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New general systems of set-valued variational inclusions involving relative (A,η)-maximal monotone operators in Hilbert spaces

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Dedicated to professor Shih-sen Chang on the occasion of his 80th birthday.

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Abstract

The purpose of this paper is to introduce and study a class of new general systems of set-valued variational inclusions involving relative (A,η)-maximal monotone operators in Hilbert spaces. By using the generalized resolvent operator technique associated with relative (A,η)-maximal monotone operators, we also construct some new iterative algorithms for finding approximation solutions to the general systems of set-valued variational inclusions and prove the convergence of the sequences generated by the algorithms. The results presented in this paper improve and extend some known results in the literature.

Keywords: general system of set-valued variational inclusions; relative (A,η)-maximal monotone operator; generalized resolvent operator technique; relative relaxed cocoercive; iterative algorithm; convergence criteria

1 Introduction

Recently, some systems of variational inequalities, variational inclusions, complementarity problems, and equilibrium problems have been studied by many authors because of their close relations to some problems arising in economics, mechanics, engineering science and other pure and applied sciences. Among these methods, the resolvent operator technique is very important. Huang and Fang [1] introduced a system of order complementarity problems and established some existence results for the system using fixed point theory. Verma [2] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of the systems of variational inequalities. Cho *et al.* [3] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces. Further, the authors proved some existence and uniqueness theorems of solutions for the systems, and also constructed some iterative algorithms for approximating the solution of the systems of nonlinear variational inequalities, respectively.

Moreover, Fang *et al.* [4], Yan *et al.* [5], Fang and Huang [6] introduced and studied some new systems of variational inclusions involving H-monotone operators and (H,η)-monotone operators in Hilbert space, respectively. Using the corresponding resolvent operator technique associated with H-monotone operators, (H,η)-monotone op-

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erators, the authors proved the existence of solutions for the variational inclusion systems and constructed some algorithms for approximating the solutions of the systems and discussed convergence of the iteration sequences generated by the algorithms, respectively. Very recently, Lan *et al.* [7] introduced and studied a new system of nonlinear A-monotone multivalued variational inclusions in Hilbert spaces. By using the concept and properties of A-monotone operators, and the resolvent operator technique associated with A-monotone operators due to Verma [8], the authors constructed a new iterative algorithm for solving this system of nonlinear multivalued variational inclusions with A-monotone operators in Hilbert spaces and proved the existence of solutions for the nonlinear multivalued variational inclusion systems and the convergence of iterative sequences generated by the algorithm. For some related work, see, for example, [1–32] and the references therein.

On the other hand, Cao [33] introduced and studied a new system of generalized quasi-variational-like-inclusions applying the η-proximal mapping technique. Further, Agarwal and Verma [34] introduced and studied relative (A,η)-maximal monotone operators and discussed the approximation solvability of a new system of nonlinear (set-valued) variational inclusions involving (A,η)-maximal relaxed monotone and relative (A,η)-maximal monotone operators in Hilbert spaces based on a generalized hybrid iterative algorithm and the general (A,η)-resolvent operator method.

Inspired and motivated by the above works, the purpose of this paper is to consider the following new general system of set-valued variational inclusions involving relative (A,η)-maximal monotone operators in Hilbert spaces: Find $(x_1^*,x_2^*,\dots,x_m^*)\in H_1\times H_2\times\cdots\times H_m$ and $u_{ij}^*\in U_{ij}(x_j^*)$ for any $i,j=1,2,\dots,m$ such that

$$0\in F_i\bigl(u_{i1}^*,u_{i2}^*,\dots,u_{im}^*\bigr)+M_i\bigl(g_i\bigl(x_i^*\bigr)\bigr), \tag{1.1}$$

where m is a given positive integer, $F_i:H_1\times H_2\times\cdots\times H_m\to H_i$, $A_i:H_i\to H_i$, $g_i:H_i\to H_i$ and $\eta_i:H_i\times H_i\to H_i$ are single-valued operators, $U_{ij}:H_j\to 2^{H_j}$ is a set-valued operator and $M_i:H_i\to 2^{H_i}$ is relative (A_i,η_i)-maximal monotone.

We note that for appropriate and suitable choices of positive integer m, the operators F_i, g_i, A_i, η_i, M_i, U_{ij}, and H_i for $i,j=1,2,\dots,m$, one can know that the problem (1.1) includes a number of known general problems of variational character, including variational inequality (system) problems, variational inclusion (system) problems as special cases. For more details, see [1–31, 35] and the following examples.

Example 1.1 For $i,j=1,2,\dots,m$, if $U_{ij}=T_{ij}$ is single-valued operator, the problem (1.1) reduces to finding $x_j\in H_j$, such that

$$0\in F_i\bigl(T_{i1}x_1^*,T_{i2}x_2^*,\dots,T_{im}x_m^*\bigr)+M_i\bigl(g_i\bigl(x_i^*\bigr)\bigr). \tag{1.2}$$

Example 1.2 For $i=1,2,\dots,m$, if $H_i=H$ and $A_i\equiv I$, an identity operator, and $M_i=\partial\varphi_i$, where $\varphi_i:H\to R\cup\{+\infty\}$ is proper and lower semi-continuous η_i-subdifferentiable functional and $\partial\varphi_i$ denotes η_i-subdifferential operator, then the problem (1.1) reduces to finding $x_i^*\in H$ and $u_{ij}^*\in U_{ij}(x_j^*)$ for $j=1,2,\dots,m$ such that

$$\bigl\langle F_i\bigl(u_{i1}^*,u_{i2}^*,\dots,u_{im}^*\bigr),\eta_i\bigl(x,g_i\bigl(x_i^*\bigr)\bigr)\bigr\rangle\ge\varphi_i\bigl(g_i\bigl(x_i^*\bigr)\bigr)-\varphi_i(x),\quad\forall x\in H. \tag{1.3}$$

The problem (1.3) is called a set-valued nonlinear generalized quasi-variational-like-inclusion system, which was considered and studied by Cao [33].

Example 1.3 When $m = 2$ and $g_i \equiv I$ for $i = 1, 2$, then the problem (1.1) is equivalent to the following nonlinear set-valued variational inclusion system problem: Find $(x_1^*, x_2^*) \in H_1 \times H_2$ and $u_1^* \in U_1(x_1^*)$, $u_2^* \in U_2(x_2^*)$ such that

$$\begin{aligned} &0 \in F_1(x_1^*, u_2^*) + M_1(x_1^*), \\ &0 \in F_2(u_1^*, x_2^*) + M_2(x_2^*), \end{aligned} \tag{1.4}$$

which was studied by Agarwal and Verma [34].

