then f has a unique fixed point.*
(3) *If T is sequentially convergent then, for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Proof Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = fx_n = f^{n+1}x_0$, for $n \geq 0$.

Step I. First, we will prove that $\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$.

Using (3.1), we obtain

$$\begin{aligned}
\psi\big(sd(Tx_{n+1}, Tx_n)\big) &= \psi\big(sd(Tfx_n, Tfx_{n-1})\big) \\
&\leq \frac{\psi(\frac{d(Tx_n,Tfx_{n-1})+d(Tx_{n-1},Tfx_n)}{s+1})}{1+\varphi(d(Tx_n, Tfx_{n-1}), d(Tx_{n-1}, Tfx_n))} \\
&= \frac{\psi(\frac{d(Tx_n,Tx_n)+d(Tx_{n-1},Tx_{n+1})}{s+1})}{1+\varphi(d(Tx_n, Tx_n), d(Tx_{n-1}, Tx_{n+1}))}.
\end{aligned} \tag{3.2}$$

Since φ is nonnegative and ψ is an increasing function and using the triangular inequality we have

$$\begin{aligned}
\psi\big(sd(Tx_{n+1}, Tx_n)\big) &\leq \psi\left(\frac{d(Tx_{n-1}, Tx_{n+1})}{s+1}\right) \\
&\leq \psi\left(\frac{s}{s+1}\big(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\big)\right).
\end{aligned}$$

Again, since ψ is increasing, we have

$$d(Tx_{n+1}, Tx_n) \leq \frac{1}{s+1}\big(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\big),$$

wherefrom

$$d(Tx_{n+1}, Tx_n) \leq \frac{1}{s} d(Tx_n, Tx_{n-1}) \leq d(Tx_n, Tx_{n-1}).$$

Thus, $\{d(Tx_{n+1}, Tx_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r \geq 0$. From the above argument we have

$$\begin{aligned} sd(Tx_{n+1}, Tx_n) &\leq \frac{1}{s+1} d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \frac{s}{s+1}\bigl(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\bigr) \\ &\leq \frac{s}{2}\bigl(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\bigr). \end{aligned}$$

On taking the limit $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}) = s(s+1)r.$$

From (3.2), we have

$$\psi\bigl(sd(Tx_{n+1}, Tx_n)\bigr) \leq \frac{\psi\bigl(\frac{0+d(Tx_{n-1}, Tx_{n+1})}{s+1}\bigr)}{1+\varphi(0, d(Tx_{n-1}, Tx_{n+1}))}.$$

On letting $n \to \infty$ and using the continuity of ψ and the properties of φ we get

$$\psi(sr) \leq \frac{\psi(sr)}{1+\varphi(0, s(s+1)r)},$$

and consequently, $\psi(sr) = 0$. Hence using the properties of ψ, we have

$$r = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{3.3}$$

Step II. Now in next step we will show that $\{Tx_n\}$ is a b-Cauchy sequence.

Suppose that $\{Tx_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m_k}\}$ and $\{Tx_{n_k}\}$ of $\{Tx_n\}$ with n_k is the smallest index for which $n_k > m_k > k$ such that

$$d(Tx_{m_k}, Tx_{n_k}) \geq \varepsilon \tag{3.4}$$

and

$$d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon. \tag{3.5}$$

From (3.4), (3.5), and using the triangular inequality, we have

$$\begin{aligned} &\varepsilon \leq d(Tx_{m_k}, Tx_{n_k}) \leq s\bigl[d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k})\bigr] \\ &\quad < s\varepsilon + sd(Tx_{n_k-1}, Tx_{n_k}). \end{aligned}$$

On letting $k \to \infty$, and using (3.3), we obtain

$$\varepsilon \leq \limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \leq s\varepsilon. \tag{3.6}$$

Further, we have

$$d(Tx_{m_k}, Tx_{n_k}) \leq s\big[d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k})\big].$$

Now using (3.3) and (3.5), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) \leq \varepsilon. \tag{3.7}$$

