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On the Hölder continuity of solution maps to parametric generalized vector quasi-equilibrium problems via nonlinear scalarization

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This paper is dedicated to Professor Shih-sen Chang on his 80th birthday

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Abstract
In this paper, by using a nonlinear scalarization technique, we obtain sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem with set-valued mappings. The results are different from the recent ones in the literature.

Keywords: parametric generalized vector quasi-equilibrium problem; solution mapping; Hölder continuity; nonlinear scalarization

1 Introduction

The generalized vector quasi-equilibrium problem is a unified model of several problems, namely generalized vector quasi-variational inequalities, vector quasi-optimization problems, traffic network problems, fixed point and coincidence point problems, *etc.* (see, for example, [1, 2] and the references therein). It is well known that the stability analysis of a solution mapping for equilibrium problems is an important topic in optimization theory and applications. Stability may be understood as lower or upper semicontinuity, continuity, and Lipschitz or Hölder continuity. There have been many papers to discuss the stability of solution mapping for equilibrium problems when they are perturbed by parameters (also known the parametric (generalized) equilibrium problems). Last decade, many authors intensively studied the sufficient conditions of upper (lower) semicontinuity of various solution mappings for parametric (generalized) equilibrium problems, see [3–10]. Let us begin now, Yen [11] obtained the Hölder continuity of the unique solution of a classic perturbed variational inequality by the metric projection method. Mansour and Riahi [12] proved the Hölder continuity of the unique solution for a parametric equilibrium problem under the concepts of strong monotonicity and Hölder continuity. Bianchi and Pini [13] introduced the concept of strong pseudomonotonicity and got the Hölder continuity of the unique solution of a parametric equilibrium problem. Anh and Khanh [14] generalized the main results of [13] to two classes of perturbed generalized equilibrium problems with set-valued mappings. Anh and Khanh [15] further discussed the uniqueness and Hölder continuity of the solutions for perturbed equilibrium problems with set-valued mappings. Anh and Khanh [16] extended the results of [15] to the case of perturbed quasi-equilibrium problems with set-valued mappings and obtained the Hölder continuity of the unique solutions. Li *et al.* [17] introduced an assumption, which is weaker than the corresponding ones of [13, 14], and established the Hölder continuity of the set-valued solution mappings for two classes of parametric generalized vector quasi-equilibrium problems in general metric spaces. Li *et al.* [18] extended the results of [17] to perturbed generalized vector quasi-equilibrium problems.

Among many approaches for dealing with the lower semicontinuity, continuity and Hölder continuity of the solution mapping for a parametric vector equilibrium problem in general metric spaces, the scalarization method is of considerable interest. The classical scalarization method using linear functionals has been already used for studying the lower semicontinuity of the solution mapping [19–21] and the Hölder continuity [22] of the solution mapping to parametric vector equilibrium problems. Wang *et al.* [23] established the lower semicontinuity and upper semicontinuity of the solution set to a parametric generalized strong vector equilibrium problem by using a scalarization method and a density result. Recently, by using this method, Peng [24] established the sufficient conditions for the Hölder continuity of the solution mapping to a parametric generalized vector quasi-equilibrium problem with set-valued mappings.

On the other hand, a useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of nonconvex vector optimization problems. The nonlinear scalarization function ξ_q, commonly known as the Gerstewitz function in the theory of vector optimization [25, 26], has been also used to study the lower semicontinuity of the set-valued solution mapping to a parametric vector variational inequality [27]. Using this method, Bianchi and Pini [28] obtained the Hölder continuity of the single-valued solution mapping to a parametric vector equilibrium problem. Recently, Chen and Li [29] studied Hölder continuity of the solution mapping for both set-valued and single-valued cases to parametric vector equilibrium problems. The key role in their paper is a globally Lipschitz property of the Gerstewitz function. Very recently, by using the idea in [29], Chen [30] obtained Hölder continuity of the unique solution to a parametric vector quasi-equilibrium problem based on nonlinear scalarization approach under three different kinds of monotonicity hypotheses. It is natural to raise and give an answer to the following question.

Question Can one establish the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings by using a nonlinear scalarization method?

Motivated and inspired by Peng [24] and Chen [30] and research going on in this direction, in this paper we aim to give positive answers to the above question. We first establish the sufficient conditions which guarantee the Hölder continuity of a solution mapping to the parametric generalized vector quasi-equilibrium problem with set-valued mappings by using a nonlinear scalarization method. We further study several kinds of the monotonicity conditions to obtain the Hölder continuity of the solution mapping. The main results of this paper are different from the corresponding results in Peng [24] and Chen [30]. These results improve the corresponding ones in recent literature.

The structure of the paper is as follows. Section 2 presents the parametric generalized vector quasi-equilibrium problem and materials used in the rest of this paper. We establish, in Section 3, a sufficient condition for the Hölder continuity of the solution mapping to a parametric generalized vector quasi-equilibrium problem.

2 Preliminaries

Throughout the paper, unless otherwise specified, we denote by $\|\cdot\|$ and $d(\cdot,\cdot)$ the norm and the metric on a normed space and a metric space, respectively. A closed ball with center $0 \in X$ and radius $\delta > 0$ is denoted by $B(0,\delta)$. We always consider X, Λ, M as metric spaces, and Y as a linear normed space with its topological dual space Y^*. For any $y^* \in Y^*$, we define $\|y^*\| := \sup\{\|\langle y^*, y\rangle\| : \|y\| = 1\}$, where $\langle y^*, y\rangle$ denotes the value of y^* at y. Let $C \subset Y$ be a pointed, closed and convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ stands for the interior of C. Let

$$C^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\}$$

be the dual cone of C. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then

$$B_e^* := \{y^* \in C^* : \langle y^*, e\rangle = 1\}$$

is a weak*-compact base of C^*. Clearly, C^q is a weak*-compact base of C^*, that is, C^q is convex and weak*-compact such that $0 \notin C^q$ and $C^* = \bigcup_{t \geq 0} tC^q$.

Let $q \in \operatorname{int} C$, the *nonlinear scalarization function* [25, 26] $\xi_q : Y \to \mathbb{R}$ is defined by

$$\xi_q = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

It is well known that ξ_q is a continuous, positively homogeneous, subadditive and convex function on Y, and it is monotone (that is, $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (that is, $y_2 - y_1 \in -\operatorname{int} C \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [25, 26]). In case, $Y = R^l$, $C = R^l_+$ and $q = (1, 1, \ldots, 1) \in \operatorname{int} R^l_+$, the nonlinear scalarization function can be expressed in the following equivalent form [25, Corollary 1.46]:

$$\xi_q(y) = \max_{1 \leq i \leq l}\{y_i\}, \quad \forall y = (y_1, y_2, \ldots, y_l) \in \mathbb{R}^l. \tag{1}$$

Lemma 2.1 [25, Proposition 1.43] *For any fixed $q \in \operatorname{int} C$, $y \in Y$ and $r \in \mathbb{R}$,*

(i) $\xi_q < r \Leftrightarrow y \in rq - \operatorname{int} C$ (*that is*, $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \operatorname{int} C$);
(ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - C$;
(iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$, *where ∂C denotes the boundary of C*;
(iv) $\xi_q(rq) = r$.

