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Lower semicontinuity of approximate solution mappings for parametric generalized vector equilibrium problems

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Abstract

In this paper, we obtain sufficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, we obtain the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness.

Keywords: lower semicontinuity; approximate solution mapping; parametric generalized vector equilibrium problems; scalarization method

1 Introduction

The vector equilibrium problem is a unified model of several problems, for example, the vector optimization problem, the vector variational inequality problem, the vector complementarity problem and the vector saddle point problem. In the literature, existence results for various types of vector equilibrium problems have been investigated intensively, *e.g.*, see [1–4] and the references therein. The stability analysis of the solution mappings for VEP is an important topic in vector equilibrium theory. Recently, the semicontinuity, especially the lower semicontinuity, of solution mappings to parametric vector equilibrium problems has been studied in the literature, see [5–16]. In the mentioned results, the lower semicontinuity of solution mappings to parametric generalized strong vector equilibrium problems is established under the assumptions of monotonicity and compactness. Very recently, Han and Gong [17] studied the lower semicontinuity of solution mappings to parametric generalized strong vector equilibrium problems without the assumptions of monotonicity and compactness.

On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently 'regular'. Moreover, these mathematical models are solved usually by numerical methods which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is of interest in both practical applications and computations. Anh and Khanh [18] considered two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems and established the sufficient conditions for their Hausdorff semicontinuity (or Berge semicontinuity). Among many approaches for dealing with the lower semicontinuity and continuity of solution mappings for parametric vector variational inequalities and parametric vector equilibrium problems, the scalarization method is of considerable interest. By using a scalarization method, Li and Li [19] discussed the Berge lower semicontinuity and Berge continuity of an approximate solution mapping for a parametric vector equilibrium problem.

Motivated by the work reported in [17–19], in this paper we aim to establish efficient conditions for the lower semicontinuity of an approximate solution mapping for a parametric generalized vector equilibrium problem involving set-valued mappings. By using a scalarization method, we obtain the lower semicontinuity of an approximate solution mapping for such a problem without the assumptions of monotonicity and compactness.

2 Preliminaries

Throughout this paper, let X and Y be real Hausdorff topological vector spaces, and let Z be a real topological space. We also assume that C is a pointed closed convex cone in Y with its interior $\operatorname{int} C \neq \emptyset$. Let Y^* be the topological dual space of Y. Let $C^* := \{\xi \in Y^* : \langle \xi, y\rangle \geq 0, \forall y \in C\}$ be the dual cone of C, where $\langle \xi, y\rangle$ denotes the value of ξ at y. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then $B_e^* := \{\xi \in C^* : \langle \xi, e\rangle = 1\}$ is a weak* compact base of C^*.

Suppose that K is a nonempty subset of X and $F : K \times K \to 2^Y \backslash \{\emptyset\}$ is a set-valued mapping. We consider the following generalized vector equilibrium problem (GVEP) of finding $x_0 \in K$ such that

$$F(x_0, y) \subset Y \backslash -\operatorname{int} C, \quad \forall y \in K. \tag{2.1}$$

When the set K and the mapping F are perturbed by a parameter μ which varies over a set M of Z, we consider the following parametric generalized vector equilibrium problem (PGVEP) of finding $x_0 \in K(\mu)$ such that

$$F(x_0, y, \mu) \subset Y \backslash -\operatorname{int} C, \quad \forall y \in K(\mu), \tag{2.2}$$

where $K : M \to 2^X \backslash \{\emptyset\}$ is a set-valued mapping, $F : B \times B \times M \subset X \times X \times Z \to 2^Y \backslash \{\emptyset\}$ is a set-valued mapping with $K(M) = \bigcup_{\mu \in M} K(\mu) \subset B$. For each $\varepsilon > 0$ and $\mu \in M$, the approximate solution set of (PGVEP) is defined by

$$\widetilde{S}(\varepsilon, \mu) := \bigl\{x \in K(\mu) : F(x, y, \mu) + \varepsilon e \subset Y \backslash -\operatorname{int} C, \forall y \in K(\mu)\bigr\},$$

where $e \in \operatorname{int} C$. For each $\xi \in B_e^*$ and $(\varepsilon, \mu) \in \mathbb{R}^+ \times M$, by $\widetilde{S}_\xi(\varepsilon, \mu)$ we denote the ξ-approximate solution set of (PGVEP), *i.e.*,

$$\widetilde{S}_\xi(\varepsilon, \mu) := \Bigl\{x \in K(\mu) : \inf_{z \in F(x,y,\mu)} \xi(z) + \varepsilon \geq 0, \forall y \in K(\mu)\Bigr\}.$$