Consider

$$d(Tx_{m_k}, Tx_{n_k}) \leq s\big[d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k})\big]$$

and

$$d(Tx_{m_k-1}, Tx_{n_k}) \leq s\big[d(Tx_{m_k-1}, Tx_{m_k}) + d(Tx_{m_k}, Tx_{n_k})\big].$$

Using (3.3) and (3.6), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) \leq s^2\varepsilon. \tag{3.8}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) \leq \varepsilon \tag{3.9}$$

and

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) \leq s^2\varepsilon. \tag{3.10}$$

Since $\frac{s^2+1}{s+1} \leq s$ and using (3.1) and (3.7)-(3.10), we have

$$\begin{aligned}
\psi(s\varepsilon) &\leq \psi\Big(s\limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k})\Big) \\
&= \psi\Big(s\limsup_{k\to\infty} d(Tfx_{m_k-1}, Tfx_{n_k-1})\Big) \\
&\leq \frac{\limsup_{k\to\infty} \psi(\frac{d(Tx_{m_k-1}, Tfx_{n_k-1}) + d(Tx_{n_k-1}, Tfx_{m_k-1})}{s+1})}{1 + \liminf_{k\to\infty} \varphi(d(Tx_{m_k-1}, Tfx_{n_k-1}), d(Tx_{n_k-1}, Tfx_{m_k-1}))} \\
&\leq \frac{\psi(\limsup_{k\to\infty} \frac{d(Tx_{m_k-1}, Tx_{n_k}) + d(Tx_{n_k-1}, Tx_{m_k})}{s+1})}{1 + \varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))} \\
&\leq \frac{\psi(\frac{s^2\varepsilon+\varepsilon}{s+1})}{1 + \varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))} \\
&\leq \frac{\psi(s\varepsilon)}{1 + \varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))}.
\end{aligned}$$

Hence, we obtain

$$\varphi\Big(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k})\Big) \leq 0.$$

By our assumption about φ, we have

$$\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) = \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) = 0,$$

which contradicts (3.9) and (3.10).

Since (X,d) is b-complete and $\{Tx_n\} = \{Tf^n x_0\}$ is a b-Cauchy sequence, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v. \tag{3.11}$$

Step III. Now in the last step, first we will prove that f has a unique fixed point by assuming that T is subsequentially convergent.

As T is subsequentially convergent, $\{f^n x_0\}$ has a b-convergent subsequence. Hence, there exist $u \in X$ and a subsequence $\{n_i\}$ such that

$$\lim_{i\to\infty} f^{n_i} x_0 = u; \tag{3.12}$$

using (3.12) and continuity of T, we obtain

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu. \tag{3.13}$$

From (3.11) and (3.13) we have $Tu = v$.

From Lemma 2.1 and using (3.1), we have

$$\begin{aligned}
\psi\left(s \cdot \frac{1}{s} d(Tfu, Tu)\right) &\leq \psi\left(\limsup_{n\to\infty} sd\big(Tfu, Tf^{n+1}x_0\big)\right) \\
&= \psi\left(\limsup_{n\to\infty} sd\big(Tfu, Tfx_n\big)\right) \\
&\leq \frac{\psi(\limsup_{n\to\infty} \frac{d(Tu,Tfx_n)+d(Tx_n,Tfu)}{s+1})}{1 + \liminf_{n\to\infty} \varphi(d(Tu, Tfx_n), d(Tx_n, Tfu))} \\
&\leq \frac{\psi(\frac{sd(Tu,Tu)+sd(Tu,Tfu)}{s+1})}{1 + \varphi(\liminf_{n\to\infty} d(Tu, Tfx_n), \liminf_{n\to\infty} d(Tx_n, Tfu))} \\
&\leq \frac{\psi(d(Tu, Tfu))}{1 + \varphi(0, \liminf_{n\to\infty} d(Tx_n, Tfu))}.
\end{aligned}$$

Using the properties of $\varphi \in \Phi$, we have $\liminf_{n\to\infty} d(Tx_n, Tfu) = 0$. By the triangular inequality we get

$$d(Tfu, Tu) \leq s\big[d(Tfu, Tx_n) + d(Tx_n, Tu)\big].$$

On letting $n \to \infty$ in above inequality, we have $d(Tfu, Tu) = 0$. Hence, $Tfu = Tu$. As T is one-to-one, $fu = u$. Therefore, f has a fixed point.