The property (i) of Lemma 2.1 plays an essential role in scalarization. From the definition of ξ_q, property (iv) in Lemma 2.1 could be strengthened as

$$\xi_q(y + rq) = \xi_q(y) + r, \quad \forall y \in Y, r \in \mathbb{R}. \tag{2}$$

For any $q \in \operatorname{int} C$, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q\rangle = 1\}$$

is a weak*-compact set of Y^* (see [19, Lemma 5.1]). The following equivalent form of ξ_q can be deduced from [31, Corollary 2.1] or [32, Proposition 2.2] ([25, Proposition 1.53]).

Proposition 2.2 [30, Proposition 2.2] *Let $q \in \operatorname{int} C$. Then, for $y \in Y$,*

$$\xi_q(y) = \max_{y^* \in C^q} \langle y^*, y\rangle.$$

Proposition 2.3 [30, Proposition 2.3] *ξ_q is Lipschitz on Y, and its Lipschitz constant is*

$$L := \sup_{y^* \in C^q} \|y^*\| \in \left[\frac{1}{\|q\|}, +\infty\right).$$

The following example can be found in [30, Example 2.1].

Example 2.4

(i) If $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then the Lipschitz constant of ξ_q is $L = \frac{1}{q}$ $(q > 0)$. Indeed, $|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|$ for all $x, y \in \mathbb{R}$.

(ii) If $Y = \mathbb{R}^2$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \leq y_2 \leq 2y_1\}$. Take $q = (2, 3) \in \operatorname{int} C$, then

$$C^q := \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1, y_1 \in [-0.1, 2]\},$$

and the Lipschitz constant is $L = \sup_{y^* \in C^q} \|y^*\| = \|(-2, 1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| \leq \sqrt{5}\|y - y'\|, \quad \forall y, y' \in \mathbb{R}^2.$$

Now we recall some basic definitions and their properties which will be used in the sequel.

Definition 2.5 (Classical notion) Let $l \geq 0$ and $\alpha > 0$. A set-valued mapping $G : \Lambda \to 2^X$ is said to be *$l \cdot \alpha$-Hölder continuous at λ_0 on a neighborhood $N(\lambda_0)$* of λ_0 if and only if

$$G(\lambda_1) \subseteq G(\lambda_2) + lB_X(0, d^\alpha(\lambda_1, \lambda_2)), \quad \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{3}$$

When X is a normed space, we say that the vector-valued mapping $g : \Lambda \to X$ is $l \cdot \alpha$-Hölder continuous at λ_0 on a neighborhood $N(\lambda_0)$ of λ_0 iff

$$\|g(\lambda_1) - g(\lambda_2)\| \leq ld^\alpha(\lambda_1, \lambda_2), \quad \forall \lambda_1, \lambda_2 \in N(\lambda_0). \tag{4}$$

Definition 2.6 Let $l_1, l_2 \geq 0$ and $\alpha_1, \alpha_2 > 0$. A set-valued mapping $G : X \times \Lambda \to 2^X$ is said to be *$(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-Hölder continuous at x_0, λ_0 on neighborhoods* $N(x_0)$ and $N(\lambda_0)$ of x_0 and λ_0 if and only if

$$G(x_1, \lambda_1) \subseteq G(x_2, \lambda_2) + (l_1 d_X^{\alpha_1}(x_1, x_2) + l_2 d_\Lambda^{\alpha_2}(\lambda_1, \lambda_2))B_X(0, 1) \tag{5}$$

for all $x_1, x_2 \in N(x_0)$, $\forall \lambda_1, \lambda_2 \in N(\lambda_0)$.

3 Main results

By using a nonlinear scalarization technique, we present the sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem.

Let $N(\lambda_0) \subset \Lambda$ and $N(\mu_0) \subset M$ be neighborhoods of λ_0 and μ_0, respectively, and let $K : X \times \Lambda \to 2^X$ and $F : X \times X \times M \to 2^Y$ be set-valued mappings. For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, we consider the following parametric generalized vector quasi-equilibrium problem (PGVQEP):

Find $x_0 \in K(x_0, \lambda)$ such that

$$F(x_0, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x_0, \lambda). \tag{6}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X | x \in K(x, \lambda)\}.$$

The weak solution set of (6) is denoted by

$$S_W(\lambda, \mu) := \{x \in E(\lambda) : F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \forall y \in K(x, \lambda)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$, the ξ_q-solution set of (6) is denoted by

$$S(\xi_q, \lambda, \mu) := \Big\{x \in E(\lambda) : \inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \forall y \in K(x, \lambda)\Big\}.$$

We first establish the following lemmas which will be used in the sequel.

Lemma 3.1 *For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$,*

$$S_W(\lambda, \mu) = S(\xi_q, \lambda, \mu).$$

Proof Let $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$. For any $x \in S_W(\lambda, \mu)$, we have

$$x \in E(\lambda) \quad \text{and} \quad F(x, y, \mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x, \lambda).$$

Therefore, for each $y \in K(x, \lambda)$ and each $z \in F(x, y, \mu)$, we have

$$z \notin -\operatorname{int} C = 0q - \operatorname{int} C.$$

By Lemma 2.1(i), we conclude that $\xi_q(z) \geq 0$. Since z is arbitrary, we have

$$\inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0 \quad \text{for all } y \in K(x, \lambda),$$

which gives that $S_W(\lambda, \mu) \subseteq S(\xi_q, \lambda, \mu)$.

On the other hand, for each $x \in S(\xi_q, \lambda, \mu)$, we have that

$$x \in E(\lambda) \quad \text{and} \quad \inf_{z \in F(x,y,\mu)} \xi_q(z) \geq 0, \quad \forall y \in K(x, \lambda). \tag{7}$$

Thus, for each $y \in K(x,\lambda)$ and each $z \in F(x,y,\mu)$, we have that $\xi_q(z) \geq 0$. By Lemma 2.1(i), we can obtain $z \notin -\operatorname{int} C$. Therefore, we have $z \in Y \backslash (-\operatorname{int} C)$, which implies that

$$x \in E(\lambda) \quad \text{and} \quad F(x,y,\mu) \subset Y \backslash (-\operatorname{int} C), \quad \forall y \in K(x,\lambda).$$

Hence, $S(\xi_q,\lambda,\mu) \subseteq S_W(\lambda,\mu)$. The proof is completed. □

Lemma 3.2 *Suppose that $N(\lambda_0)$ and $N(\mu_0)$ are the given neighborhoods of λ_0 and μ_0, respectively.*

(a) *If for each $x,y \in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1 \cdot \gamma_1$-Hölder continuous at $\mu_0 \in M$, then for any fixed $q \in \operatorname{int} C$, the function*

$$\psi_{\xi_q}(x,y,\cdot) = \inf_{z \in F(x,y,\cdot)} \xi_q(z)$$

is $Lm_1 \cdot \gamma_1$-Hölder continuous at μ_0.

(b) *If for each $x \in E(N(\lambda_0))$ and $\mu \in N(E(\mu_0))$, $F(x,\cdot,\mu)$ is $m_2 \cdot \gamma_2$-Hölder continuous on $E(N(\lambda_0))$, then for any fixed $q \in \operatorname{int} C$, the function*

$$\psi_{\xi_q}(x,\cdot,\mu) = \inf_{z \in F(x,\cdot,\mu)} \xi_q(z)$$

is $Lm_2 \cdot \gamma_2$-Hölder continuous on $E(N(\lambda_0))$.