Example 1.4 If $m = 2$ and $M_i(x_i) = \partial\varphi_i(x_i)$, where $\varphi_i : H_i \to R \cup \{+\infty\}$ is proper, convex, and lower semi-continuous functional and $\partial\varphi_i$ denotes the subdifferential operator of φ_i for all $x_i \in H_i$, $i = 1, 2$, then the problem (1.4) reduces to the following system of set-valued mixed variational inequalities: Find $(x_1^*, x_2^*) \in H_1 \times H_2$, $u_1^* \in U_1(x_1^*)$ and $u_2^* \in U_2(x_2^*)$ such that

$$\begin{aligned} &\langle F_1(x_1^*, u_2^*), x - x_1^*\rangle + \varphi_1(x) - \varphi_1(x_1^*) \ge 0, \quad \forall x \in H_1, \\ &\langle F_2(u_1^*, x_2^*), y - x_2^*\rangle + \varphi_2(y) - \varphi_2(x_2^*) \ge 0, \quad \forall y \in H_2. \end{aligned} \tag{1.5}$$

If $U_1 = U_2 \equiv I$, then the problem (1.5) reduces to finding $(x_1^*, x_2^*) \in H_1 \times H_2$ such that

$$\begin{aligned} &\langle F_1(x_1^*, x_2^*), x - x_1^*\rangle + \varphi_1(x) - \varphi_1(x_1^*) \ge 0, \quad \forall x \in H_1, \\ &\langle F_2(x_1^*, x_2^*), y - x_2^*\rangle + \varphi_2(y) - \varphi_2(x_2^*) \ge 0, \quad \forall y \in H_2, \end{aligned} \tag{1.6}$$

which is called the system of nonlinear variational inequalities considered by Cho *et al.* [3]. Some specializations of the problem (1.6) are dealt by Kim and Kim [35].

Example 1.5 If $m = 2$ and $U_1 = U_2 = g_1 = g_2 \equiv I$, then the problem (1.1) reduces to the problem of finding $(x_1^*, x_2^*) \in H_1 \times H_2$ such that

$$\begin{aligned} &0 \in F_1(x_1^*, x_2^*) + M_1(x_1^*), \\ &0 \in F_2(x_1^*, x_2^*) + M_2(x_2^*), \end{aligned}$$

which was introduced and studied by Fang *et al.* [4].

Moreover, by using the generalized resolvent operator technique associated with relative (A, η)-maximal monotone operators, we also construct some new iterative algorithms for finding approximation solutions to the general systems of set-valued variational inclusions and prove convergence of the sequences generated by the algorithms.

2 Preliminaries

Throughout, let H and H_i $(i = 1, 2, \ldots, m)$ be real Hilbert spaces and endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let 2^H and $C(H)$ denote the family of all the nonempty subsets of H and the family of all closed subsets of H, respectively.

Definition 2.1 Let $T : H \to H$ be a single-valued operator. Then the map T is said to be

(i) r-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H;$$

(ii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta\|x - y\|, \quad \forall x, y \in H.$$

Definition 2.2 Let $\eta : H \times H \to H$ and $A : H \to H$ be single-valued operators, $M : H \to 2^H$ be set-valued operator. Then

(i) η is said to be t-strongly monotone, if there exists a constant $t > 0$ such that

$$\langle \eta(x, y), x - y \rangle \geq t\|x - y\|^2, \quad \forall x, y \in H;$$

(ii) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \quad \forall x, y \in H;$$

(iii) A is said to be η-monotone, if

$$\langle A(x) - A(y), \eta(x, y) \rangle \geq 0, \quad \forall x, y \in H;$$

(iv) A is said to be strictly η-monotone, if A is η-monotone and

$$\langle A(x) - A(y), \eta(x, y) \rangle = 0 \quad \text{if and only if} \quad x = y;$$

(v) A is said to be (r, η)-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle A(x) - A(y), \eta(x, y) \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H;$$

(vi) M is said to be η-monotone with respect to A (or relative (A, η)-monotone) if

$$\langle u - v, \eta(A(x), A(y)) \rangle \geq 0, \quad \forall x, y \in H, u \in M(x), v \in M(y);$$

(vii) M is said to be relative (A, η)-maximal monotone, if M is η-monotone with respect to A (or relative (A, η)-monotone) and $(A + \lambda M)(H) = H$, where $\lambda > 0$ is an arbitrary constant.

Definition 2.3 For $i, j = 1, 2, \dots, m$, let H_i be a Hilbert space, $A_j : H_j \to H_j$ be single-valued operator, $U_{ij} : H_j \to 2^{H_j}$ be set-valued operator. Then nonlinear operator $F_i : H_1 \times H_2 \times \cdots \times H_m \to H_i$ is said to be

(i) (U_{ij}, c_j, μ_j)-relaxed cocoercive with respect to A_j (or relative (U_{ij}, c_j, μ_j)-relaxed cocoercive) in the jth argument, if there exist constants $c_j, \mu_j > 0$ such that for all $x_j^1, x_j^2 \in H_j$, and for any $u_j^1 \in U_{ij}(x_j^1)$, $u_j^2 \in U_{ij}(x_j^2)$,

$$\begin{aligned}
&\langle F_i(\dots, u_j^1, \dots) - F_i(\dots, u_j^2, \dots), A_j(x_j^1) - A_j(x_j^2) \rangle \\
&\quad \geq (-c_j)\|F_i(\dots, u_j^1, \dots) - F_i(\dots, u_j^2, \dots)\|^2 + \mu_j\|x_j^1 - x_j^2\|^2;
\end{aligned}$$

(ii) ζ_{ij}-Lipschitz continuous in the jth argument, if there exists constant $\zeta_{ij} > 0$ such that for all $x_j, y_j \in H_j$,

$$\| F_i(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) - F_i(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_m) \| \leq \|x_j - y_j\|.$$

Remark 2.1

(i) When $m = 1$ and $U = I$, then (i) and (ii) of Definition 2.3 reduce to corresponding concept of the relative relaxed cocoerciveness and Lipschitz continuity, respectively.

