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Oscillation theorems for second order nonlinear neutral difference equations

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Abstract

This paper deals with the oscillation of second order neutral difference equations of the form

$$\Delta(a_n(\Delta(x_n + p_n x_{\tau(n)}))^{\alpha}) + q_n x^{\beta}_{\sigma(n)} = 0. \qquad (E)$$

The oscillation of all solutions of this equation is established via comparison theorems. Examples are provided to illustrate the main results.

MSC: 39A10

Keywords: neutral difference equation; second order; comparison theorems; oscillation

1 Introduction

Consider the second order nonlinear neutral delay difference equation of the form

$$\Delta\big(a_n\big(\Delta(x_n + p_n x_{\tau(n)})\big)^{\alpha}\big) + q_n x^{\beta}_{\sigma(n)} = 0, \quad n \ge n_0 \in \mathbb{N}, \qquad (1.1)$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, subject to the following hypotheses:

(H$_1$) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences with $\{q_n\}$ not identically zero for infinitely many values of n;

(H$_2$) $\{a_n\}$ is a positive sequence such that $R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \to \infty$ as $n \to \infty$;

(H$_3$) α and β are ratio of odd positive integers;

(H$_4$) $\{\tau(n)\}$ and $\{\sigma(n)\}$ are nondecreasing sequences of integers such that $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty$ and $\tau \circ \sigma = \sigma \circ \tau$;

(H$_5$) there is a positive constant p such that $0 \le p_n \le p < \infty$.

By a solution of equation (1.1) we mean a real sequence $\{x_n\}$ defined, and satisfying equation (1.1), for all $n \ge n_0$. We consider only those solutions $\{x_n\}$ of equation (1.1) which satisfy $\sup\{|x_n| : n \ge N\} > 0$ for all $N \ge n_0$. We assume that equation (1.1) possesses such a solution. A solution $\{x_n\}$ of equation (1.1) is oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Since the second order equations have applications in various problems in physics, biology, and economics, there is a permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of solutions of various types of second order equations; see for example [1–12], and the references cited therein.

In [3, 4] the authors proved that $0 \leq p_n < 1$ together with

$$\sum_{n=n_0}^{\infty} q_n(1 - p_{n-\sigma}) = \infty$$

guarantees the oscillation of all solutions of the neutral difference equation

$$\triangle^2(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} = 0. \tag{1.2}$$

In [13], the author discussed the oscillatory behavior of all solutions of the difference equation

$$\triangle\bigl(a_n \triangle(x_n + p_n x_{\tau(n)})\bigr) + q_n f(x_{\sigma(n)}) = 0 \tag{1.3}$$

with

$$\frac{f(x)}{x} \geq \mu > 0 \quad \text{for } x \neq 0, \qquad 0 \leq p_n < 1 \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty.$$

In [14] and [15], the author considered the equation

$$\triangle\bigl(a_n\bigl(\triangle(x_n + p_n x_{\tau(n)})\bigr)^{\alpha}\bigr) + f(n, x_{\sigma(n)}) = 0 \tag{1.4}$$

with $\frac{f(n,x)}{x^{\alpha}} \geq cq_n$ for $x \neq 0$ and studied the oscillatory behavior under the conditions $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$ and $0 \leq p_n < 1$. In [16], Saker considered the difference equation

$$\triangle\bigl(a_n\bigl(\triangle(x_n + p_n x_{n-\tau})\bigr)^{\alpha}\bigr) + f(n, x_{n-\sigma}) = 0 \tag{1.5}$$

with

$$f(n,x) \geq q_n x_n^{\alpha}, \qquad 0 \leq p_n < 1 \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$$

and presented oscillation criteria which improved the existing results for the equation.

In [17], the authors established some sufficient conditions for the oscillation of all solutions of equation (1.3) via comparison theorems. In [18], Sun and Saker considered equation (1.5) and obtained some new oscillation criteria, which improve and complement that given in [16].

Recently in [19, 20], the authors studied the oscillatory behavior of all solutions of equation (1.1) under the conditions $0 \leq p_n \leq p < 1$ and $\alpha > \beta$ or $\alpha < \beta$ or $\alpha = \beta$.

Motivated by the above observation in this paper we shall investigate the oscillatory properties of equation (1.1) without assuming any usual restrictions on $\{p_n\}$, $\{\sigma(n)\}$, and $\{\tau(n)\}$. We shall provide new comparison theorems in which we compare the second order equation (1.1) with the first order difference equations in the sense that the oscillation nature of these first order difference equations yields the oscillation of equation (1.1). In

Section 2, we present some basic lemmas, and in Section 3 we establish oscillation results. In Section 4, we provide several examples to illustrate the main results and we present our conclusion in Section 5.