Proof (a) Let $x,y \in E(N(\lambda_0))$. The $m_1 \cdot \gamma_1$-Hölder continuity of $F(x,y,\cdot)$ implies that there exists a neighborhood $N(\mu_0)$ of μ_0 such that for all $\mu_1,\mu_2 \in N(\mu_0)$,

$$F(x,y,\mu_1) \subset F(x,y,\mu_2) + m_1 d_M^{\gamma_1}(\mu_1,\mu_2) B_Y.$$

So, for any $z_1 \in F(x,y,\mu_1)$, there exist $z_2 \in F(x,y,\mu_2)$ and $e \in B_Y$ such that

$$z_1 = z_2 + m_1 d_M^{\gamma_1}(\mu_1,\mu_2) e.$$

By using Proposition 2.3, we obtain

$$\begin{aligned} \left|\xi_q(z_1) - \xi_q(z_2)\right| &\leq L\|z_1 - z_2\| \\ &= Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2)\|e\| \\ &\leq Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \end{aligned} \tag{8}$$

which gives that

$$-Lm_1 d^{\gamma_1}(\mu_1,\mu_2) \leq \xi_q(z_1) - \xi_q(z_2).$$

Since z_1 is arbitrary and $\xi_q(z_2) \geq \inf_{z \in F(x,y,\mu_2)} \xi_q(z)$, we have

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \leq \inf_{z \in F(x,y,\mu_1)} \xi_q(z) - \inf_{z \in F(x,y,\mu_2)} \xi_q(z).$$

Applying the symmetry between μ_1 and μ_2, we arrive at

$$-Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) \le \inf_{z\in F(x,y,\mu_2)} \xi_q(z) - \inf_{z\in F(x,y,\mu_1)} \xi_q(z).$$

It follows from the last two inequalities that

$$\left|\psi_{\xi_q}(x,y,\mu_1) - \psi_{\xi_q}(x,y,\mu_2)\right| \le Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2), \quad \forall \mu_1,\mu_2 \in N(\mu_0).$$

Therefore, we conclude that $\psi_{\xi_q}(x,y,\cdot) = \inf_{z\in F(x,y,\cdot)} \xi_q(z)$ is $Lm_1 \cdot \gamma_1$-Hölder continuous at μ_0.

(b) It follows by a similar argument as in part (a). The proof is completed. □

Now, by using the nonlinear scalarization technique, we propose some sufficient conditions for Hölder continuity of the solution mapping for (PGVQEP).

Theorem 3.3 *For each fixed $q \in \operatorname{int} C$, let $S(\xi_q,\lambda,\mu)$ be nonempty in a neighborhood $N(\lambda_0)\times N(\mu_0)$ of $(\lambda_0,\mu_0)\in \Lambda\times M$. Assume that the following conditions hold.*

(i) *$K(\cdot,\cdot)$ is $(l_1\cdot\alpha_1, l_2\cdot\alpha_2)$-Hölder continuous on $E(N(\lambda_0))\times N(\lambda_0)$;*

(ii) *For each $x,y\in E(N(\lambda_0))$, $F(x,y,\cdot)$ is $m_1\cdot\gamma_1$-Hölder continuous at $\mu_0\in M$;*

(iii) *For each $x\in E(N(\lambda_0))$ and $\mu\in N(\mu_0)$, $F(x,\cdot,\mu)$ is $m_2\cdot\gamma_2$-Hölder continuous on $E(N(\lambda_0))$;*

(iv) *$F(\cdot,\cdot,\mu)$ is $h\cdot\beta$-Hölder strongly monotone with respect to ξ_q, that is, there exist constants $h>0$, $\beta>0$ such that for every $x,y\in E(N(\lambda_0))$, $x\neq y$,*

$$hd_X^{\beta}(x,y) \le d\Big(\inf_{z\in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z\in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Big);$$

(v) *$\beta = \alpha_1\gamma_2$, $h > 2m_2 L l_1^{\gamma_1}$, where $L := \sup_{\lambda\in C^q}\|\lambda\| \in [\frac{1}{\|q\|},+\infty)$ is the Lipschitz constant of ξ_q on Y.*

Then, for every $(\lambda,\mu)\in N(\lambda_0)\times N(\mu_0)$, the solution $x(\lambda,\mu)$ of (PVQGEP) *is unique, and $x(\lambda,\mu)$ as a function of λ and μ satisfies the Hölder condition*: *for all $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\lambda_0)\times N(\mu_0)$,*

$$\begin{aligned} d_X\big(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\big) &\le \left(\frac{2m_2 L l_2^{\gamma_2}}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2) \\ &\quad + \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2), \end{aligned}$$

where $x(\lambda_i,\mu_i)\in S_W(\lambda_i,\mu_i)$, $i=1,2$.

Proof Let $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N(\lambda_0)\times N(\mu_0)$. The proof is divided into the following three steps based on the fact that

$$d_X\big(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\big) \le d_X\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) + d_X\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big),$$

where $x(\lambda_i,\mu_i)\in S_W(\lambda_i,\mu_i)$, $i=1,2$.

Step 1: We prove that

$$d_1 := d_X\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \le \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2) \tag{9}$$

for all $x(\lambda_1,\mu_1) \in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2) \in S_W(\lambda_1,\mu_2)$.

If $x(\lambda_1,\mu_1) = x(\lambda_1,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_1) \neq x(\lambda_1,\mu_2)$. Since $x(\lambda_1,\mu_1) \in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x(\lambda_1,\mu_2) \in K(x(\lambda_1,\mu_2),\lambda_1)$, by the $l_1 \cdot \alpha_1$-Hölder continuity of $K(\cdot,\lambda_1)$, there exist $x_1 \in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x_2 \in K(x(\lambda_1,\mu_2),\lambda_1)$ such that

$$d_X\big(x(\lambda_1,\mu_1),x_2\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) = l_1 d_1^{\alpha_1} \tag{10}$$

and

$$d_X\big(x(\lambda_1,\mu_2),x_1\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) = l_1 d_1^{\alpha_1}. \tag{11}$$

Since $x(\lambda_1,\mu_1) \in S_W(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2) \in S_W(\lambda_1,\mu_2)$, by Lemma 3.1, we obtain

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big) := \inf_{z\in F(x(\lambda_1,\mu_1),x_1,\mu_1)} \xi_q(z) \ge 0 \tag{12}$$

and

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big) := \inf_{z\in F(x(\lambda_1,\mu_2),x_2,\mu_2)} \xi_q(z) \ge 0. \tag{13}$$

By virtue of (iv), we have

$$\begin{aligned}
hd_1^\beta &= hd_X^\beta\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\
&\le d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big),\mathbb{R}_+\big) + d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big),\mathbb{R}_+\big).
\end{aligned}$$

By combining (12) and (13) with the last inequality, we have

$$\begin{aligned}
hd_1^\beta &\le \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big)\big| \\
&\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big)\big| \\
&\le \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_1),x_1,\mu_1\big)\big| \\
&\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2\big)\big| \\
&\quad + \big|\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2\big) - \psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_2,\mu_2\big)\big| \\
&\le Lm_2 d_X^{\gamma_2}\big(x(\lambda_1,\mu_2),x_1\big) + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) + Lm_2 d_X^{\gamma_2}\big(x(\lambda_1,\mu_1),x_2\big) \\
&\le Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\
&\quad + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2) + Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \\
&= 2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) + Lm_1 d_M^{\gamma_1}(\mu_1,\mu_2).
\end{aligned} \tag{14}$$

Whence, assumption (iv) implies that

$$d_X\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big) \le \left(\frac{Lm_1}{h-Lm_2 l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2).$$

Step 2: We prove that

$$d_2 := d_X\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \le \left(\frac{2Lm_2 l_2^{\gamma_2}}{h-2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2) \tag{15}$$

for all $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$.