(ii) If $U_{ij} = T_{ij}$ is single-valued operator for $i, j = 1, 2, \ldots, m$, then F_i is (U_{ij}, c_j, μ_j)-relaxed cocoercive with respect to A_j in the jth argument reduce to (T_{ij}, c_j, μ_j)-relaxed cocoercive with respect to A_j in the jth argument, that is, if there exist constants $c_j, \mu_j > 0$ such that for all $x_j^1, x_j^2 \in H_j$,

$$\begin{aligned} &\langle F_i(\ldots, T_{ij}x_j^1, \ldots) - F_i(\ldots, T_{ij}x_j^2, \ldots), A_j(x_j^1) - A_j(x_j^2) \rangle \\ &\quad \geq (-c_j) \| F_i(\ldots, T_{ij}x_j^1, \ldots) - F_i(\ldots, T_{ij}x_j^2, \ldots) \|^2 + \mu_j \| x_j^1 - x_j^2 \|^2. \end{aligned}$$

Lemma 2.1 ([34]) *Let $\eta : H \times H \to H$ be a single-valued mapping, $A : H \to H$ be a strictly η-monotone mapping and $M : H \to 2^H$ be a relative (A, η)-maximal monotone mapping. Then the mapping $(A + \lambda M)$ is single-valued, where $\lambda > 0$ is arbitrary constant.*

Definition 2.4 Let $\eta : H \times H \to H$ be a single-valued mapping, $A : H \to H$ be a strictly η-monotone mapping and $M : H \to 2^H$ be a relative (A, η)-maximal monotone mapping. Then generalized resolvent operator $R_{M,\lambda}^{A,\eta} : H \to H$ is defined by

$$R_{M,\lambda}^{A,\eta}(z) = (A + \lambda M)^{-1}(z), \quad \forall z \in H,$$

where $\lambda > 0$ is a constant.

Lemma 2.2 ([34]) *Let $\eta : H \times H \to H$ be a t-strongly monotone and τ-Lipschitz continuous mapping, $A : H \to H$ be an r-strongly monotone mapping, and $M : H \to 2^H$ be a relative (A, η)-maximal monotone mapping. Then generalized resolvent operator $R_{M,\lambda}^{A,\eta} : H \to H$ is $\frac{\tau}{rt}$-Lipschitz continuous, that is,*

$$\| R_{M,\lambda}^{A,\eta}(x) - R_{M,\lambda}^{A,\eta}(y) \| \leq \frac{\tau}{rt} \|x - y\|, \quad \forall x, y \in H.$$

Definition 2.5 A set-valued operator $U : H \to 2^H$ is said to be D-γ-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$D(U(x), U(y)) \leq \gamma \|x - y\|, \quad \forall x, y \in H,$$

where $D : C(H) \times C(H) \to R \cup \{+\infty\}$ is called the Hausdorff pseudo-metric defined as follows:

$$D(U, V) = \max \Big\{ \sup_{x \in U} \inf_{y \in V} \|x - y\|, \sup_{y \in V} \inf_{x \in U} \|x - y\| \Big\}, \quad \forall U, V \in C(H).$$

Furthermore, the Hausdorff pseudo-metric D reduces to the Hausdorff metric when $C(H)$ is restricted to closed bounded subsets of the family $CB(H)$.

Lemma 2.3 *Let $\theta \in (0,1)$ be a constant. Then function $f(\lambda) = 1 - \lambda + \lambda\theta$ for $\lambda \in [0,1]$ is nonnegative and strictly decrease and $f(\lambda) \in [0,1]$. Further, if $\lambda \neq 0$, then $f(\lambda) \in (0,1)$.*

Lemma 2.4 ([36]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying*

$$a_{n+1} \leq \theta a_n + b_n$$

with $0 < \theta < 1$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

3 Iterative algorithm and convergence analysis

In this section, we construct a class of new iterative algorithms for finding approximate solutions of the problems (1.1) and (1.2), respectively. Then the convergence criterion for the algorithms is also discussed.

Lemma 3.1 *Let $(x_1^*, x_2^*, \ldots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$ for $i,j = 1,2,\ldots,m$, then $(x_1^*, x_2^*, \ldots, x_m^*, u_{11}^*, \ldots, u_{1m}^*, \ldots, u_{m1}^*, \ldots, u_{mm}^*)$ (denoted by $(*)$) is a solution of the problem (1.1) if and only if $(*)$ satisfy*

$$g_i(x_i^*) = R_{M_i,\rho_i}^{A_i,\eta_i}\big[A_i\big(g_i(x_i^*)\big) - \rho_i F_i\big(u_{i1}^*, \ldots, u_{ii-1}^*, u_{ii}^*, u_{ii+1}^*, \ldots, u_{im}^*\big)\big], \tag{3.1}$$

where $R_{M_i,\rho_i}^{A_i,\eta_i} = (A_i + \rho_i M_i)^{-1}$ and $\rho_i > 0$ is a constant for $i = 1,2,\ldots,m$.

Proof It follows from the definition of generalized resolvent operator $R_{M_i,\rho_i}^{A_i,\eta_i}$ that the proof can be obtained directly, and so it is omitted. □

Algorithm 3.1

Step 1. Setting $(x_1^0, x_2^0, \ldots, x_m^0) \in H_1 \times H_2 \times \cdots \times H_m$ and choose $u_{ij}^0 \in U_{ij}(x_j^0)$ for $i,j = 1,2,\ldots,m$.

Step 2. Let

$$\begin{aligned} x_i^{n+1} = (1-\lambda)x_i^n + \lambda\big\{x_i^n - g_i\big(x_i^n\big) + R_{M_i,\rho_i}^{A_i,\eta_i}\big[A_i\big(g_i\big(x_i^n\big)\big) \\ - \rho_i F_i\big(u_{i1}^n, \ldots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \ldots, u_{im}^n\big)\big]\big\} \end{aligned} \tag{3.2}$$

for all $i = 1,2,\ldots,m$ and $n = 0,1,2,\ldots$, where $\lambda \in (0,1]$ is a constant.

Step 3. By the results of Nadler [37], we can choose $u_{ij}^{n+1} \in U_{ij}(x_j^{n+1})$ such that

$$\big\|u_{ij}^{n+1} - u_{ij}^n\big\| \leq \left(1 + \frac{1}{n+1}\right) D_j\big(U_{ij}\big(x_j^{n+1}\big), U_{ij}\big(x_j^n\big)\big), \tag{3.3}$$

where $D_j(\cdot,\cdot)$ is the Hausdorff pseudo-metric on $C(H_j)$ and $i,j = 1,2,\ldots,m$.

Step 4. If x_i^{n+1} and u_{ij}^{n+1} for $i,j = 1,2,\ldots,m$ satisfy (3.2) to sufficient accuracy, stop. Otherwise, set $n := n+1$ and return to Step 2.

Remark 3.1 If $R_{M_i,\rho_i}^{A_i,\eta_i}$ reduces to $J_\rho^{\varphi_i} = (I + \rho\partial\varphi_i)^{-1}$, where $\varphi_i : H_i \to R \cup \{+\infty\}$ is proper and lower semi-continuous η_i-subdifferentiable functional, $H_i \equiv H$ for $i = 1,2,\ldots,m$ and $\lambda = 1$, then Algorithm 3.1 reduces to Algorithm (I) of Cao [33].