Now assume that w is another fixed point of f. From inequality (3.1), we have

$$\begin{aligned}
\psi\bigl(sd(Tu,Tw)\bigr) &= \psi\bigl(sd(Tfu,Tfw)\bigr) \\
&\leq \frac{\psi(\frac{d(Tu,Tfw)+d(Tw,Tfu)}{s+1})}{1+\varphi(d(Tu,Tfw),d(Tw,Tfu))} \\
&= \frac{\psi(\frac{d(Tu,Tw)+d(Tw,Tu)}{s+1})}{1+\varphi(d(Tu,Tw),d(Tw,Tu))} \\
&\leq \frac{\psi(sd(Tu,Tw))}{1+\varphi(d(Tu,Tw),d(Tw,Tu))},
\end{aligned}$$

since $\frac{2}{s+1} \leq s$ and ψ is increasing. Hence, $d(Tu,Tw) = 0$. As T is one-to-one, $u = w$. Therefore, f has a unique fixed point.

Finally, if mapping T is sequentially convergent, replacing $\{n\}$ with $\{n_i\}$ we conclude that $\lim_{n\to\infty} f^n x_0 = u$. □

Theorem 3.2 *Let (X,d) be a complete b-metric space with parameter $s \geq 1$, $T,f : X \to X$ be such that, for some $\psi,\varphi \in \Psi$, $l > 1$ and all $x,y \in X$,*

$$\psi\bigl(sd(Tfx,Tfy)\bigr) \leq \left(\psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tx,Tfy),d(Ty,Tfx))}} - l, \tag{3.14}$$

and let T be one-to-one and continuous. Then:

(1) *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
(2) *If T is subsequentially convergent then f has a unique fixed point.*
(3) *If T is sequentially convergent then, for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Proof Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = fx_n = f^{n+1}x_0$, for $n \geq 0$.

Step I. First, we will prove that $\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$.

Using (3.14), we obtain

$$\begin{aligned}
&\psi\bigl(sd(Tx_{n+1},Tx_n)\bigr) \\
&\quad = \psi\bigl(sd(Tfx_n,Tfx_{n-1})\bigr) \\
&\quad \leq \left(\psi\left(\frac{d(Tx_n,Tfx_{n-1})+d(Tx_{n-1},Tfx_n)}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tx_n,Tfx_{n-1}),d(Tx_{n-1},Tfx_n))}} - l \\
&\quad = \left(\psi\left(\frac{d(Tx_n,Tx_n)+d(Tx_{n-1},Tx_{n+1})}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tx_n,Tx_n),d(Tx_{n-1},Tx_{n+1}))}} - l.
\end{aligned} \tag{3.15}$$

Since φ is nonnegative and ψ is an increasing function and using the triangular inequality we have

$$\begin{aligned}
\psi\bigl(sd(Tx_{n+1},Tx_n)\bigr) &\leq \psi\left(\frac{d(Tx_{n-1},Tx_{n+1})}{s+1}\right) \\
&\leq \psi\left(\frac{s}{s+1}\bigl(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\bigr)\right).
\end{aligned}$$

Again, since ψ is increasing, we have

$$d(Tx_{n+1}, Tx_n) \le \frac{1}{s+1}\big(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\big),$$

wherefrom

$$d(Tx_{n+1}, Tx_n) \le \frac{1}{s} d(Tx_n, Tx_{n-1}) \le d(Tx_n, Tx_{n-1}).$$

Thus, $\{d(Tx_{n+1}, Tx_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r \ge 0$. Using similar steps to Theorem 3.1, we obtain

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}) = s(s+1)r.$$

From (3.15), we have

$$\psi\big(sd(Tx_{n+1}, Tx_n)\big) \le \left(\psi\left(\frac{0 + d(Tx_{n-1}, Tx_{n+1})}{s+1}\right) + l\right)^{\frac{1}{1+\varphi(0,d(Tx_{n-1},Tx_{n+1}))}} - l.$$

On letting $n \to \infty$ and using the continuity of ψ and the properties of φ we have

$$\psi(sr) \le \big(\psi(sr) + l\big)^{\frac{1}{1+\varphi(0,s(s+1)r)}} - l,$$

and consequently, $\psi(sr) = 0$. Hence using the properties of ψ, we have

$$r = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{3.16}$$

Step II. Now in next step we will show that $\{Tx_n\}$ is a b-Cauchy sequence.