If $x(\lambda_1,\mu_2) = x(\lambda_2,\mu_2)$, then we are done. So, we assume that $x(\lambda_1,\mu_2)\neq x(\lambda_2,\mu_2)$. Since $x(\lambda_1,\mu_2)\in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x(\lambda_2,\mu_2)\in K(x(\lambda_2,\mu_2),\lambda_2)$, by the $l_2\cdot\alpha_2$-Hölder continuity of $K(x(\lambda_1,\mu_2),\cdot)$ and $K(x(\lambda_2,\mu_2),\cdot)$, there exist $x_1'\in K(x(\lambda_2,\mu_2),\lambda_1)$ and $x_2'\in K(x(\lambda_1,\mu_2),\lambda_2)$ such that

$$d_X\big(x(\lambda_1,\mu_2),x_2'\big) \le l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2) \tag{16}$$

and

$$d_X\big(x(\lambda_2,\mu_2),x_1'\big) \le l_2 d_\Lambda^{\alpha_2}(\lambda_1,\lambda_2). \tag{17}$$

Again, by the Hölder continuity of $K(\cdot,\cdot)$, there exist $x_1''\in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x_2''\in K(x(\lambda_2,\mu_2),\lambda_2)$ such that

$$d_X\big(x_1',x_1''\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) = l_1 d_2^{\alpha_1} \tag{18}$$

and

$$d_X\big(x_2',x_2''\big) \le l_1 d_X^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) = l_1 d_2^{\alpha_1}. \tag{19}$$

Since $x(\lambda_1,\mu_2)\in S_W(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)$, by Lemma 3.1, we obtain the following:

$$\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x_1'',\mu_2\big) := \inf_{z\in F(x(\lambda_1,\mu_2),x_1'',\mu_2)} \xi_q(z) \ge 0 \tag{20}$$

and

$$\psi_{\xi_q}\big(x(\lambda_2,\mu_2),x_2'',\mu_2\big) := \inf_{z\in F(x(\lambda_2,\mu_2),x_2'',\mu_2)} \xi_q(z) \ge 0. \tag{21}$$

By virtue of (iv), we have

$$\begin{aligned} hd_2^\beta &= hd_X^\beta\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \\ &\le d\big(\psi_{\xi_q}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\big),\mathbb{R}_+\big) + d\big(\psi_{\xi_q}\big(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\big),\mathbb{R}_+\big). \end{aligned}$$

By combining (20) and (21) with the last inequality, we have

$$\begin{aligned}
hd_2^\beta &\le \bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1'',\mu_2\bigr)\bigr| \\
&\quad + \bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2'',\mu_2\bigr)\bigr| \\
&\le \bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1',\mu_2\bigr)\bigr| \\
&\quad + \bigl|\psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1',\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_1,\mu_2),x_1'',\mu_2\bigr)\bigr| \\
&\quad + \bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2',\mu_2\bigr)\bigr| \\
&\quad + \bigl|\psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2',\mu_2\bigr) - \psi_{\xi_q}\bigl(x(\lambda_2,\mu_2),x_2'',\mu_2\bigr)\bigr| \\
&\le Lm_2 d_X^{\gamma_2}\bigl(x(\lambda_2,\mu_2),x_1'\bigr) + Lm_2 d_X^{\gamma_2}\bigl(x_1',x_1''\bigr) \\
&\quad + Lm_2 d_X^{\gamma_2}\bigl(x(\lambda_1,\mu_2),x_2',\bigr) + Lm_2 d_X^{\gamma_2}\bigl(x_2',x_2''\bigr).
\end{aligned} \tag{22}$$

By virtue of (16), (17), (18) and (19), we get

$$\begin{aligned}
&hd_X^\beta\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\
&\quad\le Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2) + Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\
&\qquad + Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2) + Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\
&\quad= 2Lm_2 l_2^{\gamma_2} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2) + 2Lm_2 l_1^{\gamma_2} d_X^{\alpha_1\gamma_2}\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr).
\end{aligned} \tag{23}$$

Whence, condition (v) implies that

$$d_X^\beta\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \le \left(\frac{2Lm_2 l_2^{\gamma_2}}{h - 2Lm_2 l_1^{\gamma_2}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2}(\lambda_1,\lambda_2).$$

Step 3: Let $x(\lambda_1,\mu_1) \in S_W(\lambda_1,\mu_1)$ and $x(\lambda_2,\mu_2) \in S_W(\lambda_2,\mu_2)$. It follows from (9) and (15) that

$$\begin{aligned}
&d\bigl(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\bigr) \\
&\quad\le d\bigl(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\bigr) + d\bigl(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\bigr) \\
&\quad\le \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2) + \left(\frac{2Lm_2 l_2^{\gamma_2}}{h - 2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\rho\bigl(S_W(\lambda_1,\mu_1),S_W(\lambda_2,\mu_2)\bigr) \\
&\quad= \sup_{x(\lambda_1,\mu_1)\in S_W(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\in S_W(\lambda_2,\mu_2)} d_X\bigl(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\bigr) \\
&\quad\le \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1/\beta}(\mu_1,\mu_2) + \left(\frac{2Lm_2 l_2^{\gamma_2}}{h - 2Lm_2 l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_\Lambda^{\alpha_2\gamma_2/\beta}(\lambda_1,\lambda_2).
\end{aligned}$$

Taking $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$, we see that the diameter of $S(\lambda_1,\mu_1)$ is 0, that is, this set is a singleton $\{x(\lambda_1,\mu_1)\}$. This implies that the (PGVQEP) has a unique solution in a neighborhood of (λ_0,μ_0). The proof is completed. □

Definition 3.4 Let $F: X \times X \times M \to 2^Y$ be a set-valued mapping. A set-valued mapping $F(\cdot,\cdot,\mu) \mapsto 2^Y$ is said to be

(A) *$h \cdot \beta$-Hölder strongly monotone with respect to ξ_q* if there exist $q \in \operatorname{int} C$ and $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda))$ with $x \neq y$,

$$\inf_{z \in F(x,y,\mu)} \xi_q(z) + \inf_{z \in F(y,x,\mu)} \xi_q(z) + h d_X^{\beta}(x,y) \leq 0;$$

(B) *$h \cdot \beta$-Hölder strongly pseudomonotone with respect to* $q \in \operatorname{int} C$ and $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda_0))$ with $x \neq y$,

$$z \notin -\operatorname{int} C, \quad \exists z \in F(x,y,\mu) \quad \Rightarrow \quad z' + h d_X^{\beta}(x,y) q \in -C, \quad \exists z' \in F(y,x,\mu).$$

(C) *quasi-monotone on* $E(N(\lambda_0))$ if $\forall x, y \in E(N(\lambda_0))$ with $x \neq y$,

$$z \in -\operatorname{int} C, \quad \exists z \in F(x,y,\mu) \quad \Rightarrow \quad z' \notin -\operatorname{int} C, \quad \exists z' \in F(y,x,\mu).$$

The following proposition provides the relation among monotonicity conditions defined above.

Proposition 3.5

(i) (A) $\Rightarrow$ (iv).

(ii) (B) *and* (C) $\Rightarrow$ (iv).

Proof (i) From the definition of (A), we have

$$\begin{aligned} h d_X^{\beta}(x,y) &\leq -\inf_{z \in F(x,y,\mu)} \xi_q(z) - \inf_{z \in F(y,x,\mu)} \xi_q(z) \\ &\leq d\Big(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Big). \end{aligned}$$

(ii) Assume that F satisfies definitions (B) and (C). We consider two cases.