When $\lambda = 1$ and $U_{ij} = T_{ij}$ is single-valued operator for $i,j = 1,2,\ldots,m$, then Algorithm 3.1 reduces to the following algorithm for the problem (1.2).

Algorithm 3.2 For any given $(x_1^0, x_2^0, \dots, x_m^0) \in H_1 \times H_2 \times \cdots \times H_m$, we compute x_i^n as follows:

$$\begin{aligned} x_i^{n+1} = {} & x_i^n - g_i(x_i^n) + R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i(g_i(x_i^n)) \\ & - \rho_i F_i(T_{i1}x_1^n, \dots, T_{ii-1}x_{i-1}^n, T_{ii}x_i^n, T_{ii+1}x_{i+1}^n, \dots, T_{im}x_m^n)\bigr] + w_i^n \end{aligned} \tag{3.4}$$

for $n = 0, 1, 2, \dots$ and $i = 1, 2, \dots, m$, where $w_i^n \in H_i$ is error to take into account a possible inexact computation of the resolvent operator point satisfying conditions $\lim_{n\to\infty} \|w_i^n\| = 0$.

Remark 3.2

(i) Let $m = 2$, $g_i \equiv I$, $U_{ii} \equiv I$ for $i = 1, 2$, then Algorithm 3.1 reduces to Algorithm 4.3 of Agarwal and Verma [34].

(ii) If for appropriate and suitable choices of positive integer m and mappings F_i, g_i, A_i, η_i, M, U_{ij}, and H_i for $i, j = 1, 2, \dots, m$, one can know that Algorithms 3.1-3.2 are extending a number of known algorithms.

In the sequel, we provide main result concerning the problem (1.1) with respect to Algorithm 3.1.

Theorem 3.1 *For $i = 1, 2, \dots, m$, let $\eta_i : H_i \times H_i \to H_i$ be τ_i-Lipschitz continuous and t_i-strongly monotone operator, $A_i : H_i \to H_i$ be β_i-Lipschitz continuous and r_i-strongly monotone operator, $g_i : H_i \to H_i$ be ξ_i-Lipschitz continuous and δ_i-strongly monotone operator and $M_i : H_i \to 2^{H_i}$ be relative (A_i, η_i)-maximal monotone. Suppose that $U_{ij} : H_j \to CH_j$ is D_j-γ_{ij}-Lipschitz continuous, $F_i : H_1 \times H_2 \times \cdots \times H_m \to H_i$ is (U_{ii}, c_i, μ_i)-relaxed cocoercive with respect to A_i in the ith argument and ζ_{ij}-Lipschitz continuous in the jth for $i, j = 1, 2, \dots, m$. If there exists constant $\rho_i > 0$ for such that*

$$\begin{aligned} \theta_j = {} & \frac{\tau_j}{r_j t_j} \cdot \sqrt{\beta_j^2 \xi_j^2 - 2\rho_j \mu_j \delta_j^2 + 2\rho_j c_j \zeta_{jj}^2 \gamma_{jj}^2 + \rho_j^2 \zeta_{jj}^2 \gamma_{jj}^2} \\ & + \sqrt{1 - 2\delta_j + \xi_j^2} + \sum_{i=1, i\neq j}^{m} \frac{\rho_i \tau_i \zeta_{ij} \gamma_{ij}}{r_i t_i} < 1 \end{aligned} \tag{3.5}$$

for all $j = 1, 2, \dots, m$, then the problem (1.1) *admits a solution* $(*)$, *i.e.* $(x_1^*, x_2^*, \dots, x_m^*, u_{11}^*, \dots, u_{1m}^*, \dots, u_{m1}^*, \dots, u_{mm}^*)$, *where $(x_1^*, x_2^*, \dots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ and $u_{ij}^* \in U_{ij}(x_j^*)$ for $i, j = 1, 2, \dots, m$. Moreover, iterative sequences $\{x_j^n\}$ and $\{u_{ij}^n\}$ generated by Algorithm* 3.1 *strongly converge to x_j^* and u_{ij}^* for $i, j = 1, 2, \dots, m$, respectively.*

Proof For $i = 1, 2, \dots, m$, applying Algorithm 3.1 and Lemma 2.2, we have

$$\begin{aligned} & \|x_i^{n+1} - x_i^n\| \\ & \quad \le (1-\lambda)\|x_i^n - x_i^{n-1}\| + \lambda \|x_i^n - x_i^{n-1} - (g_i(x_i^n) - g_i(x_i^{n-1}))\| \\ & \qquad + \lambda \|R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i(g_i(x_i^n)) - \rho_i F_i(u_{i1}^n, \dots, u_{ii-1}^n, u_{ii}^n, u_{ii+1}^n, \dots, u_{im}^n)\bigr] \\ & \qquad - R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i(g_i(x_i^{n-1})) - \rho_i F_i(u_{i1}^{n-1}, \dots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \dots, u_{im}^{n-1})\bigr]\| \\ & \quad \le (1-\lambda)\|x_i^n - x_i^{n-1}\| + \lambda \|x_i^n - x_i^{n-1} - (g_i(x_i^n) - g_i(x_i^{n-1}))\| \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda\tau_i}{r_i t_i}\big\|A_i\big(g_i\big(x_i^n\big)\big) - A_i\big(g_i\big(x_i^{n-1}\big)\big) \\
&- \rho_i\big[F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^n,u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&- F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big)\big]\big\| \\
&+ \frac{\lambda\tau_i\rho_i}{r_i t_i}\big\|F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&- F_i\big(u_{i1}^{n-1},\ldots,u_{ii-1}^{n-1},u_{ii}^{n-1},u_{ii+1}^{n-1},\ldots,u_{im}^{n-1}\big)\big\|.
\end{aligned}
\tag{3.6}
$$

By ξ_i-Lipschitz continuity and δ_i-strongly monotonicity of g_i, we get

$$
\begin{aligned}
&\big\|x_i^n - x_i^{n-1} - \big(g_i\big(x_i^n\big) - g_i\big(x_i^{n-1}\big)\big)\big\|^2 \\
&\quad= \big\|x_i^n - x_i^{n-1}\big\|^2 - 2\big\langle g_i\big(x_i^n\big) - g_i\big(x_i^{n-1}\big), x_i^n - x_i^{n-1}\big\rangle \\
&\qquad+ \big\|g_i\big(x_i^n\big) - g_i\big(x_i^{n-1}\big)\big\|^2 \\
&\quad\le \big(1 - 2\delta_i + \xi_i^2\big)\big\|x_i^n - x_i^{n-1}\big\|^2.
\end{aligned}
\tag{3.7}
$$