Definition 2.1 Let D be a nonempty convex subset of X. A set-valued mapping $G : X \to 2^Y$ is said to be:

(i) *C-convex* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0,1]$, we have

$$tG(x_1) + (1-t)G(x_2) \subseteq G\bigl(tx_1 + (1-t)x_2\bigr) + C.$$

(ii) *C-concave* on D if, for any $x_1, x_2 \in D$ and for any $t \in [0,1]$, we have

$$G\bigl(tx_1 + (1-t)x_2\bigr) \subseteq tG(x_1) + (1-t)G(x_2) + C.$$

Definition 2.2 [17] Let M and M_1 be topological vector spaces. Let D be a nonempty subset of M. A set-valued mapping $G : M \to 2^{M_1}$ is said to be *uniformly continuous* on D if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood U_0 of $0 \in M$ such that $G(x_1) \subseteq G(x_2) + V$ for any $x_1, x_2 \in D$ with $x_1 - x_2 \in U_0$.

Definition 2.3 [20] Let M and M_1 be topological vector spaces. A set-valued mapping $G : M \to 2^{M_1}$ is said to be:

(i) *Hausdorff upper semicontinuous* (*H-u.s.c.*) at $u_0 \in M$ if, for any neighborhood V of $0 \in M_1$, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \subseteq G(u_0) + V \quad \text{for every } u \in U(u_0).$$

(ii) *Lower semicontinuous* (*l.s.c.*) at $u_0 \in M$ if, for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that

$$G(u) \cap V \neq \emptyset \quad \text{for every } u \in U(u_0).$$

The following lemma plays an important role in the proof of the lower semicontinuity of the solution mapping $\widetilde{S}(\cdot,\cdot)$.

Lemma 2.4 [21, Theorem 2] *The union* $\Gamma = \bigcup_{i\in I} \Gamma_i$ *of a family of l.s.c. set-valued mappings* Γ_i *from a topological space* X *into a topological space* Y *is also an l.s.c. set-valued mapping from* X *into* Y*, where* I *is an index set.*

3 Lower semicontinuity of the approximate solution mapping for (PGVEP)

In this section, we establish the lower semicontinuity of the approximate solution mapping for (PGVEP) at the considered point $(\varepsilon_0, \mu_0) \in \mathbb{R}^+ \times M$ with $\varepsilon_0 > 0$.

Firstly, using the same argument as in the proof given in [22, Lemma 3.1], we can prove the following useful result.

Lemma 3.1 *For each* $\varepsilon > 0$, $\mu \in M$, *if for each* $x \in K(\mu)$, $F(x, K(\mu), \mu) + C$ *is a convex set, then*

$$\widetilde{S}(\varepsilon,\mu) = \bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu) = \bigcup_{\xi\in B_e^*} \widetilde{S}_\xi(\varepsilon,\mu).$$

Proof For any $x \in \bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu)$, there exists $\xi' \in C^*\setminus\{0\}$ such that $x \in \widetilde{S}_{\xi'}(\varepsilon,\mu)$. Thus, we can obtain that $x \in K(\mu)$ and $\inf_{z\in F(x,y,\mu)} \xi'(z) + \varepsilon \geq 0$, $\forall y \in K(\mu)$. Then, for each $y \in K(\mu)$ and $z \in F(x,y,\mu)$, $\xi'(z) + \varepsilon \geq 0$, which arrives at $z \notin -\operatorname{int} C$. It then follows that, for each $z \in F(x,y,\mu)$,

$$F(x,y,\mu) + \varepsilon e \subseteq Y \setminus -\operatorname{int} C, \quad \forall y \in K(\mu),$$

which gives that $x \in \widetilde{S}(\varepsilon,\mu)$. Hence, $\bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu) \subseteq \widetilde{S}(\varepsilon,\mu)$. Conversely, let $x \in \widetilde{S}(\varepsilon,\mu)$ be arbitrary. Then $x \in K(\mu)$ and $F(x,y,\mu) + \varepsilon e \subseteq Y\setminus -\operatorname{int} C$, $\forall y \in K(\mu)$. Thus, we have

$$F\bigl(x,K(\mu),\mu\bigr) \cap (-\operatorname{int} C) = \emptyset,$$

and hence

$$\bigl(F\bigl(x,K(\mu),\mu\bigr) + C\bigr) \cap (-\operatorname{int} C) = \emptyset.$$

