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Generalized Ulam-Hyers stability and well-posedness for fixed point equation via α-admissibility

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Dedicated to Professor SS Chang on the occasion of his 80th birthday.

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Abstract
In this paper, we extend the concept of contraction mappings in b-metric spaces and utilize this concept to prove the existence and uniqueness of fixed point theorems for such mappings in such a space. We also prove the generalized Ulam-Hyers stability and well-posed results for a fixed point equation employing the concept of α-admissibility in b-metric spaces. We shall construct some examples to support our novel results.

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1 Introduction

The classical Banach contraction principle is a very important tool in solving existence problems in many branches of mathematics. Over the years, it has been generalized in several different directions by several mathematicians (see [1–7]). In 1993, Czerwik [8] introduced and proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in such spaces. Subsequently several other authors [9–15] have studied and established the existence of fixed points of a contractive mapping in b-metric spaces.

The study of stability problems for various functional equations play the most important role in mathematical analysis. In the fall of 1940, Ulam [16] discussed a number of important unsolved mathematical problems. Among them, a question concerning the stability of homomorphisms seemed too abstract for anyone to reach any conclusion. In the following year, Hyers [17] gave a first affirmative partial answer to Ulam's question for Banach spaces, this type of stability is called Ulam-Hyers stability. A large number of papers have been published in connection with various generalizations of Ulam-Hyers stability results in fixed point theory and remarkable result on the stability of certain classes of functional equations via fixed point approach (see [18–29] and references therein).

On the other hand, Samet *et al.* [30] introduced the concepts of α-ψ-contractive mapping and α-admissible self-mappings. Also, they proved some fixed point results for such mappings in complete metric spaces. Naturally, many authors have started to investigate the existence of a fixed point theorem via α-admissible mappings for single valued and multivalued mappings (see [31–38]). Recently Bota *et al.* [39] considered the existence and the uniqueness of fixed point theorems and generalized Ulam-Hyers stability results via α-admissible mappings in b-metric spaces.

In this paper, we extend the concept of α-ψ-contractive mapping in b-metric spaces. By using this concept, we establish the existence and uniqueness of fixed point for some new types of contractive mappings in b-metric spaces and give an example to illustrate our main results. Moreover, we study and prove the generalized Ulam-Hyers stability and well-posed results by using fixed point method via α-admissible mappings in b-metric spaces.

2 Preliminaries

Throughout this paper, we shall use the following notation.

Definition 2.1 ([40, 41]) Let X be a nonempty set and the functional $d : X \times X \to [0,\infty)$ satisfy:

(b1) $d(x,y) = 0$ if and only if $x = y$,
(b2) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(b3) there exists a real number $s \geq 1$ such that $d(x,z) \leq s[d(x,y) + d(y,z)]$, for all $x,y,z \in X$.

Then d is called a b-metric on X and a pair (X,d) is called a b-metric space with coefficient s.

Remark 2.2 If we take $s = 1$ in above definition then b-metric spaces turns into usual metric spaces. Hence, the class of b-metric spaces is larger than the class of usual metric spaces.

Examples of b-metric spaces were given in [8, 40–43].

Example 2.3 The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the functional $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to [0,\infty)$,

$$d(x,y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$$

(where $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$) is a b-metric spaces with coefficient $s = 2^{\frac{1}{p}} > 1$. Notice that the above result holds for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach spaces.

Example 2.4 Let X be a set with the cardinal $\operatorname{card}(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $\operatorname{card}(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then the functional $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) := \begin{cases} 0, & x = y, \\ 2s, & x,y \in X_1, \\ 1, & \text{otherwise}, \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

Definition 2.5 ([42]) Let (X, d) be a b-metric spaces. Then a sequence $\{x_n\}$ in X is called

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$;
(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Lemma 2.6 ([41]) *Let (X, d) be a b-metric spaces and let $\{x_k\}_{k=0}^{n} \subset X$. Then*

$$d(x_0, x_n) \leq sd(x_0, x_1) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n).$$

Definition 2.7 ([21]) A mapping $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\psi^n(t) \to 0$ as $n \to \infty$, for any $t \in [0, \infty)$.

Lemma 2.8 ([21, 44]) *If $\psi : [0, \infty) \to [0, \infty)$ is a comparison function, then*

(1) *ψ^n is also a comparison function, where ψ^n is nth iterate of ψ;*
(2) *ψ is continuous at 0;*
(3) *$\psi(t) < t$, for any $t > 0$.*

The concept of (c)-comparison function was introduced by Berinde [44] in the following definition.