2 Some basic lemmas

In this section we present some basic lemmas which will be used to prove the main results.

Lemma 2.1 *Let* $A \geq 0$, $B \geq 0$, *and* $\alpha \geq 1$. *Then*

$$(A+B)^\alpha \leq 2^{\alpha-1}\bigl(A^\alpha + B^\alpha\bigr). \tag{2.1}$$

Proof If $A = 0$ or $B = 0$, then the result is true. Next, we assume that $0 < A \leq B$. Consider the function $f(x)$ defined by $f(x) = x^\alpha$. Then

$$f''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$$

for $x > 0$. Therefore, $f(x)$ is convex and $f(\frac{A+B}{2}) \leq \frac{f(A)+f(B)}{2}$, which implies the result (2.1). □

Lemma 2.2 *Let* $A \geq 0$, $B \geq 0$, $0 < \alpha \leq 1$. *Then*

$$(A+B)^\alpha \leq A^\alpha + B^\alpha. \tag{2.2}$$

Proof If $A = 0$ or $B = 0$, then the result is true. For $A \neq 0$, let $x = \frac{B}{A}$. Then the result (2.2) takes the form $(1+x)^\alpha \leq 1 + x^\alpha$, which is clearly true for all $x > 0$. □

Next, we present the structure of positive solutions of equation (1.1) since the opposite case is similar.

Lemma 2.3 *If* $\{x_n\}$ *is a positive solution of equation* (1.1), *then* $z_n = x_n + p_n x_{\tau(n)}$ *satisfies*

$$z_n > 0, \qquad \triangle z_n > 0, \qquad \triangle\bigl(a_n(\triangle z_n)^\alpha\bigr) < 0 \tag{2.3}$$

eventually.

Proof Assume that $x_n > 0$ is a solution of equation (1.1). Then equation (1.1) implies

$$\triangle\bigl(a_n(\triangle z_n)^\alpha\bigr) = -q_n x^\beta_{\sigma(n)} < 0.$$

Therefore, $a_n(\triangle z_n)^\alpha$ is decreasing and thus either $\triangle z_n > 0$ or $\triangle z_n < 0$ eventually for $n \geq n_1 \in \mathbb{N}$. If $\triangle z_n < 0$, then there exists a negative constant c such that

$$\triangle z_n \leq \frac{c}{a_n^{\frac{1}{\alpha}}} < 0.$$

Summing the above inequality from n_1 to $n-1$, one obtains

$$z_n \leq z_{n_1} + c\sum_{s=n_1}^{n-1}\frac{1}{a_s^{\frac{1}{\alpha}}} \to -\infty \quad \text{as } n \to \infty.$$

This contradiction proves (2.3). □

The proof of the following lemma is found in [21].

Lemma 2.4 *Let $\gamma > 1$ be a quotient of odd positive integers. Assume that k is a positive integer, $\{d_n\}$ is a positive sequence defined for all $n \geq n_0 \in \mathbb{N}$, and there exists $\lambda > \frac{1}{k}\log\gamma$ such that*

$$\lim_{n\to\infty}\inf\bigl[d_n\exp\bigl(-e^{\lambda n}\bigr)\bigr] > 0, \tag{2.4}$$

then all the solutions of the difference equation

$$\triangle y_n + d_n y_{n-k}^{\gamma} = 0$$

are oscillatory.

Next we state a result given in [22].

Lemma 2.5 *Assume that $\{\phi_n\}$ is a nonnegative real sequence for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty}\inf\sum_{j=n-l}^{n-1}\phi_j > \left(\frac{l}{l+1}\right)^{l+1},$$

where l is a positive integer, then

$$\triangle w_n + \phi_n w_{n-l} \leq 0$$

has no eventually positive solution.

Lemma 2.6 *Let γ be such that $0 < \gamma \leq 1$ be a quotient of odd positive integers and $\sigma(n) = n-k$ where k is a positive integer. Assume that $\{d_n\}$ is a positive real sequence defined for all $n \geq n_0 \in \mathbb{N}$. If*

$$\lim_{n\to\infty}\inf\sum_{s=n-k}^{n-1}d_s > \left(\frac{k}{k+1}\right)^{k+1}, \tag{2.5}$$

then every solution of the first order delay difference equation

$$\triangle y_n + d_n y_{n-k}^{\gamma} = 0 \tag{2.6}$$

is oscillatory.