Because $F(x,K(\mu),\mu)+C$ is a convex set, by the well-known Edidelheit separation theorem (see [23], Theorem 3.16), there exist a continuous linear functional $\xi \in Y^*\setminus\{0\}$ and a real number γ such that

$$\xi(\hat{c}) < \gamma \leq \xi(z + c)$$

for all $z \in F(x,K(\mu),\mu)$, $c \in C$ and $\hat{c} \in -\operatorname{int} C$. Since C is a cone, we have $\xi(\hat{c}) \leq 0$ for all $\hat{c} \in -\operatorname{int} C$. Thus, $\xi(\hat{c}) \geq 0$ for all $\hat{c} \in C$, that is, $\xi \in C^*$. Moreover, it follows from $c \in C$, $\hat{c} \in -\operatorname{int} C$ and the continuity of ξ that $\xi(z) + \varepsilon \geq 0$ for all $z \in F(x,K(\mu),\mu)$. Thus, for all $y \in K(\mu)$, we have $\inf_{z\in F(x,y,\mu)} \xi(z) + \varepsilon \geq 0$, *i.e.*, $x \in \widetilde{S}_\xi(\varepsilon,\mu) \subseteq \bigcup_{\xi\in C^*\setminus\{0\}} \widetilde{S}_\xi(\varepsilon,\mu)$. □

Theorem 3.2 *We assume that for any given $\xi \in B_e^*$, there exists $\delta > 0$ such that the ξ-approximate solution set $\widetilde{S}_\xi(\cdot,\cdot)$ exists in $[\varepsilon_0,\delta) \times N(\mu_0)$, where $N(\mu_0)$ is a neighborhood of μ_0. Assume further that the following conditions are satisfied*:

(i) *$K(\mu_0)$ is nonempty convex*;
(ii) *K is H-u.s.c. at μ_0 and l.s.c. at μ_0*;
(iii) *for any $y \in K(\mu_0)$, $F(\cdot,y,\mu_0)$ is C-concave on $K(\mu_0)$*;
(iv) *$F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$.*

Then the ξ-approximate solution mapping $\widetilde{S}_\xi : [\varepsilon_0,\delta) \times N(\mu_0) \to 2^X$ is l.s.c. at (ε_0,μ_0).

Proof Suppose to the contrary that $\widetilde{S}_\xi(\cdot,\cdot)$ is not l.s.c. at (ε_0,μ_0), then there exist $x_0 \in \widetilde{S}_\xi(\varepsilon_0,\mu_0)$ and a neighborhood W_0 of $0_X \in X$. For any neighborhoods $J(\varepsilon_0)$ and $U(\mu_0)$ of ε_0 and μ_0, respectively, there exist $\varepsilon' \in J(\varepsilon_0) \cap [\varepsilon_0,\delta)$ and $\mu' \in U(\mu_0)$ such that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon',\mu') = \emptyset$. In particular, there exist sequences $\{\varepsilon_n\} \downarrow \varepsilon_0$ and $\{\mu_n\} \to \mu_0$ such that

$$(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_n,\mu_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

For the above W_0, there exists a neighborhood W_1 of $0_X \in X$ such that

$$W_1 + W_1 \subseteq W_0. \tag{3.2}$$

We define a ξ-set-valued mapping $H_\xi : [0,\delta) \to 2^X$ by

$$H_\xi(\varepsilon) = \Bigl\{x \in K(\mu_0) : \inf_{z\in F(x,y,\mu_0)} \xi(z) + \varepsilon + \varepsilon_0 \geq 0, \forall y \in K(\mu_0)\Bigr\}, \quad \varepsilon \in [0,\delta).$$

Notice that $H_\xi(0) = \widetilde{S}_\xi(\varepsilon_0,\mu_0) \neq \emptyset$. Next, we claim that H_ξ is l.s.c. at 0. Suppose to the contrary that H_ξ is not l.s.c. at 0, then there exist $\bar{x} \in H_\xi(0)$ and a neighborhood O_0 of

$0_X \in X$. For any neighborhood U of 0, there exists $\varepsilon \in U$ such that $(\bar{x} + O_0) \cap H_\xi(\varepsilon) = \emptyset$. In particular, there exists a nonnegative sequence $\{\varepsilon'_n\} \downarrow 0$ such that

$$(\bar{x} + O_0) \cap H_\xi(\varepsilon'_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Since $H_\xi(0) \neq \emptyset$, we choose $x^* \in H_\xi(0)$. Since $\varepsilon'_n \to 0$, there exists ε'_{n_0} such that

$$\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}x^* = \bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}(x^* - \bar{x}) \in \bar{x} + O_0. \tag{3.4}$$