Suppose that $\{Tx_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m_k}\}$ and $\{Tx_{n_k}\}$ of $\{Tx_n\}$ with n_k being the smallest index for which $n_k > m_k > k$ such that

$$d(Tx_{m_k}, Tx_{n_k}) \ge \varepsilon \tag{3.17}$$

and

$$d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon. \tag{3.18}$$

From (3.17), (3.18), and using the triangular inequality, we have

$$\begin{aligned}\varepsilon \le d(Tx_{m_k}, Tx_{n_k}) &\le s\big[d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k})\big]\\ &< s\varepsilon + sd(Tx_{n_k-1}, Tx_{n_k}).\end{aligned}$$

On letting $k \to \infty$, and using (3.3), we obtain

$$\varepsilon \le \limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \le s\varepsilon. \tag{3.19}$$

Further, we have

$$d(Tx_{m_k}, Tx_{n_k}) \le s\big[d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k})\big].$$

Now using (3.16) and (3.18), we get

$$\frac{\varepsilon}{s} \le \limsup_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) \le \varepsilon. \tag{3.20}$$

Consider

$$d(Tx_{m_k}, Tx_{n_k}) \le s\big[d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k})\big]$$

and

$$d(Tx_{m_k-1}, Tx_{n_k}) \le s\big[d(Tx_{m_k-1}, Tx_{m_k}) + d(Tx_{m_k}, Tx_{n_k})\big].$$

Using (3.16) and (3.19), we get

$$\frac{\varepsilon}{s} \le \limsup_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) \le s^2\varepsilon. \tag{3.21}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \le \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) \le \varepsilon \tag{3.22}$$

and

$$\frac{\varepsilon}{s} \le \liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) \le s^2\varepsilon. \tag{3.23}$$

Since $\frac{s^2+1}{s+1} \le s$ and using (3.14) and (3.20)-(3.23), we have

$$\begin{aligned}
&\psi(s\varepsilon) \\
&\quad \le \psi\Big(s \limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k})\Big) \\
&\quad = \psi\Big(s \limsup_{k\to\infty} d(Tfx_{m_k-1}, Tfx_{n_k-1})\Big) \\
&\quad \le \left(\limsup_{k\to\infty} \psi\left(\frac{d(Tx_{m_k-1}, Tfx_{n_k-1}) + d(Tx_{n_k-1}, Tfx_{m_k-1})}{s+1}\right) + l\right)^{\frac{1}{1+\liminf_{k\to\infty}\varphi(d(Tx_{m_k-1}, Tfx_{n_k-1}), d(Tx_{n_k-1}, Tfx_{m_k-1}))}} - l \\
&\quad \le \left(\psi\left(\limsup_{k\to\infty} \frac{d(Tx_{m_k-1}, Tx_{n_k}) + d(Tx_{n_k-1}, Tx_{m_k})}{s+1}\right) + l\right)^{\frac{1}{1+\varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))}} - l \\
&\quad \le \left(\psi\left(\frac{s^2\varepsilon + \varepsilon}{s+1}\right) + l\right)^{\frac{1}{1+\varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))}} - l \\
&\quad \le \big(\psi(s\varepsilon) + l\big)^{\frac{1}{1+\varphi(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}))}} - l.
\end{aligned}$$

Hence, we have

$$\varphi\Bigl(\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}), \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k})\Bigr) \le 0.$$

By our assumption about φ, we have

$$\liminf_{k\to\infty} d(Tx_{m_k-1}, Tx_{n_k}) = \liminf_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k}) = 0,$$

which contradicts (3.22) and (3.23).

Since (X, d) is b-complete and $\{Tx_n\} = \{Tf^n x_0\}$ is a b-Cauchy sequence, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v. \tag{3.24}$$

Step III. Now, in the last step, first we will prove that f has a unique fixed point by assuming that T is subsequentially convergent.