Since A_i is β_i-Lipschitz continuous, F_i is (U_{ii}, c_i, μ_i)-relaxed cocoercive with respect to A_i in the ith argument and F_i is ζ_{ij}-Lipschitz continuous in the jth argument, then we have

$$
\begin{aligned}
&\big\|A_i\big(g_i\big(x_i^n\big)\big) - A_i\big(g_i\big(x_i^{n-1}\big)\big) - \rho_i\big[F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^n,u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big)\big]\big\|^2 \\
&\quad= \big\|A_i\big(g_i\big(x_i^n\big)\big) - A_i\big(g_i\big(x_i^{n-1}\big)\big)\big\|^2 \\
&\qquad- 2\rho_i\big\langle F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^n,u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big), A_i\big(g_i\big(x_i^n\big)\big) - A_i\big(g_i\big(x_i^{n-1}\big)\big)\big\rangle \\
&\qquad+ \rho_i^2\big\|F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^n,u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big)\big\|^2 \\
&\quad\le \beta_i^2\big\|g_i\big(x_i^n\big) - g_i\big(x_i^{n-1}\big)\big\|^2 \\
&\qquad- 2\rho_i\big[(-c_i)\big\|F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^n,u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big)\big\|^2 \\
&\qquad+ \mu_i\big\|g_i\big(x_i^n\big) - g_i\big(x_i^{n-1}\big)\big\|^2\big] + \rho_i^2\zeta_{ii}^2\big\|u_{ii}^n - u_{ii}^{n-1}\big\|^2 \\
&\quad\le \big(\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2\big)\big\|x_i^n - x_i^{n-1}\big\|^2 + \big(2\rho_i c_i\zeta_{ii}^2 + \rho_i^2\zeta_{ii}^2\big)\big\|u_{ii}^n - u_{ii}^{n-1}\big\|^2.
\end{aligned}
\tag{3.8}
$$

By D_j-γ_{ij}-Lipschitz continuity of the U_{ij} and (3.3), we get

$$
\begin{aligned}
&\big\|F_i\big(u_{i1}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^{n-1},\ldots,u_{ii-1}^{n-1},u_{ii}^{n-1},u_{ii+1}^{n-1},\ldots,u_{im}^{n-1}\big)\big\| \\
&\quad\le \big\|F_i\big(u_{i1}^n,u_{i2}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big) \\
&\qquad- F_i\big(u_{i1}^{n-1},u_{i2}^n,\ldots,u_{ii-1}^n,u_{ii}^{n-1},u_{ii+1}^n,\ldots,u_{im}^n\big)\big\|
\end{aligned}
$$

$$\begin{aligned}
&+ \cdots + \big\| F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n}, u_{ii}^{n-1}, u_{ii+1}^{n}, \ldots, u_{im}^{n}\big) \\
&- F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n}, \ldots, u_{im}^{n}\big)\big\| \\
&+ \big\| F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n}, \ldots, u_{im}^{n}\big) \\
&- F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \ldots, u_{im}^{n}\big)\big\| \\
&+ \cdots + \big\| F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \ldots, u_{im}^{n}\big) \\
&- F_i\big(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{ii-1}^{n-1}, u_{ii}^{n-1}, u_{ii+1}^{n-1}, \ldots, u_{im}^{n-1}\big)\big\| \\
\le{}& \zeta_{i1}\big\|u_{i1}^{n} - u_{i1}^{n-1}\big\| + \cdots + \zeta_{ii-1}\big\|u_{ii-1}^{n} - u_{ii-1}^{n-1}\big\| \\
&+ \zeta_{ii+1}\big\|u_{ii+1}^{n} - u_{ii+1}^{n-1}\big\| + \cdots + \zeta_{im}\big\|u_{im}^{n} - u_{im}^{n-1}\big\| \\
={}& \sum_{j=1, j\neq i}^{m} \zeta_{ij}\big\|u_{ij}^{n} - u_{ij}^{n-1}\big\| \\
\le{}& \sum_{j=1, j\neq i}^{m} \zeta_{ij}\left(1 + \frac{1}{n}\right) D_j\big(U_{ij}\big(x_j^{n}\big), U_{ij}\big(x_j^{n-1}\big)\big) \\
\le{}& \left(1 + \frac{1}{n}\right) \sum_{j=1, j\neq i}^{m} \zeta_{ij}\gamma_{ij}\big\|x_j^{n} - x_j^{n-1}\big\|
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\big\|u_{ii}^{n} - u_{ii}^{n-1}\big\| &\le \left(1 + \frac{1}{n}\right) D_i\big(U_{ii}\big(x_i^{n}\big), U_{ii}\big(x_i^{n-1}\big)\big) \\
&\le \left(1 + \frac{1}{n}\right)\gamma_{ii}\big\|x_i^{n} - x_i^{n-1}\big\|.
\end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\begin{aligned}
&\big\|A_i\big(g_i\big(x_i^{n}\big)\big) - A_i\big(g_i\big(x_i^{n-1}\big)\big) - \rho_i\big[F_i\big(u_{i1}^{n}, \ldots, u_{ii-1}^{n}, u_{ii}^{n}, u_{ii+1}^{n}, \ldots, u_{im}^{n}\big) \\
&\quad - F_i\big(u_{i1}^{n}, \ldots, u_{ii-1}^{n}, u_{ii}^{n-1}, u_{ii+1}^{n}, \ldots, u_{im}^{n}\big)\big]\big\|^2 \\
&\le \Bigg[\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2 \\
&\quad + \left(1 + \frac{1}{n}\right)^2 \gamma_{ii}^2\big(2\rho_i c_i\zeta_{ii}^2 + \rho_i^2\zeta_{ii}^2\big)\Bigg]\big\|x_i^{n} - x_i^{n-1}\big\|^2.
\end{aligned} \tag{3.11}$$