We claim that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. In fact, since $\bar{x} \in H_\xi(0)$ and $x^* \in H_\xi(0)$, for any $y \in K(\mu_0)$, we have $\inf_{t\in F(\bar{x},y,\mu_0)} \xi(t) + \varepsilon_0 \geq 0$ and $\inf_{k\in F(x^*,y,\mu_0)} \xi(k) + \varepsilon_0 \geq 0$. Then, for any $u \in F(\bar{x}, y, \mu_0)$,

$$\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\xi(u) + \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 \geq 0, \tag{3.5}$$

and for any $v \in F(x^*, y, \mu_0)$,

$$\frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\xi(v) + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 \geq 0. \tag{3.6}$$

By the C-concavity of $F(\cdot, y, \mu_0)$, we have that

$$F\left(\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}x^*, y, \mu_0\right) \subseteq \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}F(\bar{x}, y, \mu_0) + \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}F(x^*, y, \mu_0) + C.$$

It follows that, for any $w \in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*, y, \mu_0)$, there exist $\bar{z} \in F(\bar{x}, y, \mu_0)$, $z^* \in F(x^*, y, \mu_0)$ and $c' \in C$ such that $w = \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{z} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}z^* + c'$. It follows from the linearity of ξ that $\xi(w) - \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z}) - \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*) = \xi(c') \geq 0$, which gives that $\xi(w) \geq \frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\xi(\bar{z}) + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}\xi(z^*)$. For all $w \in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*, y, \mu_0)$, by (3.5) and (3.6), we have

$$\xi(w) \geq -\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 - \frac{\varepsilon'_{n_0}}{\varepsilon_0 + \varepsilon'_{n_0}}\varepsilon_0 = -\frac{\varepsilon_0}{\varepsilon_0 + \varepsilon'_{n_0}}(\varepsilon'_{n_0} + \varepsilon_0) \geq -(\varepsilon'_{n_0} + \varepsilon_0).$$

This implies that $\inf_{z\in F(\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x}+\frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^*,y,\mu_0)} \xi(z) + \varepsilon'_{n_0} + \varepsilon_0 \geq 0$, that is, $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in H_\xi(\varepsilon'_{n_0})$. By (3.4), we get that $\frac{\varepsilon_0}{\varepsilon_0+\varepsilon'_{n_0}}\bar{x} + \frac{\varepsilon'_{n_0}}{\varepsilon_0+\varepsilon'_{n_0}}x^* \in (\bar{x} + O_0) \cap H_\xi(\varepsilon'_{n_0})$, which contradicts (3.3). Therefore, H_ξ is l.s.c. at 0. Since H_ξ is l.s.c. at 0, for above $x_0 \in \widetilde{S}_\xi(\varepsilon_0, \mu_0) = H_\xi(0)$ and for above W_1, there exists a balanced neighborhood V_0 of 0 such that $(x_0 + W_1) \cap H_\xi(\varepsilon) \neq \emptyset$, $\forall \varepsilon \in V_0$. In particular, from $\{\varepsilon_n\} \downarrow \varepsilon_0$, there exits $N_0 \in \mathbb{N}$ such that $(x_0 + W_1) \cap H_\xi(\varepsilon_{N_0} - \varepsilon_0) \neq \emptyset$. Let $x' \in (x_0 + W_1) \cap H_\xi(\varepsilon_{N_0} - \varepsilon_0)$.

For any $\bar{\varepsilon} > 0$, since $e \in \operatorname{int} C$, there exists $\delta_0 > 0$ such that

$$\delta_0 B_Y + \bar{\varepsilon} e \subseteq C. \tag{3.7}$$

Since $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$, for above $\delta_0 B_Y$, there exists a neighborhood V_1 of $0 \in B$, a neighborhood U_1 of $0 \in B$ and a neighborhood N_1 of $0 \in M$, for any $(x_1, y_1, \mu_1), (x_2, y_2, \mu_2) \in K(M) \times K(M) \times N(\mu_0)$ with $x_1 - x_2 \in V_1$, $y_1 - y_2 \in U_1$ and $\mu_1 - \mu_2 \in N_1$, we have

$$F(x_1, y_1, \mu_1) \subseteq \delta_0 B_Y + F(x_2, y_2, \mu_2). \tag{3.8}$$

Since K is H-u.s.c. at μ_0, for above U_1, there exists a neighborhood $U_1(\mu_0)$ of μ_0 such that

$$K(\mu) \subseteq K(\mu_0) + U_1, \quad \forall \mu \in U_1(\mu_0). \tag{3.9}$$

We see that $x' \in K(\mu_0)$. Since K is l.s.c. at μ_0, for $V_1 \cap W_1$, there exists a neighborhood $U_2(\mu_0)$ of μ_0 such that

$$\bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu) \neq \emptyset, \quad \forall \mu \in U_2(\mu_0). \tag{3.10}$$