Case 1. $z \notin -\operatorname{int} C$, $\exists z \in F(x,y,\mu)$, then there exists $z' \in F(y,x,\mu)$ such that $z' + h d_X^{\beta}(x,y) q \in -C$. From Lemma 2.1, we have

$$\xi_q(z') + h d_X^{\beta}(x,y) = \xi_q\big(z' + h d_X^{\beta}(x,y) q\big) \leq 0,$$

which implies that $\inf_{z \in F(y,x,\mu)} \xi_q(z) \leq \xi_q(z') \leq -h d_X^{\beta}(x,y)$. Hence,

$$h d_X^{\beta}(x,y) \leq -\inf_{z \in F(y,x,\mu)} \xi_q(z) \leq d\Big(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Big).$$

Case 2. $z \in -\operatorname{int} C$, $\exists z \in F(x,y,\mu)$, then there exists $z' \in F(y,x,\mu)$ such that $z \notin -\operatorname{int} C$. By a similar argument as in the previous case, we have the desired result. □

Remark 3.6 The converse of Proposition 3.5 does not hold in general, even in the special case $X = Y = \mathbb{R}$ and $C = \mathbb{R}_+$. See, for example, Examples 1.1 and 1.2 in [15]. Therefore, Theorem 3.3 still holds when condition (iv) is replaced by condition (A) or conditions (B) and (C). We can immediately obtain the following two theorems.

Theorem 3.7 *Theorem* 3.3 *still holds when condition* (iv) *is replaced by condition* (A).

Theorem 3.8 *Theorem* 3.3 *still holds when condition* (iv) *is replaced by conditions* (B) *and* (C).

Let $f : X \times X \times M \to Y$ be a vector-valued mapping. Then (PGVQEP) becomes the following *parametric vector quasi-equilibrium problem* (PVQEP):

Find $x_0 \in K(x_0, \lambda)$ such that

$$f(x_0, y, \mu) \notin -\operatorname{int} C, \quad \forall y \in K(x_0, \lambda). \tag{24}$$

Remark 3.9 In the case of a vector-valued mapping, condition (iv) in Theorem 3.3 and condition (ii$''$) coincide. Also, condition (A) and conditions (B) and (C) are the same as conditions (ii) and (ii$'$) in [30], respectively. It is obvious that Theorems 3.3, 3.7 and 3.8 extend Theorems 3.3, 3.1 and 3.2 in [30], respectively, in the case that the vector-valued mapping $f(\cdot, \cdot, \cdot)$ is extended to a set-valued one.

4 Applications

Since the parametric generalized vector quasi-equilibrium problem (PGVQEP) contains as special cases many optimization-related problems, including quasi-variational inequalities, traffic equilibrium problems, quasi-optimization problems, fixed point and coincidence point problems, complementarity problems, vector optimization, Nash equilibria, *etc.*, we can derive from Theorem 3.3 a direct consequence for such special cases. We discuss now only some applications of our results.

4.1 Quasi-variational inequalities

In this section, we assume that X is a normed space. Let $K : X \times \Lambda \rightrightarrows X$ and $T : X \times M \rightrightarrows B^*(X, Y)$ be set-valued mappings, where $B^*(X, Y)$ denotes the space of all bounded linear mappings of X into Y. Setting $F(x, y, \mu) = \langle T(x, \mu), y - x\rangle := \bigcup_{t \in T(x,\mu)} \langle t, y - x\rangle$ in (6), we obtain parametric generalized vector quasi-variational inequalities (PGVQVI) in the case of set-valued mappings as follows:

$$\text{Find } x_0 \in K(x_0, \lambda) \text{ such that } \bigl\langle T(x_0, \mu), y - x_0\bigr\rangle \subseteq Y \setminus -\operatorname{int} C, \quad \forall y \in K(x_0, \lambda). \tag{25}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \bigl\{x \in X : x \in K(x, \lambda)\bigr\}.$$

The solution set of (25) is denoted by

$$S^V_{QVI}(\lambda, \mu) := \bigl\{x \in E(\lambda) : \bigl\langle T(x, \mu), y - x\bigr\rangle \subseteq Y \setminus -\operatorname{int} C, \forall y \in K(x, \lambda)\bigr\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $q \in \operatorname{int} C$, the ξ_q-solution set of (25) is

$$S^V_{QVI}(\xi_q, \lambda, \mu) := \Bigl\{x \in E(\lambda) : \inf_{z \in \langle T(x,\mu), y - x\rangle} \xi_q(z) \geq 0, \forall y \in K(x, \lambda)\Bigr\}.$$

Theorem 4.1 *Assume that for each fixed* $q \in \operatorname{int} C$, $S^V_{QVI}(\xi_q, \lambda, \mu)$ *is nonempty in a neighborhood* $N(\lambda_0) \times N(\mu_0)$ *of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$. *Assume further that the following conditions hold.*

(i′) *$K(\cdot,\cdot)$ is $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$-Hölder continuous on $E(N(\lambda_0)) \times N(\lambda_0)$;*

(ii′) *For each $x \in E(N(\lambda_0))$, $T(x,\cdot)$ is $m_3 \cdot \gamma_3$-Hölder continuous at $\mu_0 \in M$;*

(iii′) *$T(\cdot,\cdot)$ is bounded in $x \in E(N(\lambda_0))$, and $E(N(\lambda_0))$ is bounded;*

(iv′) *$T(\cdot,\mu)$ is $h \cdot \beta$-Hölder strongly monotone with respect to ξ_q, i.e., there exist constants $h > 0$, $\beta > 0$ such that for every $x, y \in E(N(\lambda_0))$: $x \neq y$,*

$$h\|x-y\|^{\beta} \le d\Big(\inf_{z\in\langle T(x,\mu),y-x\rangle} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z\in\langle T(y,\mu),x-y\rangle} \xi_q(z), \mathbb{R}_+\Big);$$

(v′) *$\beta = \alpha_1$, $h > 2MLl_1^{\gamma_1}$, where $L := \sup_{\lambda\in C^q} \|\lambda\| \in [\frac{1}{\|q\|}, +\infty)$ is the Lipschitz constant of ξ_q on Y.*

Then, for every $(\lambda,\mu) \in N(\lambda_0) \times N(\mu_0)$, the solution of (PGVQVI) *is unique, $x(\lambda,\mu)$, and this function satisfies the Hölder condition: for all $(\lambda_1,\mu_1), (\lambda_2,\mu_2) \in N(\lambda_0) \times N(\mu_0)$,*

$$\begin{aligned} d_X\big(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\big) \le{}& \left(\frac{2MLl_2}{h - 2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_\lambda^{\alpha_2/\beta}(\lambda_1,\lambda_2) \\ &+ \left(\frac{Nm_3L}{h - 2MLl_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_M^{\gamma_3/\beta}(\mu_1,\mu_2), \end{aligned}$$

where $x(\lambda_i,\mu_i) \in S_{QVI}(\lambda_i,\mu_i)$, $i = 1, 2$.