It follows from (3.6)-(3.9), and (3.11), that

$$\begin{aligned}
&\big\|x_i^{n+1} - x_i^{n}\big\| \\
&\le \left(1 - \lambda + \lambda\sqrt{1 - 2\delta_i + \xi_i^2}\right)\big\|x_i^{n} - x_i^{n-1}\big\| \\
&\quad + \frac{\lambda\tau_i}{r_i t_i}\Bigg[\sqrt{\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2 + \big(1 + n^{-1}\big)^2\gamma_{ii}^2\big(2\rho_i c_i\zeta_{ii}^2 + \rho_i^2\zeta_{ii}^2\big)}\,\big\|x_i^{n} - x_i^{n-1}\big\| \\
&\quad + \left(1 + \frac{1}{n}\right)\rho_i \sum_{j=1, j\neq i}^{m} \zeta_{ij}\gamma_{ij}\big\|x_j^{n} - x_j^{n-1}\big\|\Bigg],
\end{aligned}$$

which implies that

$$\begin{aligned}
\sum_{j=1}^{m}\|x_j^{n+1}-x_j^n\| &= \sum_{i=1}^{m}\|x_i^{n+1}-x_i^n\| \\
&\le \sum_{i=1}^{m}\Bigg[\Big(1-\lambda+\lambda\sqrt{1-2\delta_i+\xi_i^2}\Big)\|x_i^n-x_i^{n-1}\| \\
&\quad + \frac{\lambda\tau_i}{r_it_i}\Bigg(\sqrt{\beta_i^2\xi_i^2-2\rho_i\mu_i\delta_i^2+\Big(1+\frac{1}{n}\Big)^2\gamma_{ii}^2\big(2\rho_ic_i\zeta_{ii}^2+\rho_i^2\zeta_{ii}^2\big)}\|x_i^n-x_i^{n-1}\| \\
&\quad + \Big(1+\frac{1}{n}\Big)\rho_i\sum_{j=1,j\neq i}^{m}\zeta_{ij}\gamma_{ij}\|x_j^n-x_j^{n-1}\|\Bigg)\Bigg] \\
&= \sum_{i=1}^{m}\Bigg[\Big(1-\lambda+\lambda\sqrt{1-2\delta_i+\xi_i^2}\Big) \\
&\quad + \frac{\lambda\tau_i}{r_it_i}\sqrt{\beta_i^2\xi_i^2-2\rho_i\mu_i\delta_i^2+\Big(1+\frac{1}{n}\Big)^2\gamma_{ii}^2\big(2\rho_ic_i\zeta_{ii}^2+\rho_i^2\zeta_{ii}^2\big)}\Bigg]\|x_i^n-x_i^{n-1}\| \\
&\quad + \Big(1+\frac{1}{n}\Big)\lambda\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\frac{\rho_i\tau_i\zeta_{ij}\gamma_{ij}}{r_it_i}\|x_j^n-x_j^{n-1}\| \\
&= \sum_{j=1}^{m}\Bigg[\Big(1-\lambda+\lambda\sqrt{1-2\delta_j+\xi_j^2}\Big) \\
&\quad + \frac{\lambda\tau_j}{r_jt_j}\sqrt{\beta_j^2\xi_j^2-2\rho_j\mu_j\delta_j^2+\Big(1+\frac{1}{n}\Big)^2\gamma_{jj}^2\big(2\rho_jc_j\zeta_{jj}^2+\rho_j^2\zeta_{jj}^2\big)}\Bigg]\|x_j^n-x_j^{n-1}\| \\
&\quad + \Big(1+\frac{1}{n}\Big)\lambda\sum_{j=1}^{m}\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}\gamma_{ij}}{r_it_i}\|x_j^n-x_j^{n-1}\| \\
&= \sum_{j=1}^{m}\Bigg[(1-\lambda)+\lambda\Bigg(\sqrt{1-2\delta_j+\xi_j^2} \\
&\quad + \frac{\tau_j}{r_jt_j}\sqrt{\beta_j^2\xi_j^2-2\rho_j\mu_j\delta_j^2+\Big(1+\frac{1}{n}\Big)^2\gamma_{jj}^2\big(2\rho_jc_j\zeta_{jj}^2+\rho_j^2\zeta_{jj}^2\big)}\Bigg) \\
&\quad + \Big(1+\frac{1}{n}\Big)\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}\gamma_{ij}}{r_it_i}\Bigg]\|x_j^n-x_j^{n-1}\| \\
&= \sum_{j=1}^{m}\big[1-\lambda+\lambda\theta_j^n\big]\|x_j^n-x_j^{n-1}\| \le f_n(\lambda)\sum_{j=1}^{m}\|x_j^n-x_j^{n-1}\|,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\theta_j^n &= \frac{\tau_j}{r_jt_j}\sqrt{\beta_j^2\xi_j^2-2\rho_j\mu_j\delta_j^2+\Big(1+\frac{1}{n}\Big)^2\gamma_{jj}^2\big(2\rho_jc_j\zeta_{jj}^2+\rho_j^2\zeta_{jj}^2\big)} \\
&\quad + \sqrt{1-2\delta_j+\xi_j^2}+\Big(1+\frac{1}{n}\Big)\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}\gamma_{ij}}{r_it_i}
\end{aligned}$$

and

$$f_n(\lambda) = \max_{1\le j\le m}\left\{1-\lambda+\lambda\theta_j^n\right\}.$$

By condition (3.5), we know that sequence $\{\theta_j^n\}$ is monotone decreasing and $\theta_j^n \to \theta_j$ as $n\to\infty$. Thus,

$$f(\lambda) = \lim_{n\to\infty} f_n(\lambda) = \max_{1\le j\le m}\{1-\lambda+\lambda\theta_j\}.$$

Since $0<\theta_j<1$ for $j=1,2,\ldots,m$, we get $\theta = \max_{1\le j\le m}\{\theta_j\}\in(0,1)$, by Lemma 2.3, we have $f(\lambda)=1-\lambda+\lambda\theta\in(0,1)$. From (3.12), it follows that $\{x_j^n\}$ is a Cauchy sequence and there exists $x_j^*\in H_j$ such that $x_j^n\to x_j^*$ as $n\to\infty$ for $j=1,2,\ldots,m$.

Next, we show that $u_{ij}^n \to u_{ij}^*\in U_{ij}(x_j^*)$ as $n\to\infty$ for $i,j=1,2,\ldots,m$.