Proof Assume that $\{y_n\}$ is a positive solution of equation (2.6). First, observe that condition (2.5) implies

$$\sum_{n=n_1}^{\infty} d_n = \infty. \tag{2.7}$$

Since $\{y_n\}$ is decreasing, there exists l such that $\lim_{n\to\infty} y_n = l \geq 0$. If $l > 0$, then summing the equation (2.6) from n_1 to $n-1$, we obtain

$$y_{n_1} \geq \sum_{s=n_1}^{n-1} d_s y_{s-k}^{\gamma} \geq l^{\gamma} \sum_{s=n_1}^{n-1} d_s \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction, and therefore, we conclude that $\lim_{n\to\infty} y_n = 0$, and also $0 < y_n < 1$, eventually. Therefore, $y_{n-k}^{\gamma} \geq y_{n-k}$. Substituting this into the equation (2.6) we deduce that $\{y_n\}$ is a positive solution of the difference inequality

$$\triangle y_n + d_n y_{n-k} \leq 0. \tag{2.8}$$

But this contradicts Lemma 2.5 according to which condition (2.5) ensures that (2.8) has no positive solution. The proof is now complete. □

We conclude this section with the following lemma.

Lemma 2.7 *Assume that the difference inequality*

$$\triangle w_n + \phi_n w_{n-k}^{\gamma} \leq 0 \tag{2.9}$$

has an eventually positive solution, where $\phi_n \geq 0$, ϕ_n *is not identically zero, and* $\gamma > 0$ *is a ratio of odd positive integers. Then the difference equation*

$$\triangle w_n + \phi_n w_{n-k}^{\gamma} = 0 \tag{2.10}$$

also has an eventually positive solution.

Proof Let $\{w_n\}$ be an eventually positive solution of (2.9). Define a set $S = \{u_n : 0 \leq u_n \leq w_n, n \geq N \in \mathbb{N}\}$. Then define a mapping Γ on S as follows:

$$(\Gamma u)_n = \begin{cases} \sum_{s=n}^{\infty} \phi_s u_{s-k}^{\gamma}, & n \geq N, \\ (\Gamma u)_N + w_n - w_N, & n_0 \leq n \leq N. \end{cases}$$

Define a sequence $\{u_n^{(k)}\}$, $k = 1, 2, \ldots$, as follows:

$$u_n^{(1)} = w_n,$$
$$u_n^{(k+1)} = \left(\Gamma u^{(k)}\right)_n, \quad k = 1, 2, \ldots.$$

Since $\{w_n\}$ is a solution of (2.9), we obtain

$$\triangle w_n + \phi_n w_{n-k}^{\gamma} \leq 0.$$

Summing the last inequality from n to n_1 and letting $n_1 \to \infty$, we obtain

$$-w_n + \sum_{s=n}^{\infty} \phi_s w_{s-k}^{\gamma} \leq 0$$

or

$$u_n^{(2)} = (\Gamma w)_n = \sum_{s=n}^{\infty} \phi_s w_{s-k}^{\gamma} \leq w_n = u_n^{(1)}.$$

By induction, we see that

$$0 \leq u_n^{(k)} \leq u_n^{(k-1)} \leq \cdots \leq u_n^{(1)} = w_n, \quad n \geq n_0.$$

Hence, $\lim_{k\to\infty} u_n^{(k)} = u_n$ exists with $0 \leq u_n \leq w_n$. Then we can apply the Lebesgue dominated convergence theorem to show that $u = \Gamma u$, that is,

$$u_n = \sum_{s=n}^{\infty} \phi_n u_{n-k}^{\gamma}, \quad n \geq N.$$

Clearly, $\{u_n\}$ is an eventually positive solution of (2.10) for all $n \geq N$. Since $u_n > 0$ for all $n_0 \leq n \leq N$, it follows that $u_n > 0$ for all $n \geq n_0$. Hence equation (2.10) has an eventually positive solution for all $n \geq n_0$. This completes the proof. □

3 Oscillation results

In this section, we establish some new oscillation criteria for equation (1.1). To simplify our notation, let us denote

$$Q_n = \min\{q_n, q_{\tau(n)}\}, \qquad Q_n^{\star} = Q_n \left(\sum_{s=n_1}^{\sigma(n)-1} \frac{1}{a_s^{\frac{1}{\alpha}}} \right)^{\beta},$$

where n_1 is sufficiently large. We first study the case $\alpha = \beta = 1$.