Definition 2.9 A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(1) ψ is increasing;
(2) there exist $n_0 \in \mathbb{N}$, $k \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} \epsilon_n$ such that $\psi^{n+1}(t) \leq k\psi^n(t) + \epsilon_n$, for $n \geq n_0$ and any $t \in [0, \infty)$.

Here we recall the definitions of the following class of (b)-comparison function as given by Berinde [45] in order to extend some fixed point results to the class of b-metric spaces.

Definition 2.10 ([45]) Let $s \geq 1$ be a real number. A mapping $\psi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

(1) ψ is increasing;
(2) there exist $n_0 \in \mathbb{N}$, $k \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} \epsilon_n$ such that $s^{n+1}\psi^{n+1}(t) \leq ks^n\psi^n(t) + \epsilon_n$, for $n \geq n_0$ and any $t \in [0, \infty)$.

In this work, we denote by Ψ_b the class of (b)-comparison function $\psi : [0, \infty) \to [0, \infty)$. It is evident that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

Lemma 2.11 ([43]) *If $\psi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following*:

(i) *the series $\sum_{n=0}^{\infty} s^n\psi^n(t)$ converges for any $t \in [0, \infty)$;*
(ii) *the function $S : [0, \infty) \to [0, \infty)$, defined by $S(t) = \sum_{n=0}^{\infty} s^n\psi^n(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.*

Next, we will present the concept of α-admissible mappings introduced by Samet *et al.* [30].

Definition 2.12 ([30]) Let X be a nonempty set, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that f is an α-admissible mapping if it satisfies the following condition:

$$\text{for } x, y \in X \text{ for which } \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha\bigl(f(x), f(y)\bigr) \geq 1.$$

Example 2.13 Let $X = (0, \infty)$. Define $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$f(x) = \sinh(x) \quad \text{for all } x \in X$$

and

$$\alpha(x,y) = \begin{cases} \frac{x+y+1}{|x-y|+1}, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Then f is α-admissible.

Example 2.14 Let $X = \mathbb{R}$. Define $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$f(x) = \begin{cases} \cosh(2x+1), & \text{if } x > 1, \\ 1 - \frac{1}{2^{|x|}}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then f is α-admissible.

3 Fixed point theorems for α-admissible mapping in b-metric spaces

In this section, we prove the existence and uniqueness of fixed point theorems in a b-metric space.

Definition 3.1 Let (X, d) be a b-metric space with coefficient s. A mapping $f : X \to X$ is said to be a generalized α-ψ-contraction in b-metric space if there exist functions $\psi \in \Psi_b$ and $\alpha : X \times X \to [0, \infty)$ such that the following condition holds:

$$\text{for } x, y \in X \text{ with } \alpha\big(x, f(x)\big)\alpha\big(y, f(y)\big) \geq 1 \quad \Longrightarrow \quad d\big(f(x), f(y)\big) \leq \psi\big(d(x,y)\big). \tag{3.1}$$

Theorem 3.2 *Let (X, d) be a complete b-metric space with coefficient s and f be a generalized α-ψ-contraction. Suppose that the following conditions hold*:

(a) *f is an α-admissible*;
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$*;
(c) *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, f(x_n)) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x, f(x)) \geq 1$.*

Then f has a unique fixed point x^* in X such that $\alpha(x^*, f(x^*)) \geq 1$.

Proof Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ (from condition (b)). We define the sequence $\{x_n\}$ in X such that

$$x_n = f(x_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

Since f is an α-admissible and

$$\alpha(x_0, x_1) = \alpha\bigl(x_0, f(x_0)\bigr) \geq 1 \tag{3.2}$$

we deduce that

$$\alpha\bigl(x_1, f(x_1)\bigr) = \alpha\bigl(f(x_0), f(x_1)\bigr) \geq 1. \tag{3.3}$$