It follows from $\mu_n \to \mu_0$ that there exists a positive integer $N_0' \geq N_0$ such that $\mu_{N_0'} \in U_1(\mu_0) \cap U_2(\mu_0) \cap U(\mu_0) \cap (\mu_0 + N_1)$. Noting that (3.9) and (3.10), we obtain

$$K(\mu_{N_0'}) \subseteq K(\mu_0) + U_1 \tag{3.11}$$

and

$$\bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu_{N_0'}) \neq \emptyset. \tag{3.12}$$

By (3.12), we choose

$$x'' \in \bigl(x' + V_1 \cap W_1\bigr) \cap K(\mu_{N_0'}). \tag{3.13}$$

Next, we prove that $x'' \in \widetilde{S}_\xi(\varepsilon_{N_0'}, \mu_{N_0'})$. For any $y' \in K(\mu_{N_0'})$, by (3.11), there exists $y_0 \in K(\mu_0)$ such that $y' - y_0 \in U_1$. It follows from (3.13) that $x'' - x' \in V_1$. Noting that $\mu_{N_0'} \in U(\mu_0) \cap (\mu_0 + N_1)$ and (3.8), we have

$$F\bigl(x'', y', \mu_{N_0'}\bigr) \subseteq \delta_0 B_Y + F\bigl(x', y_0, \mu_0\bigr).$$

By (3.7), we have

$$F\bigl(x'', y', \mu_{N_0'}\bigr) \subseteq C - \bar{\varepsilon} e + F\bigl(x', y_0, \mu_0\bigr). \tag{3.14}$$

Hence, for any $y \in K(\mu_{N_0'})$ and $z'' \in F(x'', y', \mu_{N_0'})$, there exist $c'' \in C$ and $z' \in F(x', y, \mu_0)$ such that

$$z'' = c'' - \bar{\varepsilon} e + z'.$$

It follows from the linearity of ξ that $\xi(z'') + \bar{\varepsilon} \geq \xi(z')$ for all $\bar{\varepsilon} > 0$. This leads to $\xi(z'') \geq \xi(z')$. Thus

$$\xi\bigl(z''\bigr) + \varepsilon_{N_0'} \geq \xi\bigl(z'\bigr) + \varepsilon_{N_0'} = \xi\bigl(z'\bigr) + (\varepsilon_{N_0'} - \varepsilon_0) + \varepsilon_0 \geq 0.$$

Hence $x'' \in \widetilde{S}_\xi(\varepsilon_{N'_0}, \mu_{N'_0})$. Also, since $x' \in (x_0 + W_1)$ and by (3.2) and (3.13), we have

$$x'' \in x' + V_1 \cap W_1 \subseteq x_0 + W_1 + W_1 \subseteq x_0 + W_0.$$

This means that $(x_0 + W_0) \cap \widetilde{S}_\xi(\varepsilon_{N'_0}, \mu_{N'_0}) \neq \emptyset$, which contradicts (3.1). This completes the proof. □

Theorem 3.3 *We assume that for any given $\xi \in B_e^*$, there exists $\delta > 0$ such that the approximate solution set $\widetilde{S}_\xi(\cdot,\cdot)$ exists in $[\varepsilon_0, \delta) \times N(\mu_0)$. Suppose that conditions* (i)-(iv) *as in Theorem* 3.2 *are satisfied. Assume further that for each $x \in K(\mu_0)$, $F(x, K(\mu_0), \mu_0) + C$ is a convex set. Then the approximate solution mapping $\widetilde{S} : [\varepsilon_0, \delta) \times N(\mu_0) \to 2^X$ is l.s.c. at (ε_0, μ_0).*

Proof Since $F(x, K(\mu_0), \mu_0) + C$ is a convex set for each $x \in K(\mu_0)$, by virtue of Lemma 3.1, it holds that $\widetilde{S}(\varepsilon_0, \mu_0) = \bigcup_{\xi \in B_e^*} \widetilde{S}_\xi(\varepsilon_0, \mu_0)$. It follows from Theorem 3.2 that for each $\xi \in B_e^*$, $\widetilde{S}_\xi(\cdot,\cdot)$ is l.s.c. at (ε_0, μ_0). Thus, in view of Lemma 2.4, we obtain that $\widetilde{S}(\cdot,\cdot)$ is l.s.c. at (ε_0, μ_0). □

The following example illustrates all of the assumptions in Theorem 3.3.