Theorem 3.1 *Let $\alpha = \beta = 1$ in equation* (1.1). *Assume that the first order neutral difference inequality*

$$\triangle(y_n + p y_{\tau(n)}) + Q_n(R_{\sigma(n)} - R_{n_1}) y_{\sigma(n)} \leq 0 \tag{3.1}$$

has no positive solution, then every solution of equation (1.1) *is oscillatory.*

Proof Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then the corresponding function z_n satisfies

$$z_{\sigma(n)} = x_{\sigma(n)} + p_{\sigma(n)} x_{\tau(\sigma(n))} \leq x_{\sigma(n)} + p x_{\sigma(\tau(n))}, \tag{3.2}$$

where we have used the hypotheses (H_4) and (H_5). From equation (1.1), we have

$$\triangle(a_n \triangle z_n) + q_n x_{\sigma(n)} = 0 \tag{3.3}$$

and

$$0 = p\triangle(a_{\tau(n)} \triangle z_{\tau(n)}) + p q_{\tau(n)} x_{\tau(\sigma(n))}. \tag{3.4}$$

Combining (3.3) and (3.4), we are led to

$$\triangle(a_n\triangle z_n + pa_{\tau(n)}\triangle z_{\tau(n)}) + Q_n z_{\sigma(n)} \leq 0. \tag{3.5}$$

It follows from Lemma 2.3 that $y_n = a_n\triangle z_n > 0$ is decreasing, and then

$$z_n \geq \sum_{s=n_1}^{n-1}\frac{1}{a_s}(a_s\triangle z_s) \geq y_n\sum_{s=n_1}^{n-1}\frac{1}{a_s} = y_n(R_n - R_{n_1}). \tag{3.6}$$

Therefore, (3.6) together with (3.5) ensures that $\{y_n\}$ is a positive solution of inequality (3.1). This contradicts our assumption and therefore the proof is complete. □

Remark 3.1 The condition $\tau \circ \sigma = \sigma \circ \tau$ of the hypothesis (H_4) is satisfied, for example $\tau(n) = n - \tau$ and $\sigma(n) = n - \sigma$.

Remark 3.2 When studying the oscillatory properties of neutral type equations one usually assumes $\sigma(n) \leq \tau(n)$, $\sigma(n) \leq n$, $\tau(n) \leq n$, $0 \leq p_n < 1$, *etc.* In Theorem 3.1 no such constraint is involved and what is more we do not impose is $\tau(n)$ is delayed or advanced and accordingly $\sigma(n)$ can be delayed or advanced. Hence, our result is of high generality and extends and complements the known ones.

Theorem 3.2 *Let* $\alpha = \beta = 1$ *in equation* (1.1), *and assume that*

$$\tau(n) \geq n. \tag{3.7}$$

If the first order difference inequality

$$\triangle w_n + \frac{Q_n}{(1+p)}(R_{\sigma(n)} - R_{n_1})w_{\sigma(n)} \leq 0 \tag{3.8}$$

has no positive solution, then every solution of equation (1.1) *is oscillatory.*

Proof We assume that $\{x_n\}$ is a positive solution of equation (1.1). It follows from Lemma 2.3 and the proof of Theorem 3.1 that $y_n = a_n\triangle z_n > 0$ is decreasing and it satisfies (3.1). Let us denote $w_n = y_n + py_{\tau(n)}$.

It follows from (3.7) that

$$w_n \leq y_n(1+p).$$

Substituting this term into (3.1), we obtain $\{w_n\}$ is a positive solution of (3.8), a contradiction. This completes the proof. □

Adding the restriction that $\sigma(n) = n - \sigma$, and using suitable criterion for the absence of positive solutions of equation (3.8) (see *e.g.* Lemma 2.5), we obtain an easily verifiable oscillation result for equation (1.1).

Corollary 3.1 *Assume that* $\alpha = \beta = 1$, *and* (3.7) *holds. If*

$$\sigma(n) = n - \sigma, \quad \sigma \text{ is a positive integer} \tag{3.9}$$

and

$$\lim_{n\to\infty}\inf\sum_{s=n-\sigma}^{n-1} Q_s R_{\sigma(s)} > (1+p)\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \tag{3.10}$$

then every solution of equation (1.1) *is oscillatory.*

Proof It is easy to see that if (3.10) holds, then

$$\lim_{n\to\infty}\inf\sum_{s=n-\sigma}^{n-1} \frac{Q_s}{(1+p)}(R_{\sigma(s)} - R_{n_1}) > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}.$$

But this condition, according to Lemma 2.5, guarantees that (3.8) has no positive solution and the assertion now follows from Theorem 3.2. □

Now, we turn our attention to the case when $\tau(n)$ is delayed. We use the notation $\tau^{-1}(n)$ for its inverse function.