Proof We verify that all the assumptions of Theorem 3.3 are fulfilled. First, (i′), (iv′) and (v′) are the same as (i), (iv) and (v) in Theorem 3.3. We need only to verify conditions (ii) and (iii). Taking $M, \widetilde{M} > 0$ such that

$$\big\|T(x,\mu)\big\| \le M, \quad \forall (x,\mu) \in E\big(N(\lambda_0)\big) \times N(\mu_0)$$

and

$$\|x-y\| \le \widetilde{M}, \quad \forall x,y \in E\big(N(\lambda_0)\big).$$

We put $m_1 = \widetilde{M}m_3$ and $\gamma_1 = \gamma_3$. For any fixed $x, y \in E(N(\lambda_0))$, by assumption (ii′), we have

$$T(x,\mu_1) \subseteq T(x,\mu_2) + m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, \quad \forall \mu_1,\mu_2 \in N(\mu_0).$$

Then

$$\begin{aligned} \big\langle T(x,\mu_1), y-x\big\rangle &\subseteq \big\langle T(x,\mu_2) + m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, y-x\big\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + \big\langle m_3 d^{\gamma_3}(\mu_1,\mu_2) B_{B^*(X,Y)}, y-x\big\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \langle B_{B^*(X,Y)}, y-x\rangle \\ &= \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \bigcup_{g\in B_{B^*(X,Y)}} \langle g, y-x\rangle \\ &\subseteq \big\langle T(x,\mu_2), y-x\big\rangle + m_3 d^{\gamma_3}(\mu_1,\mu_2) \widetilde{M} B_Y. \end{aligned}$$

Hence

$$\langle T(x,\mu_1), y-x\rangle \subseteq \langle T(x,\mu_2), y-x\rangle + m_1 d^{\gamma_1}(\mu_1,\mu_2)\widetilde{M}B_Y.$$

Also, we put $m_2 = M$ and $\gamma_2 = 1$. We need to show that

$$\langle T(x,\mu), y_1-x\rangle \subseteq \langle T(x,\mu), y_2-x\rangle + \widetilde{M}\|y_1-y_2\|B_Y.$$

For each fixed $x \in E(N(\lambda_0))$ and $\mu \in N(\mu_0)$,

$$\begin{aligned}\langle T(x,\mu), y_1-x\rangle &= \bigcup_{t\in T(x,\mu)} \langle t, y_1-x\rangle \\ &= \bigcup_{t\in T(x,\mu)} \langle t, y_1-x+y_2-y_2\rangle \\ &= \bigcup_{t\in T(x,\mu)} \langle t, y_2-x\rangle + \bigcup_{t\in T(x,\mu)} \langle t, y_1-y_2\rangle \\ &\subseteq \langle T(x,\mu), y-x\rangle + M\|y_1-y_2\|B_Y.\end{aligned}$$

Hence, condition (iii) is verified, and so we obtain the result. □

For (PGVQVI), if we put $Y = \mathbb{R}$, $C = [0,+\infty)$, then (25) becomes the following parametric generalized quasi-variational inequality problem in the case of scalar-valued one:

$$\text{Find } x_0 \in K(x_0,\lambda) \text{ such that } \langle t, y-x_0\rangle \geq 0, \quad \forall y \in K(x_0,\lambda), \forall t \in T(x_0,\mu). \tag{26}$$

For each $\lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, let

$$E(\lambda) := \{x \in X : x \in K(x,\lambda)\}.$$

The solution set of (26) is denoted by

$$S^S_{QVI}(\lambda,\mu) := \{x \in E(\lambda) : \langle t, y-x\rangle \geq 0, \forall y \in K(x,\lambda), \forall t \in T(x,\mu)\}.$$

For each $\lambda \in N(\lambda_0)$, $\mu \in N(\mu_0)$ and fixed $1 \in \operatorname{int} C$, the ξ_q-solution set of (25) is

$$S^S_{QVI}(\xi_1,\lambda,\mu) := \Bigl\{x \in E(\lambda) : \inf_{z\in\langle T(x,\mu),y-x\rangle} \xi_1(z) \geq 0, \forall y \in K(x,\lambda)\Bigr\}.$$

It follows from Lemma 2.1 that $S^S_{QVI}(\xi_1,\lambda,\mu)$ coincides with $S^S_{QVI}(\lambda,\mu)$.

Corollary 4.2 *Assume that $S^S_{QVI}(\lambda,\mu)$ is nonempty in a neighborhood $N(\lambda_0)\times N(\mu_0)$ of the considered point $(\lambda_0,\mu_0) \in \Lambda \times M$. Assume further that conditions* (i′)-(iii′) *and* (v′) *in Corollary* 4.1 *hold. Replace* (iv′) *by* (iv″).

(iv″) *$T(\cdot,\mu)$ is $h\cdot\beta$-Hölder strongly monotone, i.e., there exist constants $h > 0$, $\beta > 0$, such that for every $x, y \in E(N(\lambda_0))$: $x \neq y$,*

$$\langle u-v, x-y\rangle \geq h\|x-y\|^\beta, \quad \forall u \in T(x), \forall v \in T(y).$$

Then, for every $(\lambda,\mu) \in N(\lambda_0) \times N(\mu_0)$, *the solution of* (PGVQVI) *is unique*, $x(\lambda,\mu)$, *and this function satisfies the Hölder condition*: *for all* $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N(\lambda_0) \times N(\mu_0)$,

$$d_X\big(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)\big) \le \left(\frac{2Ml_2}{h-2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_\lambda^{\alpha_2/\beta}(\lambda_1,\lambda_2) + \left(\frac{Nm_3}{h-2Ml_1^{\gamma_3}}\right)^{\frac{1}{\beta}} d_M^{\gamma_3/\beta}(\mu_1,\mu_2),$$

where $x(\lambda_i,\mu_i) \in S_{QVI}^S(\lambda_i,\mu_i)$, $i = 1,2$.

Proof It is not hard to show that (iv″) implies (iv′). Indeed, for any $x,y \in E(N(\lambda_0))$ with $x \neq y$,

$$\begin{aligned}
h\|x-y\|^\beta &\le \langle u-v, x-y\rangle \\
&= \langle u, x-y\rangle + \langle v, y-x\rangle \\
&\le \sup_{u\in T(x)} \langle u, x-y\rangle + \sup_{v\in T(y)} \langle v, y-x\rangle \\
&= \sup_{u\in T(x)} -\langle u, y-x\rangle + \sup_{v\in T(y)} -\langle v, x-y\rangle \\
&= -\inf_{u\in T(x)} \langle u, y-x\rangle - \inf_{v\in T(y)} \langle v, x-y\rangle \\
&\le d\Big(\inf_{u\in T(x)} \langle u, y-x\rangle, \mathbb{R}_+\Big) + d\Big(\inf_{v\in T(y)} \langle v, x-y\rangle, \mathbb{R}_+\Big).
\end{aligned}$$

Therefore, (iv′) is satisfied. □

Remark 4.3 Corollary 4.2 extends Corollary 3.1 in [33] since the mapping T is a multi-valued mapping.

4.2 Traffic equilibrium problems

The foundation of the study of traffic network problems goes back to Wardrop [34], who stated the basic equilibrium principle in 1952. Over the past decades, a large number of efforts have been devoted to the study of traffic assignment models, with emphasis on efficiency and optimality, in order to improve practicability, reduce gas emissions and contribute to the welfare of the community. The variational inequality approach to such problems begins with the seminal work of Smith [35] who proved that the user-optimized equilibrium can be expressed in terms of a variational inequality. Thus, the possibility of exploiting the powerful tools of variational analysis has led to dealing with a large variety of models, reaching valuable theoretical results and providing applications in practical situations. In this paper, we are concerned with a class of equilibrium problems which can be studied in the framework of quasi-variational inequalities, see [36, 37].