Example 3.4 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $Z = X = \mathbb{R}$. Let $B(0, \frac{1}{2})$ be the closed ball of radius 1/2 in $\mathbb{R}^2$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = \bigl(w(x, y, \mu), v(x, y, \mu)\bigr) + B(0, 1/2),$$

where $w(x, y, \mu) := y^2(2^\mu - 1) + x(y - x + 1) - 3y + 2$ and $v(x, y, \mu) := y^2(2^\mu - 1) - x^2 + 2xy + 3$. Define a set-valued mapping $K : M \to 2^X$ for all $\mu \in M$, by $K(\mu) := [-2 + \mu, 2 + \mu] \cap [-2, 2]$. We choose $e = (1, 1) \in \operatorname{int} C$, $\varepsilon_0 = 2.5$, $\mu_0 = 0$ and $\xi = (1, 0)$. We can see that $B^*_{(1,1)} = \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\}$ and $1 \in \widetilde{S}_{(1,0)}(\varepsilon_0, 0)$. Further, for any $\mu \in (-1, 1)$, there exists $\varepsilon \in [2.5, 4.5)$ such that $1 \in \widetilde{S}_{(1,0)}(\varepsilon, \mu)$. Hence, $\widetilde{S}_{(1,0)}(\cdot,\cdot)$ exists in $[2.5, 4.5) \times [-1, 1]$. It is easy to observe that for any $y \in K(0)$, $F(\cdot, y, 0)$ is C-concave on $K(0)$. Clearly, condition (ii) is true. It is obvious that $K(M) = [-2, 2]$. Let $N(\mu_0) = [-1, 1]$, we can see that $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $K(M) \times K(M) \times N(\mu_0)$. Finally, we can check that for each $x \in [-2, 2]$, $F(x, [-2, 2], 0) + C$ is a convex set. Applying Theorem 3.3, we obtain that $\widetilde{S}$ is l.s.c. at $(2.5, 0)$.

The following example illustrates that the concavity of F cannot be dropped.

Example 3.5 Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $Z = X = \mathbb{R}$. Let $B = [-2, 2]$, $M = [-1, 1]$ and the set-valued mapping $F : B \times B \times M \to 2^Y$ be defined by

$$F(x, y, \mu) = \bigl[\mu x(x - y) - 0.5, 2\bigr] \times \bigl\{x(x - y) - 0.5\bigr\}.$$

Define a set-valued mapping $K : M \to 2^X$ for all $\mu \in M$, by $K(\mu) := [0, 1]$. We choose $e = (1, 1) \in \operatorname{int} C$, $\varepsilon_0 = 0.5$, $\mu_0 = 0$. Then, all the assumptions of Theorem 3.3 are satisfied except (iii). Indeed, taking $y = 1$, $x_1 = 0$, $x_2 = 1$ and $t = 0.5$, we have

$$\begin{aligned}(-2.5,-0.25) &= (-0.5,-0.75) - 0.5(2,-0.5) - 0.5(2,-0.5) \\ &\in [-0.5,2] \times \{-0.75\} - 0.5\big([-0.5,2] \times \{-0.5\}\big) \\ &\quad - 0.5\big([-0.5,2] \times \{-0.5\}\big) \\ &\in F\big(0.5(0) + 0.5(1),1,0\big) - 0.5F(0,1,0) - 0.5F(1,1,0) \\ &= F(0.5,1,0) - 0.5F(0,1,0) - 0.5F(1,1,0),\end{aligned}$$

but $(-2.5,-0.25) \notin C$. The direct computation shows that

$$\tilde{S}(\varepsilon_0,\mu) = \begin{cases} \{0,1\} & \text{if } \mu \in (0,1], \\ [0,1] & \text{if } \mu = 0, \\ \{0\} & \text{if } \mu \in [-1,0). \end{cases} \tag{3.15}$$

Clearly, we see that $\tilde{S}(\cdot,\cdot)$ is even not l.s.c. at (ε_0,μ_0) since $F(\cdot,y,\mu_0)$ is not C-concave on $K(\mu_0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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