Theorem 3.3 *Let* $\alpha = \beta = 1$ *in equation* (1.1), *and assume that*

$$\tau(n) \le n. \tag{3.11}$$

If the difference inequality

$$\triangle w_n + \frac{Q_n}{(1+p)}(R_{\sigma(n)} - R_{n_1}) w_{\tau^{-1}\sigma(n)} \le 0 \tag{3.12}$$

has no positive solution, then every solution of equation (1.1) *is oscillatory.*

Proof Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then $y_n = a_n \triangle z_n > 0$ is decreasing solution of (3.1). We denote $w_n = y_n + p y_{\tau(n)}$. Then by (3.11), we have $w_n \le y_{\tau(n)}(1+p)$. Substituting this into (3.1), we see that $\{w_n\}$ is a positive solution of inequality (3.12), a contradiction. This completes the proof. □

Corollary 3.2 *Assume that* $\alpha = \beta = 1$, $\tau(n) = n - \tau$, $\sigma(n) = n - \sigma$ *with*

$$n - \sigma \le n - \tau, \quad \sigma \text{ and } \tau \text{ are nonnegative integers} \tag{3.13}$$

and

$$\lim_{n\to\infty}\inf\sum_{s=n+\tau-\sigma}^{n-1} Q_s R_s > (1+p)\left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}, \tag{3.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof The proof is very similar to that of Corollary 3.1 and hence it is omitted. □

Next, we consider the case $0 < \beta \le 1$, and $\sigma(n) \le n$ in equation (1.1).

Theorem 3.4 *Let* $0 < \beta \leq 1$. *If the difference inequality*

$$\triangle\left(w_n + p^\beta w_{\tau(n)}\right) + Q_n^\star w_{\sigma(n)}^{\frac{\beta}{\alpha}} \leq 0 \tag{3.15}$$

has no positive solution, then every solution of equation (1.1) *is oscillatory.*

Proof Let $\{x_n\}$ be a positive solution of equation (1.1). Then from equation (1.1), we have

$$0 = \triangle\left(a_n(\triangle z_n)^\alpha\right) + q_n x_{\sigma(n)}^\beta \tag{3.16}$$

and

$$0 = p^\beta \triangle\left(a_{\tau(n)}(\triangle z_{\tau(n)})^\alpha\right) + p^\beta q_{\tau(n)} x_{\sigma(\tau(n))}. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\triangle\left(a_n(\triangle z_n)^\alpha\right) + p^\beta \triangle\left(a_{\tau(n)}(\triangle z_{\tau(n)})^\alpha\right) + Q_n\left(x_{\sigma(n)}^\beta + p^\beta x_{\sigma(\tau(n))}^\beta\right) \leq 0. \tag{3.18}$$

By Lemma 2.2, we have

$$\begin{aligned} z_{\sigma(n)}^\beta &= (x_{\sigma(n)} + p_{\sigma(n)} x_{\sigma(\tau(n))})^\beta \\ &\leq x_{\sigma(n)}^\beta + p^\beta x_{\sigma(\tau(n))}^\beta. \end{aligned} \tag{3.19}$$

Using (3.19) in (3.18), we obtain

$$\triangle\left(a_n(\triangle z_n)^\alpha\right) + p^\beta \triangle\left(a_{\tau(n)}(\triangle z_{\tau(n)})^\alpha\right) + Q_n z_{\sigma(n)}^\beta \leq 0. \tag{3.20}$$

It follows from Lemma 2.3 that $w_n = a_n(\triangle z_n)^\alpha > 0$ is decreasing and so

$$z_n \geq \sum_{s=n_1}^{n-1}\left(a_s(\triangle z_s)^\alpha\right)^{\frac{1}{\alpha}} a_s^{-\frac{1}{\alpha}} \geq w_n^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{a_s^{\frac{1}{\alpha}}}.$$

Using the last inequality in (3.20), we see that $\{w_n\}$ is a positive solution of

$$\triangle\left(w_n + p^\beta w_{\tau(n)}\right) + Q_n^* w_{\sigma(n)}^{\frac{\beta}{\alpha}} \leq 0,$$

which is a contradiction. This completes the proof. □

Next, we shall deduce new sufficient conditions for inequality (3.15) to have no positive solutions, to obtain new oscillation criteria for equation (1.1). We shall discuss both the cases when $\tau(n)$ is delayed and advanced.