As T is subsequentially convergent, $\{f^n x_0\}$ has a b-convergent subsequence. Hence, there exist $u \in X$ and a subsequence $\{n_i\}$ such that

$$\lim_{i\to\infty} f^{n_i} x_0 = u; \tag{3.25}$$

using (3.25) and continuity of T, we obtain

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu. \tag{3.26}$$

From (3.24) and (3.26) we have $Tu = v$.

From Lemma 2.1 and using (3.14), we have

$$\begin{aligned}
&\psi\left(s\cdot\frac{1}{s}d(Tfu, Tu)\right) \\
&\quad\le \psi\Bigl(\limsup_{n\to\infty} sd\bigl(Tfu, Tf^{n+1}x_0\bigr)\Bigr) \\
&\quad= \psi\Bigl(\limsup_{n\to\infty} sd(Tfu, Tfx_n)\Bigr) \\
&\quad\le \left(\psi\left(\limsup_{n\to\infty} \frac{d(Tu, Tfx_n) + d(Tx_n, Tfu)}{s+1}\right) + l\right)^{\frac{1}{1+\liminf_{n\to\infty}\varphi(d(Tu,Tfx_n),d(Tx_n,Tfu))}} - l \\
&\quad\le \left(\psi\left(\frac{sd(Tu, Tu) + sd(Tu, Tfu)}{s+1}\right) + l\right)^{\frac{1}{1+\varphi(\liminf_{n\to\infty} d(Tu,Tfx_n),\liminf_{n\to\infty} d(Tx_n,Tfu))}} - l \\
&\quad\le \bigl(\psi\bigl(d(Tu, Tfu)\bigr) + l\bigr)^{\frac{1}{1+\varphi(0,\liminf_{n\to\infty} d(Tx_n,Tfu))}} - l.
\end{aligned}$$

Using the properties of $\varphi \in \Phi$, we have $\liminf_{n\to\infty} d(Tx_n, Tfu) = 0$. By the triangular inequality we have

$$d(Tfu, Tu) \le s\bigl[d(Tfu, Tx_n) + d(Tx_n, Tu)\bigr].$$

On letting $n \to \infty$ in above inequality, we have $d(Tfu, Tu) = 0$. Hence, $Tfu = Tu$. As T is one-to-one, $fu = u$. Therefore, f has a fixed point.

Now assume that w is another fixed point of f. From inequality (3.14), we have

$$\begin{aligned}\psi\big(sd(Tu,Tw)\big) &= \psi\big(sd(Tfu,Tfw)\big)\\ &\leq \left(\psi\left(\frac{d(Tu,Tfw)+d(Tw,Tfu)}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tu,Tfw),d(Tw,Tfu))}}-l\\ &= \left(\psi\left(\frac{d(Tu,Tw)+d(Tw,Tu)}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tu,Tw),d(Tw,Tu))}}-l\\ &\leq \big(\psi\big(sd(Tu,Tw)\big)+l\big)^{\frac{1}{1+\varphi(d(Tu,Tw),d(Tw,Tu))}}-l,\end{aligned}$$

since $\frac{2}{s+1} \leq s$ and ψ is increasing. Hence, $d(Tu,Tw) = 0$. As T is one-to-one, $u = w$. Therefore, f has a unique fixed point.

Finally, if T is sequentially convergent, replacing $\{n\}$ with $\{n_i\}$ we conclude that $\lim_{n\to\infty} f^n x_0 = u$. □

If we take $\psi(t) = t$ and $\varphi(t,u) = \frac{s}{(s+1)a} - 1$, $a \in (0,1)$, in Theorem 3.1, we obtain the following result which is an extended Chatterjea fixed point theorem in the setting of b-metric spaces.

Corollary 3.1 *Let (X,d) be a complete b-metric space and $T,f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a \in (0,1)$ and*

$$d(Tfx,Tfy) \leq \frac{a}{s(s+1)}\big(d(Tx,Tfy)+d(Ty,Tfx)\big),$$

for all $x,y \in X$, then f has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in X$ the sequence of iterates $f^n x_0$ converges to this fixed point.