It follows from (3.9) and (3.10) that $\{u_{ij}^n\}$ are also Cauchy sequences. Hence, there exists $u_{ij}^*\in H_j$ such that $u_{ij}^n\to u_{ij}^*$ as $n\to\infty$ for $i,j=1,2,\ldots,m$. Furthermore,

$$\begin{aligned} d\bigl(u_{ij}^*, U_{ij}\bigl(x_j^*\bigr)\bigr) &= \inf\bigl\{\bigl\|u_{ij}^*-t\bigr\| : t\in U_{ij}\bigl(x_j^*\bigr)\bigr\} \\ &\le \bigl\|u_{ij}^*-u_{ij}^n\bigr\| + d\bigl(u_{ij}^n, U_{ij}\bigl(x_j^*\bigr)\bigr) \\ &\le \bigl\|u_{ij}^*-u_{ij}^n\bigr\| + D_j\bigl(U_{ij}\bigl(x_j^n\bigr), U_{ij}\bigl(x_j^*\bigr)\bigr) \\ &\le \bigl\|u_{ij}^*-u_{ij}^n\bigr\| + \gamma_{ij}\bigl\|x_j^n-x_j^*\bigr\| \to 0 \quad (n\to\infty). \end{aligned}$$

Since $U_{ij}(x_j^*)$ is closed for $i,j=1,2,\ldots,m$, we have $u_{ij}^*\in U_{ij}(x_j^*)$ for $i,j=1,2,\ldots,m$. Using continuity, $(x_1^*,x_2^*,\ldots,x_m^*)\in H_1\times H_2\times\cdots\times H_m$ and $u_{ij}^*\in U_{ij}(x_j^*)$ for $i,j=1,2,\ldots,m$ satisfy (3.1) and so in light of Lemma 3.1, $(*)$ is a solution to the problem (1.1). This completes the proof. □

Remark 3.3 If the generalized resolvent operator $R_{M_i,\rho_i}^{A_i,\eta_i}$ reduces to $J_\rho^{\varphi_i} = (I+\rho\partial\varphi_i)^{-1}$, where $\varphi_i : H_i \to R\cup\{+\infty\}$ is proper and lower semi-continuous η_i-subdifferentiable functional, $H_i = H$ for $i=1,2,\ldots,m$, $\lambda=1$ and (U_{ii},c_i,μ_i)-relaxed cocoerciveness with respect to A_i in the ith argument of F_i reduces to μ_i-(U_{ii},A_i)-strongly monotonicity (right now, $c_i=0$, $A_i\equiv g_i$), then Theorem 3.1 reduces to Theorem 3.1 of Cao [33].

Theorem 3.2 *Assume that η_i, A_i, g_i, M_i are the same as in the Theorem* 3.1 *for $i=1,2,\ldots,m$. Suppose that $T_{ij}: H_j\to H_j$ is γ_{ij}-Lipschitz continuous, $F_i : H_1\times H_2\times\cdots\times H_m\to H_i$ is (T_{ii},c_i,μ_i)-relaxed cocoercive with respect to A_i in the ith argument and ζ_{ij}-Lipschitz continuous in the jth for $i,j=1,2,\ldots,m$. If there exists constant $\rho_i>0$ for such that*

$$\begin{aligned} \theta_j = {}& \frac{\tau_j}{r_j t_j}\cdot\sqrt{\beta_j^2\xi_j^2 - 2\rho_j\mu_j\delta_j^2 + 2\rho_j c_j\zeta_{jj}^2\gamma_{jj}^2 + \rho_j^2\zeta_{jj}^2\gamma_{jj}^2} \\ &+ \sqrt{1-2\delta_j+\xi_j^2} + \sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}\gamma_{ij}}{r_i t_i} < 1 \end{aligned}$$

for $j=1,2,\ldots,m$, then the problem (1.2) *has a unique solution $(x_1^*,x_2^*,\ldots,x_m^*)\in H_1\times H_2\times\cdots\times H_m$. Moreover, the iterative sequences $\{x_j^n\}$ generated by Algorithm* 3.2 *strongly converge to x_j^* for $j=1,2,\ldots,m$.*

Proof Define the norm $\|\cdot\|_*$ on product space $H_1 \times H_2 \times \cdots \times H_m$ by

$$\|(x_1, x_2, \ldots, x_m)\|_* = \sum_{j=1}^{m} \|x_j\|, \quad \forall (x_1, x_2, \ldots, x_m) \in H_1 \times H_2 \times \cdots \times H_m.$$

It is easy to see that $(H_1 \times H_2 \times \cdots \times H_m, \|\cdot\|_*)$ is a Banach space. Set

$$\begin{aligned} y_i = {} & x_i - g_i(x_i) + R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i(g_i(x_i)) \\ & - \rho_i F_i(T_{i1}x_1, \ldots, T_{ii-1}x_{i-1}, T_{ii}x_i, T_{ii+1}x_{i+1}, \ldots, T_{im}x_m)\bigr]. \end{aligned}$$

Let $G : H_1 \times H_2 \times \cdots \times H_m \to H_1 \times H_2 \times \cdots \times H_m$ be defined by

$$G(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots, y_m), \quad \forall (x_1, x_2, \ldots, x_m) \in H_1 \times H_2 \times \cdots \times H_m.$$

For any $(x_1^1, x_2^1, \ldots, x_m^1), (x_1^2, x_2^2, \ldots, x_m^2) \in H_1 \times H_2 \times \cdots \times H_m$, it follows from Lemma 2.2 that

$$\begin{aligned} & \bigl\|G(x_1^1, x_2^1, \ldots, x_m^1) - G(x_1^2, x_2^2, \ldots, x_m^2)\bigr\|_* \\ & \quad = \sum_{i=1}^{m} \|y_i^1 - y_i^2\| \\ & \quad \le \sum_{i=1}^{m} \bigl\{ \bigl\|x_i^1 - x_i^2 - \bigl(g_i(x_i^1) - g_i(x_i^2)\bigr)\bigr\| + \bigl\|R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i\bigl(g_i(x_i^1)\bigr) \\ & \qquad - \rho_i F_i\bigl(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\bigr)\bigr] \\ & \qquad - R_{M_i,\rho_i}^{A_i,\eta_i}\bigl[A_i\bigl(g_i(x_i^2)\bigr) \\ & \qquad - \rho_i F_i\bigl(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\bigr)\bigr]\bigr\| \bigr\} \\ & \quad \le \sum_{i=1}^{m} \biggl\{ \bigl\|x_i^1 - x_i^2 - \bigl(g_i(x_i^1) - g_i(x_i^2)\bigr)\bigr\| \\ & \qquad + \frac{\tau_i}{r_i t_i} \bigl\|A_i\bigl(g_i(x_i^1)\bigr) - A_i\bigl(g_i(x_i^2)\bigr) \\ & \qquad - \rho_i\bigl[F_i\bigl(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\bigr) \\ & \qquad - F_i\bigl(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^2, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\bigr)\bigr]\bigr\| \\ & \qquad + \frac{\tau_i \rho_i}{r_i t_i} \bigl\|F_i\bigl(T_{i1}x_1^1, \ldots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^2, T_{ii+1}x_{i+1}^1, \ldots, T_{im}x_m^1\bigr) \\ & \qquad - F_i\bigl(T_{i1}x_1^2, \ldots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \ldots, T_{im}x_m^2\bigr)\bigr\| \biggr\}. \end{aligned} \tag{3.13}$$