Theorem 3.5 *Let* $0 < \beta \leq 1$ *and* $\tau(n) \geq n$. *If the difference equation*

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{\sigma(n)}^{\frac{\beta}{\alpha}} = 0 \tag{3.21}$$

is oscillatory, then so is equation (1.1).

Proof We assume that $\{x_n\}$ is a positive solution of equation (1.1). Then it follows from the proof of Theorem 3.4 that $w_n = a_n(\triangle z_n)^\alpha > 0$ is decreasing and it satisfies (3.15). We denote

$$y_n = w_n + p^\beta w_{\tau(n)}. \tag{3.22}$$

Then

$$y_n \le w_n\bigl(1 + p^\beta\bigr).$$

Substituting this into (3.15), we see that y_n is a positive solution of the difference inequality

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{\sigma(n)}^{\frac{\beta}{\alpha}} \le 0.$$

It follows from Lemma 2.7 that the associated difference equation (3.21) also has a positive solution, which contradicts the oscillatory nature of equation (3.21). □

Theorem 3.6 *Let* $0 < \beta \le 1$ *and* $\sigma(n) \le \tau(n) \le n$. *If the difference equation*

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{\tau^{-1}(\sigma(n))}^{\frac{\beta}{\alpha}} = 0 \tag{3.23}$$

is oscillatory, then so is equation (1.1).

Proof Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then it follows from (3.22) that

$$y_n \le w_{\tau(n)}\bigl(1 + p^\beta\bigr)$$

or

$$w_{\sigma(n)}^{\frac{\beta}{\alpha}} \ge \frac{1}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{\tau^{-1}(\sigma(n))}^{\frac{\beta}{\alpha}}.$$

Using the above inequality in (3.15), we see that $\{y_n\}$ is a positive solution of the difference inequality

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} y_{\tau^{-1}(\sigma(n))}^{\frac{\beta}{\alpha}} \le 0.$$

By Lemma 2.7, the associated difference equation (3.23) also has a positive solution, which is a contradiction. This completes the proof. □

Now, we derive a criterion for equations (3.21) and (3.23) to be oscillatory. Employing this criterion, one can easily verify sufficient conditions for the oscillation of all solutions of equation (1.1).

Applying condition (2.6) to equations (3.21) and (3.23) in view of Theorem 3.5 and Theorem 3.6, immediately we obtain the following oscillatory criteria for equation (1.1).

Corollary 3.3 *Let the equation* (1.1) *with* $0 < \beta \le 1$, $\beta \le \alpha$, $\tau(n) \ge n$, *and* (3.9). *If*

$$\lim_{n\to\infty} \inf \sum_{s=n-\sigma}^{n-1} Q_s^\star > \left(1+p^\beta\right)^{\frac{\beta}{\alpha}} \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \tag{3.24}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 3.4 *Let the equation* (1.1) *with* $0 < \beta \le 1$, $\beta \le \alpha$, $\tau(n) = n - \tau$, *and* (3.9) *with* $\sigma > \tau > 0$. *If*

$$\lim_{n\to\infty} \inf \sum_{s=n+\tau-\sigma}^{n-1} Q_s^\star > \left(1+p^\beta\right)^{\frac{\beta}{\alpha}} \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}, \tag{3.25}$$

then every solution of (1.1) *is oscillatory.*

In view of Lemma 2.4, and Theorems 3.5 and 3.6, we have the following oscillation criteria for equation (1.1).

Corollary 3.5 *Let the equation* (1.1) *with* $0 < \beta \le 1$, $\beta > \alpha$, $\tau(n) = n + \tau$ *with* $\tau > 0$, *and* (3.9). *Assume that there exists* $\lambda > \frac{1}{\sigma} \log \frac{\beta}{\alpha}$ *such that*

$$\lim_{n\to\infty} \inf\left[Q_n^\star \exp\left(-e^{\lambda n}\right)\right] > 0, \tag{3.26}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 3.6 *Let the equation* (1.1) *with* $0 < \beta \le 1$, $\beta > \alpha$, $\tau(n) = n - \tau$, *and* (3.9) *with* $\sigma > \tau > 0$. *Assume that there exists* $\lambda > \frac{1}{\sigma-\tau} \log \frac{\beta}{\alpha}$ *such that*

$$\lim_{n\to\infty} \inf\left[Q_n^\star \exp\left(-e^{\lambda n}\right)\right] > 0, \tag{3.27}$$

then every solution of equation (1.1) *is oscillatory.*

Next, we turn our attention to the case $\beta \ge 1$ and we rewrite our previous results to cover this case.