By continuing this process, we get $\alpha(x_{n-1}, f(x_{n-1})) \geq 1$ for all $n \in \mathbb{N}$. This implies that

$$\alpha\bigl(x_{n-1}, f(x_{n-1})\bigr)\alpha\bigl(x_n, f(x_n)\bigr) \geq 1$$

for all $n \in \mathbb{N}$. From (3.1), we obtain

$$d(x_n, x_{n+1}) = d\bigl(f(x_{n-1}), f(x_n)\bigr) \leq \psi\bigl(d(x_{n-1}, x_n)\bigr)$$

for all $n \in \mathbb{N}$. By repeating the above process, we get

$$d(x_n, x_{n+1}) \leq \psi^n\bigl(d(x_0, x_1)\bigr)$$

for all $n \in \mathbb{N}$. Next, we show that $\{x_n\}$ is a Cauchy sequence in X. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n-2} d(x_{m-3}, x_{m-2}) \\ &\quad + s^{m-n-1} d(x_{m-2}, x_{m-1}) + s^{m-n} d(x_{m-1}, x_m) \\ &\leq s\psi^n\bigl(d(x_0, x_1)\bigr) + s^2\psi^{n+1}\bigl(d(x_0, x_1)\bigr) + \cdots + s^{m-n-2}\psi^{m-3}\bigl(d(x_0, x_1)\bigr) \\ &\quad + s^{m-n-1}\psi^{m-2}\bigl(d(x_0, x_1)\bigr) + s^{m-n}\psi^{m-1}\bigl(d(x_0, x_1)\bigr) \\ &= \frac{1}{s^{n-1}}\bigl[s^n\psi^n\bigl(d(x_0, x_1)\bigr) + s^{n+1}\psi^{n+1}\bigl(d(x_0, x_1)\bigr) + \cdots + s^{m-2}\psi^{m-2}\bigl(d(x_0, x_1)\bigr) \\ &\quad + s^{m-1}\psi^{m-1}\bigl(d(x_0, x_1)\bigr)\bigr]. \end{aligned}$$

Denote $S_n = \sum_{i=0}^{n} s^i\psi^i(d(x_0, x_1))$ for all $n \in \mathbb{N}$. This implies that

$$d(x_n, x_m) \leq \frac{1}{s^{n-1}}[S_{m-1} - S_{n-1}] \quad \text{for all } n, m \in \mathbb{N}.$$

By Lemma 2.11 we know that the series $\sum_{i=0}^{\infty} s^i\psi^i(d(x_0, x_1))$ converges. Therefore, $\{x_n\}$ is Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Using condition (c), we get $\alpha(x^*, f(x^*)) \geq 1$. Also, we have $\alpha(x_{n-1}, f(x_{n-1}))\alpha(x^*, f(x^*)) \geq 1$ for all $n \in \mathbb{N}$. From the assumption (3.1), we have

$$\begin{aligned} d\bigl(f(x^*), x^*\bigr) &\leq s\bigl[d\bigl(f(x^*), x_n\bigr) + d\bigl(x_n, x^*\bigr)\bigr] \\ &= s\bigl[d\bigl(f(x^*), f(x_{n-1})\bigr) + d\bigl(x_n, x^*\bigr)\bigr] \\ &\leq s\bigl[\psi\bigl(d(x^*, x_{n-1})\bigr) + d\bigl(x_n, x^*\bigr)\bigr]. \end{aligned}$$

Letting $n \to \infty$, it follows that $d(f(x^*), x^*) = 0$, that is, x^* is a fixed point of f such that $\alpha(x^*, f(x^*)) \geq 1$.

Next, we prove the uniqueness of the fixed point of f. Let y^* be another fixed point of f such that

$$\alpha\bigl(y^*, f\bigl(y^*\bigr)\bigr) \geq 1.$$

Therefore, we get

$$\alpha\bigl(x^*, f\bigl(x^*\bigr)\bigr)\alpha\bigl(y^*, f\bigl(y^*\bigr)\bigr) \geq 1.$$

It follows that

$$d\bigl(x^*, y^*\bigr) = d\bigl(f\bigl(x^*\bigr), f\bigl(y^*\bigr)\bigr) < d\bigl(x^*, y^*\bigr),$$

which is a contradiction. Therefore, x^* is the unique fixed point of f such that $\alpha(x^*, f(x^*)) \geq 1$. This completes the proof. □

In view of Theorem 3.2, we have the following corollary.

Corollary 3.3 *Let (X, d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0, \infty)$, and $\psi \in \Psi_b$ be three mappings. Suppose that the following conditions hold*:

(a) *f is an α-admissible*;

(b) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$*;

(c) *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, f(x_n)) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x, f(x)) \geq 1$*;

(d) *f satisfies the following condition*:

$$\alpha\bigl(x, f(x)\bigr)\alpha\bigl(y, f(y)\bigr)d\bigl(f(x), f(y)\bigr) \leq \psi\bigl(d(x, y)\bigr) \tag{3.4}$$

for all $x, y \in X$.