Let a set N of nodes, a set L of links, a set $W := (W_1,\dots,W_l)$ of origin-destination pairs (O/D pairs for short) be given. Assume that there are $r_j \ge 1$ paths connecting the pairs W_j, $j = 1,\dots,l$, whose set is denoted by P_j. Set $m := r_1 + \cdots + r_l$; *i.e.*, there are in whole m paths in the traffic network. Let $F := (F_1,\dots,F_m)$ stand for the path flow vector. Assume that the travel cost of the path R_s, $s = 1,\dots,m$, is a set $T_s(F) \subset \mathbb{R}_+$. So, we have a multifunction $T : \mathbb{R}_+^m \rightrightarrows \mathbb{R}_+^m$ with $T(F) := (T_1(F),\dots,T_m(F))$. Let the capacity restriction be

$$F \in A := \big\{F \in \mathbb{R}_+^m : F_s \le \Gamma_s, s = 1,\dots,m\big\},$$

where Γ_s are given real numbers. Extending the Wardrop definition to the case of multi-valued costs, we propose the following definition.

A path flow vector H is said to be a ***weak*** *equilibrium flow* vector if

$$\begin{aligned}
&\forall W_j, \forall R_q \in P_j, R_s \in P_j, \text{ there exists } t \in T(H) \text{ such that} \\
&\quad t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0,
\end{aligned} \tag{27}$$

where $j = 1, \dots, l$ and $q, s \in \{1, \dots, m\}$ are among r_j indices corresponding to P_j.

A path flow vector H is said to be a ***strong*** *equilibrium flow* vector if

$$\forall W_j, \forall R_q \in P_j, R_s \in P_j, \text{ for all } t \in T(H) \text{ such that } t_q < t_s \Rightarrow H_q = \Gamma_q \text{ or } H_s = 0. \tag{28}$$

Suppose that the travel demand ρ_j of the O/D pair $W_j, j = 1, \dots, l$, depends on the weak (or strong) equilibrium problem flow H. So, considering all the O/D pairs, we have a mapping $\rho : \mathbb{R}^m_+ \to \mathbb{R}^l_+$. We use the Kronecker notation

$$\phi_{js} = \begin{cases} 1 & \text{if } s \in P_j, \\ 0 & \text{if } s \notin P_j. \end{cases}$$

Then the matrix

$$\phi = \{\phi_{js}\}, \quad j = 1, \dots, l, s = 1, \dots, m,$$

is called an O/D pair/path incidence matrix. The path flow vectors meeting the travel demands are called the feasible path flow vectors and form the constraint set, for a given weak (or strong) equilibrium flow H,

$$K(H, \lambda) := \{F \in A : \phi F = \rho(H, \lambda)\}.$$

Assume further that the path costs are also perturbed, *i.e.*, depend on a perturbation parameter μ of a metric space M: $T_s(F, \mu)$, $s = 1, \dots, m$.

Our traffic equilibrium problem is equivalent to a quasi-variational inequality as follows (see [38]).

Lemma 4.4 *A path vector flow $H \in K(H, \lambda)$ is a* ***weak*** *equilibrium flow if and only if it is a solution of the following quasi-variational inequality*:

$$\begin{aligned}
&\textit{Find } H \in K(H, \lambda) \textit{ such that there exists } t \in T(H, \lambda) \textit{ satisfying } \langle t, F - H \rangle \geq 0, \\
&\quad \forall F \in K(H, \lambda).
\end{aligned}$$

Lemma 4.5 *A path vector flow $H \in K(H, \lambda)$ is a* ***strong*** *equilibrium flow if and only if it is a solution of the following quasi-variational inequality*:

$$\begin{aligned}
&\textit{Find } H \in K(H, \lambda) \textit{ such that for all } t \in T(H, \lambda) \textit{ it satisfies } \langle t, F - H \rangle \geq 0, \\
&\quad \forall F \in K(H, \lambda).
\end{aligned}$$

Corollary 4.6 *Assume that solutions of the traffic network equilibrium problem exist and all the assumptions of Corollary* 4.2 *are satisfied. Then, in a neighborhood of* (λ_0, μ_0), *the solution is unique and satisfies the same Hölder condition as in Corollary* 4.2.

4.3 Quasi-optimization problem

For the normed linear space Y and pointed, closed and convex cone C with nonempty interior, we denote the ordering induced by C as follows:

$$x \leq y \quad \text{iff} \quad y - x \in C;$$
$$x < y \quad \text{iff} \quad y - x \in \operatorname{int} C.$$

The orderings $\geq$ and $>$ are defined similarly. Let $g : X \times M \to Y$ be a vector-valued mapping. For each $(\lambda, \mu) \in \Lambda \times M$, consider the problem of parametric quasi-optimization problem (PQOP) finding $x_0 \in K(x_0, \lambda)$ such that

$$g(x_0, \mu) = \min_{y \in K(x_0, \lambda)} g(y, \mu). \tag{29}$$

Since the constraint set depends on the minimizer x_0, this is a quasi-optimization problem. Setting $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, (PVQEP) becomes a special case of (PQOP).

The following results are derived from Theorem 3.8 (Theorem 3.3 cannot be applied since $f(x, y, \mu) + f(y, x, \mu) = 0$, $\forall x, y \in A$ and $\mu \in M$).

Theorem 4.7 *For* (PQOP), *assume that the solution exists in a neighborhood* $N(\lambda_0) \times N(\mu_0)$ *of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$. *Assume further that the following conditions hold.*

(i) $K(\cdot, \cdot)$ *is* $(l_1 \cdot \alpha_1, l_2 \cdot \alpha_2)$*-Hölder continuous on* $E(N(\lambda_0)) \times N(\lambda_0)$;
(ii) *For each* $x, y \in E(N(\lambda_0))$, $F(x, y, \cdot)$ *is* $m_1 \cdot \gamma_1$*-Hölder continuous at* $\mu_0 \in M$;
(iii) *For each* $x \in E(N(\lambda_0))$ *and* $\mu \in N(\mu_0)$, $F(x, \cdot, \mu)$ *is* $m_2 \cdot \gamma_2$*-Hölder continuous on* $E(N(\lambda_0))$;
(iv) $F(\cdot, \cdot, \mu)$ *is* $h \cdot \beta$*-Hölder strongly monotone with respect to* ξ_q, *i.e., there exist constants* $h > 0$, $\beta > 0$ *such that for every* $x, y \in E(N(\lambda_0))$: $x \neq y$,

$$h d_X^{\beta}(x, y) \leq d\Big(\inf_{z \in F(x,y,\mu)} \xi_q(z), \mathbb{R}_+\Big) + d\Big(\inf_{z \in F(y,x,\mu)} \xi_q(z), \mathbb{R}_+\Big);$$

(v) $\beta = \alpha_1 \gamma_2$, $h > 2m_2 L l_1^{\gamma_1}$, *where* $L := \sup_{\lambda \in C^q} \|\lambda\| \in [\frac{1}{\|q\|}, +\infty)$ *is the Lipschitz constant of* ξ_q *on* Y.

Then, for every $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$, *the solution of* (PVQGEP) *is unique,* $x(\lambda, \mu)$, *and this function satisfies the Hölder condition*:

for all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N(\lambda_0) \times N(\mu_0)$,

$$\begin{aligned} d_X\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \leq{}& \left(\frac{2m_2 L l_2^{\gamma_2}}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_{\Lambda}^{\alpha_2 \gamma_2 / \beta}(\lambda_1, \lambda_2) \\ &+ \left(\frac{m_1 L}{h - 2m_2 L l_1^{\gamma_1}}\right)^{\frac{1}{\beta}} d_M^{\gamma_1 / \beta}(\mu_1, \mu_2), \end{aligned}$$

where $x(\lambda_i, \mu_i) \in S_W(\lambda_i, \mu_i)$, $i = 1, 2$.