Remark 3.1

(1) If we take $Tx = x$, in Corollary 3.1, then we obtain the result of Jovanovic [16, Corollary 3.8.3°] (the case $g = f$), which is Chatterjea's Theorem [2] in the framework of b-metric spaces.

(2) By taking $Tx = x$ and $\psi(t) = t$ in Theorem 3.1, we obtain an extension of Choudhury's [27] main result to the setup of b-metric spaces.

(3) If $s = 1$, in Theorem 3.1, we obtain the corresponding result of [30].

Example 3.1 Let $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, and let $d(x,y) = (x-y)^2$ for $x,y \in X$. Then d is a b-metric with the parameter $s = 2$ and (X,d) is a complete b-metric space. Consider the mappings $f, T : X \to X$ given by

$$f(0) = 0, \qquad f\left(\frac{1}{n}\right) = \frac{1}{n+1}, \qquad T(0) = 0, \qquad T\left(\frac{1}{n}\right) = \frac{1}{n^n}, \quad n \in \mathbb{N}.$$

We will show that the mappings f, T satisfy the conditions of Corollary 3.1 with $\alpha = \frac{2}{9} < \frac{1}{3} = \frac{1}{s+1}$. Indeed, for $m, n \in \mathbb{N}$, $m > n$, we have

$$d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) = \left[\frac{1}{(n+1)^{n+1}} - \frac{1}{(m+1)^{m+1}}\right]^2 < \left[\frac{1}{(n+1)^{n+1}}\right]^2.$$

It is easy to prove that, for $n \in \mathbb{N}$,

$$\frac{1}{(n+1)^{n+1}} < \frac{1}{3}\left[\frac{1}{n^n} - \frac{1}{(n+2)^{n+2}}\right].$$

It follows that

$$d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) < \frac{1}{9}\left[\frac{1}{n^n} - \frac{1}{(n+2)^{n+2}}\right]^2.$$

Now, $m > n$ implies that $m \geq n+1$ and $n+2 \leq m+1$. It follows that $1/(n+2)^{n+2} \geq 1/(m+1)^{m+1}$, and hence

$$\begin{aligned} d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) &< \frac{1}{9}\left[\frac{1}{n^n} - \frac{1}{(m+1)^{m+1}}\right]^2 \\ &\leq \frac{\alpha}{s}\left[d\left(T\frac{1}{n}, Tf\frac{1}{m}\right) + d\left(T\frac{1}{m}, TF\frac{1}{n}\right)\right]. \end{aligned}$$

If one of the points is equal to 0, the proof is even simpler.

By Corollary 3.1, it follows that f has a unique fixed point (which is $u = 0$).

Theorem 3.3 *Let (X, d) be a complete b-metric space with the parameter $s \geq 1$, $T, f : X \to X$ be such that for some $\psi \in \Psi$, $\varphi \in \Phi$, and all $x, y \in X$,*

$$\psi\big(d(Tfx, Tfy)\big) \leq \frac{\psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{s+1}\right)}{1 + \varphi(d(Tx, Tfx), d(Ty, Tfy))}, \tag{3.27}$$

and let T be one-to-one and continuous. Then:

1. *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
2. *If T is subsequentially convergent then f has a unique fixed point.*
3. *If T is sequentially convergent then, for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Proof Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = fx_n = f^{n+1}x_0$, $n \geq 0$.

In the first step, we will prove that $\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$.

Using (3.27), we obtain

$$\begin{aligned} \psi\big(d(Tx_{n+1}, Tx_n)\big) = \psi\big(d(Tfx_n, Tfx_{n-1})\big) &\leq \frac{\psi\left(\frac{d(Tx_n,Tfx_n)+d(Tx_{n-1},Tfx_{n-1})}{s+1}\right)}{1 + \varphi(d(Tx_n, Tfx_n), d(Tx_{n-1}, Tfx_{n-1}))} \\ &= \frac{\psi\left(\frac{d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)}{s+1}\right)}{1 + \varphi(d(Tx_n, Tfx_n), d(Tx_{n-1}, Tfx_{n-1}))}. \end{aligned} \tag{3.28}$$