Then f has a unique fixed point x^* in X such that $\alpha(x^*, f(x^*)) \geq 1$.

Corollary 3.4 *Let (X, d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0, \infty)$, and $\psi \in \Psi_b$ be three mappings. Suppose that the following conditions hold*:

(a) *f is an α-admissible*;

(b) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$*;

(c) *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, f(x_n)) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x, f(x)) \geq 1$*;

(d) *f satisfies the following condition*:

$$\bigl[d\bigl(f(x), f(y)\bigr) + \xi\bigr]^{\alpha(x, f(x))\alpha(y, f(y))} \leq \psi\bigl(d(x, y)\bigr) + \frac{\xi}{s} \tag{3.5}$$

for all $x, y \in X$, where $\xi \geq 1$.

Then f has a unique fixed point x^* in X such that $\alpha(x^*, f(x^*)) \geq 1$.

Corollary 3.5 *Let (X,d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0,\infty)$, and $\psi \in \Psi_b$ be three mappings. Suppose that the following conditions hold*:

(a) *f is an α-admissible*;
(b) *there exists $x_0 \in X$ such that $\alpha(x_0,f(x_0)) \geq 1$*;
(c) *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,f(x_n)) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x,f(x)) \geq 1$*;
(d) *f satisfies the following condition*:

$$\bigl(\alpha\bigl(x,f(x)\bigr)\alpha\bigl(y,f(y)\bigr) - 1 + \xi\bigr)^{d(f(x),f(y))} \leq \xi^{\psi(d(x,y))} \tag{3.6}$$

for all $x,y \in X$, where $\xi > 1$.

Then f has a unique fixed point x^* in X such that $\alpha(x^*,f(x^*)) \geq 1$.

If we set $\alpha(x,y) = 1$ for all $x,y \in X$ in Theorem 3.2, we get the following results.

Corollary 3.6 *Let (X,d) be a complete b-metric space with coefficient s and $f : X \to X$ be a mapping. Suppose that f satisfies*

$$d\bigl(f(x),f(y)\bigr) \leq \psi\bigl(d(x,y)\bigr) \tag{3.7}$$

for all $x,y \in X$, where $\psi \in \Psi_b$. Then f has a unique fixed point in X.

If the coefficient $s = 1$ in Corollary 3.6, we immediately get the following result.

Corollary 3.7 [46] *Let (X,d) be a complete metric space and $\psi : [0,\infty) \to [0,\infty)$ be (c)-comparison function. Suppose that $f : X \to X$ be a mapping satisfies*

$$d\bigl(f(x),f(y)\bigr) \leq \psi\bigl(d(x,y)\bigr) \tag{3.8}$$

for all $x,y \in X$. Then f has a unique fixed point in X.

Remark 3.8 If $\psi(t) = kt$, where $k \in (0,1)$ in Corollary 3.7, we get the Banach contraction principle.

Next, we give an example showing that the contractive conditions in our results are independent. Also, our results are real generalizations of the Banach contraction principle in b-metric spaces and several results in literature.

Example 3.9 Let $X = [0,\infty)$ and $d(x,y) = |x-y|^2$ for all $x,y \in X$. Then d is a complete b-metric space on X with coefficient $s = 2$. Define $f : X \to X$ by

$$f(x) = \begin{cases} 0.2, & x = 0, \\ \frac{4x}{\cosh x}, & x \in (0,1), \\ \frac{x+1}{2}, & [1,\infty). \end{cases}$$

Also, define $\alpha : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 0, & x,y \in [0,1), \\ 1, & x,y \in [1,\infty), \end{cases}$$

and $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$.

Now, we show that f is a generalized α-ψ-contraction mapping. For $x,y \in X$ with

$$\alpha\big(x,f(x)\big)\alpha\big(y,f(y)\big) \geq 1,$$

we get $x,y \in [1,\infty)$. Then we have

$$\begin{aligned} d\big(f(x),f(y)\big) &= \left|\frac{x+1}{2} - \frac{y+1}{2}\right|^2 \\ &= \frac{1}{4}|x-y|^2 \\ &\leq \frac{1}{2}d(x,y) \\ &= \psi\big(d(x,y)\big). \end{aligned}$$

It is easy to see that f is an α-admissible mapping. There exists $x_0 = 2 \in X$ such that

$$\alpha\big(x_0,f(x_0)\big) = \alpha\big(2,f(2)\big) = \alpha(2,1.5) = 2 \geq 1.$$

Also, we can easily to prove that condition (c) in Theorem 3.2 holds. Therefore, all of conditions in Theorem 3.2 hold. In this example, we have 1 is a unique fixed point of f and $\alpha(1,f(1)) \geq 1$.