Theorem 3.7 *Let* $\beta \ge 1$ *and* $\sigma(n) \le n$. *If the difference inequality*

$$\triangle\left(w_n + p^\beta w_{\tau(n)}\right) + 2^{1-\beta} Q_n^\star w_{\sigma(n)}^{\frac{\beta}{\alpha}} \le 0 \tag{3.28}$$

has no positive solution, then every solution of equation (1.1) *is oscillatory.*

Proof This theorem can be proved exactly as Theorem 3.4; we need to replace the inequality (3.19) by

$$\begin{aligned} z_{\sigma(n)}^\beta &= (x_{\sigma(n)} + p_{\sigma(n)} x_{\tau(\sigma(n))})^\beta \\ &\le 2^{\beta-1}\left(x_{\sigma(n)}^\beta + p^\beta x_{\sigma(\tau(n))}^\beta\right), \end{aligned}$$

which follows from Lemma 2.1. □

The following results are equivalent to Theorems 3.5 and 3.6 and the proofs are omitted.

Theorem 3.8 *Let $\beta \geq 1$ and $\tau(n) \geq n$. If the difference equation*

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} 2^{1-\beta} y_{\sigma(n)}^{\frac{\beta}{\alpha}} = 0 \tag{3.29}$$

is oscillatory, then every solution of equation (1.1) *is oscillatory.*

Theorem 3.9 *Let $\beta \geq 1$ and $\sigma(n) \leq \tau(n) \leq n$. If the difference equation*

$$\triangle y_n + \frac{Q_n^\star}{(1+p^\beta)^{\frac{\beta}{\alpha}}} 2^{1-\beta} y_{\tau^{-1}(\sigma(n))}^{\frac{\beta}{\alpha}} = 0 \tag{3.30}$$

is oscillatory, then solution of equation (1.1) *is oscillatory.*

Combining Lemma 2.6 with Theorems 3.8 and 3.9, we have the following oscillation criteria for equation (1.1).

Corollary 3.7 *Let $\alpha \geq \beta \geq 1$, $\tau(n) \geq n$, and $\sigma(n) = n - \sigma$ in equation* (1.1). *If*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} Q_s^\star > 2^{\beta-1}\left(1+p^\beta\right)^{\frac{\beta}{\alpha}} \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \tag{3.31}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 3.8 *Let $\alpha \geq \beta \geq 1$, $\tau(n) = n - \tau$, $\sigma(n) = n - \sigma$ with $\sigma > \tau$ in equation* (1.1). *If*

$$\liminf_{n\to\infty} \sum_{s=n+\tau-\sigma}^{n-1} Q_s^\star > 2^{\beta-1}\left(1+p^\beta\right)^{\frac{\beta}{\alpha}} \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}, \tag{3.32}$$

then every solution of equation (1.1) *is oscillatory.*

Combining Lemma 2.4 with Theorems 3.8 and 3.9, we have following results.

Corollary 3.9 *Let $\beta > \alpha \geq 1$, $\tau(n) \geq n$, and $\sigma(n) = n - \sigma$ in equation* (1.1). *Assume that there exists $\lambda > \frac{1}{\sigma}\log\frac{\beta}{\alpha}$ such that*

$$\liminf_{n\to\infty} \left[Q_n^\star \exp\left(-e^{\lambda n}\right)\right] > 0,$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 3.10 *Let $\beta > \alpha \geq 1$, $\tau(n) = n - \tau$, $\sigma(n) = n - \sigma$ with $\sigma > \tau$ in equation* (1.1). *Assume that there exists $\lambda > \frac{1}{\sigma-\tau}\log\frac{\beta}{\alpha}$ such that*

$$\liminf_{n\to\infty} \left[Q_n^\star \exp\left(-e^{\lambda n}\right)\right] > 0,$$

then every solution of equation (1.1) *is oscillatory.*

4 Examples

In this section, we present some examples to illustrate the main results.

Example 4.1 Consider the difference equation

$$\triangle\left(\frac{1}{n}\triangle(x_n + px_{\tau(n)})\right) + \frac{b}{n(n+1)}x_{n-2} = 0, \quad n \geq 1, \tag{4.1}$$

where $0 < p < \infty$, $\tau(n) = an$, $a \geq 1$ is an integer and $b > 0$. Here $R(n) = \frac{n(n-1)}{2}$, $Q_n = \frac{b}{an(an+1)}$. Condition (3.10) reduces to

$$\frac{27b}{8a^2} > (1 + p). \tag{4.2}$$

If the condition (4.2) is satisfied, then by Corollary 3.1, every solution of equation (4.1) is oscillatory.