5 Conclusions

In this paper, by using a nonlinear scalarization technique, we obtain sufficient conditions for Hölder continuity of the solution mapping for a parametric generalized vector quasi-equilibrium problem in the case where the mapping F is a general set-valued one. As applications, we derived this Hölder continuity for some quasi-variational inequalities, traffic network problems and quasi-optimization problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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References

1. Ansari, QH: Vector equilibrium problems and vector variational inequalities. In: Giannessi, F (ed.) Vector Variational Inequalities and Vector Equilibria Mathematical Theories, pp. 1-16. Kluwer Academic, Dordrecht (2000)
2. Ansari, QH, Yao, JC: An existence result for the generalized vector equilibrium problem. Appl. Math. Lett. **12**, 53-56 (1999)
3. Chuong, TD, Yao, JC: Isolated and proper efficiencies in semi-infinite vector optimization problems. J. Optim. Theory Appl. **162**, 447-462 (2014)
4. Xue, XW, Li, SJ, Liao, CM, Yao, JC: Sensitivity analysis of parametric vector set-valued optimization problems via coderivatives. Taiwan. J. Math. **15**, 2533-2554 (2011)
5. Chuong, TD, Yao, JC, Yen, ND: Further results on the lower semicontinuity of efficient point multifunctions. Pac. J. Optim. **6**, 405-422 (2010)
6. Gong, XH, Kimura, K, Yao, JC: Sensitivity analysis of strong vector equilibrium problems. J. Nonlinear Convex Anal. **9**, 83-94 (2008)
7. Kimura, K, Yao, JC: Semicontinuity of solution mappings of parametric generalized vector equilibrium problems. J. Optim. Theory Appl. **138**, 429-443 (2008)
8. Kimura, K, Yao, JC: Semicontinuity of solution mappings of parametric generalized strong vector equilibrium problems. J. Ind. Manag. Optim. **4**, 167-181 (2008)
9. Kimura, K, Yao, JC: Sensitivity analysis of solution mappings of parametric vector quasiequilibrium problems. J. Glob. Optim. **41**, 187-202 (2008)
10. Wangkeeree, R, Wangkeeree, R, Preechasilp, P: Continuity of the solution mappings of parametric generalized vector equilibrium problems. Appl. Math. Lett. **29**, 42-45 (2014)
11. Yen, ND: Hölder continuity of solutions to parametric variational inequalities. Appl. Math. Optim. **31**, 245-255 (1995)
12. Mansour, MA, Riahi, H: Sensitivity analysis for abstract equilibrium problems. J. Math. Anal. Appl. **306**, 684-691 (2005)
13. Bianchi, M, Pini, R: A note on stability for parametric equilibrium problems. Oper. Res. Lett. **31**, 445-450 (2003)
14. Anh, LQ, Khanh, PQ: On the Hölder continuity of solutions to multivalued vector equilibrium problems. J. Math. Anal. Appl. **321**, 308-315 (2006)
15. Anh, LQ, Khanh, PQ: Uniqueness and Hölder continuity of solutions to multivalued vector equilibrium problems in metric spaces. J. Glob. Optim. **37**, 349-465 (2007)
16. Anh, LQ, Khanh, PQ: Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions. J. Glob. Optim. **42**, 515-531 (2008)
17. Li, SJ, Li, XB, Teo, KL: The Hölder continuity of solutions to generalized vector equilibrium problems. Eur. J. Oper. Res. **199**, 334-338 (2011)
18. Li, SJ, Chen, CR, Li, XB, Teo, KL: Hölder continuity and upper estimates of solution to vector quasiequilibrium problems. Eur. J. Oper. Res. **210**, 148-157 (2011)
19. Chen, CR, Li, SJ, Teo, KL: Solution semicontinuity of parametric generalized vector equilibrium problems. J. Glob. Optim. **45**, 309-318 (2009)
20. Gong, XH: Continuity of the solution set to parametric weak vector equilibrium problems. J. Glob. Optim. **139**, 3-46 (2008)
21. Gong, XH, Yao, JC: Lower semicontinuity of the set of efficient solutions for generalized systems. J. Optim. Theory Appl. **138**, 197-205 (2008)
22. Li, SJ, Li, XB: Hölder continuity of solutions to parametric weak generalized Ky Fan inequality. J. Optim. Theory Appl. **149**, 540-553 (2011)
23. Wang, QL, Lin, Z, Li, XX: Semicontinuity of the solution set to a parametric generalized strong vector equilibrium problem. Positivity (2014). doi:10.1007/s11117-014-0273-9

24. Peng, ZY: Hölder continuity of solutions to parametric generalized vector quasiequilibrium problems. Abstr. Appl. Anal. (2012). doi:10.1155/2012/236413
25. Chen, GY, Huang, XX, Yang, XQ: Vector Optimization: Set-Valued and Variational Analysis. Springer, Berlin (2005)
26. Luc, DT: Theory of Vector Optimization. Springer, Berlin (1989)
27. Chen, CR, Li, SJ: Semicontinuity of the solution set map to a set-valued weak vector variational inequality. J. Ind. Manag. Optim. **3**, 519-528 (2007)
28. Bianchi, M, Pini, R: Sensitivity of parametric vector equilibria. Optimization **55**, 221-230 (2006)
29. Chen, CR, Li, MH: Hölder continuity of solutions to parametric vector equilibrium problem with nonlinear scalarization. Numer. Funct. Anal. Optim. **35**, 685-707 (2014)
30. Chen, CR: Hölder continuity of the unique solution to parametric vector quasiequilibrium problems via nonlinear scalarization. Positivity **17**, 133-150 (2013)
31. Chen, GY, Yang, XQ, Yu, H: A nonlinear scalarization function and generalized quasi-vector equilibrium problems. J. Glob. Optim. **32**, 451-466 (2005)
32. Li, SJ, Yang, XQ, Chen, GY: Nonconvex vector optimization of set-valued mappings. J. Math. Anal. Appl. **283**, 337-350 (2003)
33. Anh, LQ, Khanh, PQ: Hölder continnuity of the unique solution to quasequilibrium problems in metric spaces. J. Optim. Theory Appl. **141**, 37-54 (2009)
34. Wardrop, JG: Some theoretical aspects of road traffic research. In: Proc. Inst. Civil Eng., Part II (1952)
35. Smith, MJ: The existence, uniqueness and stability of traffic equilibrium. Transp. Res., Part B, Methodol. **13**, 295-304 (1979)
36. De Luca, M, Maugeri, A: Quasi-variational inequality and applications to equilibrium problems with elastic demands. In: Clarke, FM, Demyanov, VF, Giannessi, F (eds.) Nonsmooth Optimization and Related Topic, pp. 61-77. Plenum, New York (1989)
37. De Luca, M: Existence of solutions for a time-dependent quasi-variational inequality. Rend. Circ. Mat. Palermo Suppl. **48**, 101-106 (1997)
38. Khanh, PQ, Luu, LM: On the existence of solution to vector quasivariational inequalities and quasicomplementarity with applications to traffic equilibria. J. Optim. Theory Appl. **123**, 1533-1548 (2004)

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