Since φ is nonnegative and ψ is increasing, we have

$$d(Tx_{n+1}, Tx_n) \leq \frac{d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)}{s+1},$$

that is,

$$d(Tx_{n+1}, Tx_n) \leq \frac{1}{s} d(Tx_n, Tx_{n-1}) \leq d(Tx_n, Tx_{n-1}).$$

Thus, $\{d(Tx_{n+1}, Tx_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r$. On letting $n \to \infty$ in (3.28) and using the properties of ψ and φ we obtain

$$\psi(r) \leq \frac{\psi(\frac{2r}{s+1})}{1 + \varphi(r,r)} \leq \frac{\psi(r)}{1 + \varphi(r,r)},$$

which is possible only if

$$r = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0.$$

Now, we will show that $\{Tx_n\}$ is a b-Cauchy sequence.

Suppose that this is not true. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m_k}\}$ and $\{Tx_{n_k}\}$ of $\{Tx_n\}$ with n_k is the smallest index for which $n_k > m_k > k$ such that $d(Tx_{m_k}, Tx_{n_k}) \geq \varepsilon$. This means that

$$d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon.$$

Again, as in Step II of Theorem 3.1 one can prove that

$$\varepsilon \leq \limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \leq s\varepsilon. \tag{3.29}$$

Using (3.27) we have

$$\begin{aligned}
\psi\bigl(d(Tx_{m_k}, Tx_{n_k})\bigr) &= \psi\bigl(d(Tfx_{m_k-1}, Tfx_{n_k-1})\bigr) \\
&\leq \frac{\psi(\frac{d(Tx_{m_k-1}, Tfx_{m_k-1}) + d(Tx_{n_k-1}, Tfx_{n_k-1})}{s+1})}{1 + \varphi(d(Tx_{m_k-1}, Tfx_{m_k-1}), d(Tx_{n_k-1}, Tfx_{n_k-1}))} \\
&= \frac{\psi(\frac{d(Tx_{m_k-1}, Tx_{m_k}) + d(Tx_{n_k-1}, Tx_{n_k})}{s+1})}{1 + \varphi(d(Tx_{m_k-1}, Tx_{m_k}), d(Tx_{n_k-1}, Tx_{n_k}))}.
\end{aligned}$$

Passing to the upper limit as $k \to \infty$ in the above inequality and using (3.29), we have

$$\psi(\varepsilon) \leq \frac{\psi(0)}{1 + \varphi(0,0)} = 0,$$

and so $\psi(\varepsilon) = 0$. By our assumptions about ψ, we have $\varepsilon = 0$, which is a contradiction. Therefore as in Step II of Theorem 3.1, we obtain $\{Tx_n\}$ is a b-Cauchy sequence.

Since (X,d) is b-complete and $\{Tx_n\} = \{Tf^n x_0\}$ is a b-Cauchy sequence, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v. \tag{3.30}$$

Now, if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence. Hence, there exist a point $u \in X$ and a sequence $\{n_i\}$ such that

$$\lim_{i\to\infty} f^{n_i} x_0 = u. \tag{3.31}$$

Using (3.31) and continuity of T, we obtain

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu. \tag{3.32}$$

By using (3.30) and (3.32), we obtain $Tu = v$.

Using Lemma 2.1 and inequality (3.27), we have

$$\begin{aligned}
\psi\left(\frac{1}{s} d(Tfu, Tu)\right) &\leq \psi\left(\limsup_{n\to\infty} d\bigl(Tfu, Tf^{n+1}x_0\bigr)\right) \\
&= \psi\left(\limsup_{n\to\infty} d(Tfu, Tfx_n)\right) \\
&\leq \frac{\psi(\limsup_{n\to\infty} \frac{d(Tu,Tfu)+d(Tx_n,Tfx_n)}{s+1})}{1+\liminf_{n\to\infty} \varphi(d(Tu,Tfu), d(Tx_n,Tfx_n))} \\
&= \frac{\psi(\frac{d(Tu,Tfu)+0}{s+1})}{1+\varphi(d(Tu,Tfu),0)} \\
&\leq \frac{\psi(\frac{d(Tu,Tfu)}{s})}{1+\varphi(d(Tu,Tfu),0)}.
\end{aligned}$$

Using the properties of $\varphi \in \Phi$, $d(Tu, Tfu) = 0$. As T is one-to-one, $fu = u$. Therefore, f has a fixed point.