If $\tau(n) = n - 1$, then condition (3.10) reduces to

$$2b > (1 + p). \tag{4.3}$$

If the condition (4.3) holds, then by Corollary 3.2, every solution of equation (4.1) is oscillatory. Hence, we have covered the oscillation of equation (4.1) when $\tau(n)$ is delayed or advanced.

Example 4.2 Consider the difference equation

$$\triangle\left(\frac{1}{n}\left(\triangle(x_n + px_{\tau(n)})\right)^{\frac{1}{3}}\right) + \frac{b}{n^{\frac{2}{3}}(n+1)^{\frac{2}{3}}}x_{n-2}^{\frac{1}{3}} = 0, \quad n \geq 1, \tag{4.4}$$

where $0 < p < \infty$, $b > 0$, and $\tau(n) = an$, $a \geq 1$ is an integer, and $\alpha = \beta = \frac{1}{3}$. Here $R(n) = \frac{n(n-1)}{2}$, $Q_n = \frac{b}{(an)^{\frac{2}{3}}(an+1)^{\frac{2}{3}}}$. Then condition (3.24) reduces to

$$\frac{2^{\frac{1}{3}}b}{a^{\frac{4}{3}}} > \frac{8}{27}\left(1 + p^{\frac{1}{3}}\right). \tag{4.5}$$

Therefore, by Corollary 3.3, every solution of equation (4.4) is oscillatory if condition (4.5) holds.

If $\tau(n) = n - 1$, then condition (3.25) reduces to

$$2^{\frac{4}{3}}b > 1 + p^{\frac{1}{3}}. \tag{4.6}$$

Therefore, by Corollary 3.4, every solution of equation (4.4) is oscillatory if condition (4.6) holds.

Example 4.3 Consider the difference equation

$$\triangle\left(\frac{1}{n^{\frac{1}{5}}}\left(\triangle(x_n + px_{n+1})\right)^{\frac{1}{5}}\right) + \frac{be^{e^{2n}}}{(n+1)^{\frac{2}{3}}}x_{n-1}^{\frac{1}{3}} = 0, \quad n \geq 1, \tag{4.7}$$

where $b > 0$, $\alpha = \frac{1}{5}$, $\beta = \frac{1}{3}$, $\tau = 1$, $\sigma = 1$. Choose $\lambda = 2$, then condition (3.26) reduces to $\frac{b}{2^{\frac{1}{3}}} > 0$. Therefore, by Corollary 3.5 every solution of equation (4.7) is oscillatory.

Example 4.4 Consider the difference equation

$$\triangle\left(\frac{1}{n^5}\left(\triangle(x_n + px_{\tau(n)})\right)^5\right) + \frac{\mu}{(n+1)^6}x_{n-2}^3 = 0, \quad n \geq 1, \tag{4.8}$$

where $\mu > 0$, $0 < p < \infty$, $\alpha = 5$, $\beta = 3$, $\tau(n) = an$, $a \geq 1$ is an integer. Here $Q_n^\star = \frac{\mu}{(an+1)^6}(\frac{(n-2)(n-3)}{2})^3$. Then condition (3.31) reduces to

$$\frac{27\mu}{128a^6} > \left(1 + p^3\right)^{\frac{3}{5}}. \tag{4.9}$$

Therefore, by Corollary 3.7, every solution of equation (4.8) is oscillatory if condition (4.9) holds.

Further, if $\tau(n) = n - 1$, then condition (3.32) reduces to

$$\mu > 8\left(1 + p^3\right)^{\frac{3}{5}}. \tag{4.10}$$

Therefore, by Corollary 3.8, every solution of equation (4.8) is oscillatory if condition (4.10) holds.

5 Conclusions

In this paper, we have introduced new comparison theorems for investigation of the oscillation of equation (1.1). The established comparison principles reduce the study of the oscillation of the second order neutral difference equations to a study of the oscillation properties of various types of first order difference inequalities, which clearly simplifies the investigation of the oscillation of equation (1.1). Further, the method used here permits us to relax the restrictions usually imposed on the coefficients of equation (1.1). So the results obtained here are of high generality and easily may be applicable, as illustrated with suitable examples.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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