Remark 3.10 We observe that the contractive condition in Corollary 3.4 cannot be applied to this example. Indeed, for $x = 1$ and $y = 2$, we obtain

$$\big[d\big(f(x),f(y)\big) + \xi\big]^{\alpha(x,f(x))\alpha(y,f(y))} > \psi\big(d(x,y)\big) + \frac{\xi}{s},$$

where $\xi = 1$ and $s = 2$. Therefore, Corollary 3.4 cannot be applied to this case. Also, by a similar method, we can show that Corollary 3.5 cannot be applied to this case.

Also, we can see that the fixed point result for Banach contraction principle in b-metric spaces cannot be applied to this case. Indeed, for $x = 0.4$ and $y = 0.5$, we get

$$d\big(f(x),f(y)\big) = \left|\frac{4(0.4)}{\cosh 0.4} - \frac{4(0.5)}{\cosh 0.5}\right|^2 > 0.07 > 0.01 = |0.4 - 0.5|^2 = d(x,y).$$

4 The generalized Ulam-Hyers stability in *b*-metric spaces

In this section, we prove the generalized Ulam-Hyers stability in b-metric spaces for which Theorem 3.2 holds.

Let (X,d) be a b-metric spaces with coefficient s and $f : X \to X$ be an operator. Let us consider the fixed point equation

$$x = f(x), \quad x \in X \tag{4.1}$$

and the inequality

$$d(v,f(v)) \leq \varepsilon, \quad \text{where } \varepsilon > 0. \tag{4.2}$$

Theorem 4.1 *Let (X,d) be a complete b-metric space with coefficient s. Suppose that all the hypotheses of Theorem* 3.2 *hold and also that the function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) := t - s\psi(t)$ is strictly increasing and onto. If $\alpha(u^*,f(u^*)) \geq 1$ for all $u^* \in X$ which is an ε-solution, then the fixed point equation* (4.1) *is generalized Ulam-Hyers stable.*

Proof By Theorem 3.2, we have $f(x^*) = x^*$, that is, $x^* \in X$ is a solution of the fixed point equation (4.1). Let $\varepsilon > 0$ and $y^* \in X$ is an ε-solution, that is,

$$d(y^*,f(y^*)) \leq \varepsilon.$$

Since $x^*, y^* \in X$ are ε-solution, we have

$$\alpha(x^*,f(x^*)) \geq 1 \quad \text{and} \quad \alpha(y^*,f(y^*)) \geq 1.$$

Also, we have

$$\alpha(x^*,f(x^*))\alpha(y^*,f(y^*)) \geq 1.$$

Now, we obtain

$$\begin{aligned} d(x^*,y^*) &= d(f(x^*),y^*) \\ &\leq s[d(f(x^*),f(y^*)) + d(f(y^*),y^*)] \\ &\leq s[\psi(d(x^*,y^*)) + d(f(y^*),y^*)] \\ &\leq s\psi(d(x^*,y^*)) + s\varepsilon. \end{aligned}$$

It follows that

$$d(x^*,y^*) - s(\psi(d(x^*,y^*))) \leq s\varepsilon.$$

Since $\varphi(t) := t - s\psi(t)$, we have

$$\varphi(d(x^*,y^*)) = d(x^*,y^*) - s\psi(d(x^*,y^*)).$$

It implies that

$$d(x^*,y^*) \leq \varphi^{-1}(s\varepsilon).$$

Notice that $\varphi^{-1} : [0,\infty) \to [0,\infty)$ exists, is increasing, continuous at 0 and $\varphi^{-1}(0) = 0$. Therefore, the fixed point equation (4.1) is generalized Ulam-Hyers stable. □

Corollary 4.2 *Let (X,d) be a complete b-metric space with coefficient s. Suppose that all the hypotheses of Corollary* 3.3 *hold and also that the function $\varphi : [0,\infty) \to [0,\infty)$ defined* *by $\varphi(t) := t - s\psi(t)$ is strictly increasing and onto. If $\alpha(u^*, f(u^*)) \geq 1$ for all $u^* \in X$ which is an ε-solution, then the fixed point equation* (4.1) *is generalized Ulam-Hyers stable.*