Uniqueness of the fixed point can be proved similarly to Theorem 3.1.

Finally, if T is sequentially convergent, replacing $\{n\}$ with $\{n_i\}$ we conclude that $\lim_{n\to\infty} f^n x_0 = u$. □

Taking $\psi(t) = t$ and $\varphi(t,u) = \frac{1}{(s+1)a} - 1$, $a \in (0,1)$ in Theorem 3.3, an extended Kannan fixed point theorem in the setting of b-metric spaces has been obtained.

Corollary 3.2 *Let (X,d) be a complete b-metric space with the parameter $s \geq 1$, $T,f : X \to X$ be such that for some $a \in (0, \frac{1}{s+1})$ and all $x,y \in X$,*

$$d(Tfx, Tfy) \leq a\bigl(d(Tx, Tfx) + d(Ty, Tfy)\bigr) \tag{3.33}$$

and let T be one-to-one and continuous. Then:

(1) *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*

(2) *If T is subsequentially convergent then f has a unique fixed point.*

(3) *If T is sequentially convergent then, for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Remark 3.2

(1) If we take $Tx = x$, in Corollary 3.2, then we obtain the result of Jovanović *et al.* [16, Corollary 3.8.2°] (the case $g = f$).
(2) If $s = 1$, in Corollary 3.2, then we obtain the main result of Moradi (*i.e.*, [29, Theorem 2.1]).
(3) If both of these conditions are fulfilled, we get just the classical result of Kannan [4].

Example 3.2 ([13]) Let $X = \{a, b, c\}$ and $d: X \times X \to \mathbb{R}$ be defined by $d(x, x) = 0$ for $x \in X$, $d(a, b) = d(b, c) = 1$, $d(a, c) = \frac{9}{4}$, $d(x, y) = d(y, x)$ for $x, y \in X$. It is easy to check that (X, d) is a b-metric space (with $s = \frac{9}{8} > 1$) which is not a metric space. Consider the mapping $f: X \to X$ given by

$$f = \begin{pmatrix} a & b & c \\ a & a & b \end{pmatrix}.$$

We first note that the b-metric version of the classical weak Kannan theorem is not satisfied in this example. Indeed, for $x = b$, $y = c$, we have $d(fx, fy) = d(a, b) = 1$ and $d(x, fx) + d(y, fy) = d(b, a) + d(c, b) = 2$, hence the inequality

$$\psi\left(d(fx, fy)\right) \le \psi\left(\frac{d(x, fx) + d(y, fy)}{s+1}\right) - \varphi\left(d(x, fx), d(y, fy)\right)$$

cannot hold, whatever $\psi \in \Psi$ and $\varphi \in \Phi$ are chosen.

Take now $T: X \to X$ defined by

$$T = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$$

Obviously, all the properties of T given in Corollary 3.2 are fulfilled. We will check that the contractive condition (3.33) holds true if α is chosen such that

$$\frac{4}{9} < \alpha < \frac{8}{17} = \frac{1}{s+1}.$$

Only the following cases are nontrivial:

1° $x = a$, $y = c$. Then (3.33) reduces to

$$d(Tfa, Tfc) = d(b, c) = 1 = \frac{4}{9} \cdot \frac{9}{4} < \alpha\left(d(b, b) + d(a, c)\right) = \alpha\left(d(Ta, Tfa) + d(Tc, Tfc)\right).$$

2° $x = b$, $y = c$. Then (3.33) reduces to

$$d(Tfb, Tfc) = d(b, c) = 1 < \frac{4}{9} \cdot \frac{13}{4} < \alpha\left(d(c, b) + d(a, c)\right) = \alpha\left(d(Tb, Tfb) + d(Tc, Tfc)\right).$$

All the conditions of Corollary 3.2 are satisfied and f has a unique fixed point ($u = a$).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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