Corollary 4.3 *Let (X, d) be a complete b-metric space with coefficient s. Suppose that all the hypotheses of Corollary* 3.4 *hold and also that the function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) := t - s\psi(t)$ is strictly increasing and onto. If $\alpha(u^*, f(u^*)) \geq 1$ for all $u^* \in X$ which is an ε-solution, then the fixed point equation* (4.1) *is generalized Ulam-Hyers stable.*

Corollary 4.4 *Let (X, d) be a complete b-metric space with coefficient s. Suppose that all the hypotheses of Corollary* 3.5 *hold and also that the function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) := t - s\psi(t)$ is strictly increasing and onto. If $\alpha(u^*, f(u^*)) \geq 1$ for all $u^* \in X$ which is an ε-solution, then the fixed point equation* (4.1) *is generalized Ulam-Hyers stable.*

5 Well-posedness of a function with respect to α-admissibility in b-metric spaces

In this section, we present and prove well-posedness of a function with respect to an α-admissible mapping in b-metric spaces.

Definition 5.1 Let (X, d) be a complete b-metric spaces with coefficient s and $f : X \to X$, $\alpha : X \times X \to [0,\infty)$. The fixed point problem of f is said to be well-posed with respect to α if

(i) f has a unique fixed point x^* in X such that $\alpha(x^*, f(x^*)) \geq 1$;
(ii) for sequence $\{x_n\}$ in X such that $d(x_n, f(x_n)) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.

In the following next theorems, we add a new condition to assure the well-posedness via α-admissibility.

(S) If $\{x_n\}$ is sequence in X such that $d(x_n, f(x_n)) \to 0$, as $n \to \infty$, then $\alpha(x_n, f(x_n)) \geq 1$ for all $n \in \mathbb{N}$.

Theorem 5.2 *Let (X, d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0,\infty)$, and $\psi \in \Psi_b$. Suppose that all the hypotheses of Theorem* 3.2 *and condition* (S) *hold. Then the fixed point equation* (4.1) *is well-posed with respect to α.*

Proof By Theorem 3.2, there unique exists $x^* \in X$ such that $f(x^*) = x^*$ and $\alpha(x^*, f(x^*)) \geq 1$. Let $\{x_n\}$ be sequence in X such that $d(x_n, f(x_n)) \to 0$, as $n \to \infty$. By condition (S), we get

$$\alpha\bigl(x_n, f(x_n)\bigr) \geq 1.$$

Also, we get

$$\alpha\bigl(x_n, f(x_n)\bigr)\alpha\bigl(x^*, f\bigl(x^*\bigr)\bigr) \geq 1.$$

Now, we have

$$\begin{aligned} d\bigl(x_n, x^*\bigr) &= d\bigl(x_n, f\bigl(x^*\bigr)\bigr) \\ &\leq s\bigl[d\bigl(x_n, f(x_n)\bigr) + d\bigl(f(x_n), f\bigl(x^*\bigr)\bigr)\bigr] \\ &\leq s\bigl[\psi\bigl(d\bigl(x_n, x^*\bigr)\bigr) + d\bigl(x_n, f(x_n)\bigr)\bigr]. \end{aligned}$$

ψ is continuous at 0 and $d(x_n,f(x_n)) \to 0$ as $n \to \infty$. It implies that $d(x_n,x^*) \to 0$ as $n \to \infty$, that is, $x_n \to x^*$, as $n \to \infty$. Therefore, the fixed point equation (4.1) is well-posed with respect to α. □

Corollary 5.3 *Let (X,d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0,\infty)$, and $\psi \in \Psi_b$. Suppose that all the hypotheses of Corollary* 3.3 *and condition* (S) *hold. Then the fixed point equation* (4.1) *is well-posed with respect to α.*

Corollary 5.4 *Let (X,d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0,\infty)$, and $\psi \in \Psi_b$. Suppose that all the hypotheses of Corollary* 3.4 *and condition* (S) *hold. Then the fixed point equation* (4.1) *is well-posed with respect to α.*

Corollary 5.5 *Let (X,d) be a complete b-metric space with coefficient s, $f : X \to X$, $\alpha : X \times X \to [0,\infty)$, and $\psi \in \Psi_b$. Suppose that all the hypotheses of Corollary* 3.5 *and condition* (S) *hold. Then the fixed point equation* (4.1) *is well-posed with respect to α.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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