By ξ_i-Lipschitz continuity and δ_i-strongly monotonicity of g_i, we get

$$\bigl\|x_i^1 - x_i^2 - \bigl(g_i(x_i^1) - g_i(x_i^2)\bigr)\bigr\| \le \sqrt{1 - 2\delta_i + \xi_i^2}\,\bigl\|x_i^1 - x_i^2\bigr\|. \tag{3.14}$$

Since A_i is β_i-Lipschitz continuous, F_i is (T_{ii}, c_i, μ_i)-relaxed cocoercive with respect to A_i in the ith argument and F_i is ζ_{ij}-Lipschitz continuous in the jth argument and $T_{ij} : H_j \to H_j$

is γ_{ij}-Lipschitz continuous, then we have

$$
\begin{aligned}
&\big\|A_i\big(g_i\big(x_i^1\big)\big) - A_i\big(g_i\big(x_i^2\big)\big) \\
&\quad - \rho_i\big[F_i\big(T_{i1}x_1^1, \dots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \dots, T_{im}x_m^1\big) \\
&\quad - F_i\big(T_{i1}x_1^1, \dots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^2, T_{ii+1}x_{i+1}^1, \dots, T_{im}x_m^1\big)\big]\big\|^2 \\
&\leq \beta_i^2\big\|g_i\big(x_i^1\big) - g_i\big(x_i^2\big)\big\|^2 \\
&\quad - 2\rho_i\big[(-c_i)\big\|F_i\big(T_{i1}x_1^1, \dots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^1, T_{ii+1}x_{i+1}^1, \dots, T_{im}x_m^1\big) \\
&\quad - F_i\big(T_{i1}x_1^1, \dots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^2, T_{ii+1}x_{i+1}^1, \dots, T_{im}x_m^1\big)\big\|^2 \\
&\quad + \mu_i\big\|g_i\big(x_i^1\big) - g_i\big(x_i^2\big)\big\|^2\big] + \rho_i^2\zeta_{ii}^2\big\|T_{ii}x_i^1 - T_{ii}x_i^2\big\|^2 \\
&\leq \big(\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2\big)\big\|x_i^1 - x_i^2\big\|^2 + \big(2\rho_ic_i\zeta_{ii}^2 + \rho_i^2\zeta_{ii}^2\big)\big\|T_{ii}x_i^1 - T_{ii}x_i^2\big\|^2 \\
&\leq \big(\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2 + 2\rho_ic_i\zeta_{ii}^2\gamma_{ii} + \rho_i^2\zeta_{ii}^2\gamma_{ii}\big)\big\|x_i^1 - x_i^2\big\|^2
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&\big\|F_i\big(T_{i1}x_1^1, \dots, T_{ii-1}x_{i-1}^1, T_{ii}x_i^2, T_{ii+1}x_{i+1}^1, \dots, T_{im}x_m^1\big) \\
&\quad - F_i\big(T_{i1}x_1^2, \dots, T_{ii-1}x_{i-1}^2, T_{ii}x_i^2, T_{ii+1}x_{i+1}^2, \dots, T_{im}x_m^2\big)\big\| \\
&\leq \zeta_{i1}\big\|T_{i1}x_1^1 - T_{i1}x_1^2\big\| + \cdots + \zeta_{ii-1}\big\|T_{ii-1}x_{i-1}^1 - T_{ii-1}x_{i-1}^2\big\| \\
&\quad + \zeta_{ii+1}\big\|T_{ii+1}x_{i+1}^1 - T_{ii+1}x_{ii+1}^2\big\| + \cdots + \zeta_{im}\big\|T_{im}x_m^1 - T_{im}x_m^2\big\| \\
&= \sum_{j=1, j\neq i}^{m} \zeta_{ij}\big\|T_{ij}x_j^1 - T_{ij}x_j^2\big\| \\
&\leq \sum_{j=1, j\neq i}^{m} \zeta_{ij}\gamma_{ij}\big\|x_j^1 - x_j^2\big\|.
\end{aligned}
\tag{3.16}
$$

From (3.13)-(3.16), we have

$$
\begin{aligned}
&\big\|G\big(x_1^1, x_2^1, \dots, x_m^1\big) - G\big(x_1^2, x_2^2, \dots, x_m^2\big)\big\|_* \\
&\leq \sum_{i=1}^{m}\bigg(\sqrt{1 - 2\delta_i + \xi_i^2} \\
&\quad + \frac{\tau_i}{r_it_i}\sqrt{\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2 + 2\rho_ic_i\zeta_{ii}^2\gamma_{ii}^2 + \rho_i^2\zeta_{ii}^2\gamma_{ii}^2}\bigg)\big\|x_i^1 - x_i^2\big\| \\
&\quad + \sum_{j=1, j\neq i}^{m}\frac{\rho_j\tau_j\zeta_{ij}\gamma_{ij}}{r_jt_j}\big\|x_j^1 - x_j^2\big\| \\
&= \sum_{j=1}^{m}\theta_j\big\|x_j^1 - x_j^2\big\| \\
&\leq \theta\sum_{j=1}^{m}\big\|x_j^1 - x_j^1\big\| \\
&= \theta\big\|\big(x_1^1, x_2^1, \dots, x_m^1\big) - \big(x_1^2, x_2^2, \dots, x_m^2\big)\big\|_*,
\end{aligned}
$$

where $\theta = \max_{1\leq j\leq m}\theta_j$. It follows from assumption (3.5) that $0 < \theta < 1$. This shows that $G : H_1 \times H_2 \times \cdots \times H_m \to H_1 \times H_2 \times \cdots \times H_m$ is a contractive operator, and so there exists a unique $(x_1^*, x_2^*, \ldots, x_m^*) \in H_1 \times H_2 \times \cdots \times H_m$ such that $G(x_1^*, x_2^*, \ldots, x_m^*) = (x_1^*, x_2^*, \ldots, x_m^*)$. Thus, $(x_1^*, x_2^*, \ldots, x_m^*)$ is the unique solution of the problem (1.2).

Now we prove that $x_i^n \to x_i^*$ as $n \to \infty$ for $i = 1, 2, \ldots, m$. In fact, it follows from (3.4) and Lemma 2.2 that

$$\begin{aligned}
&\|x_i^{n+1} - x^*\| \\
&\quad\leq \|x_i^n - x_i^* - (g_i(x_i^n) - g_i(x_i^*))\| \\
&\qquad + \|R_{M_i,\rho_i}^{A_i,\eta_i}[A_i(g_i(x_i^n)) \\
&\qquad - \rho_i F_i(T_{i1}x_1^n, \ldots, T_{ii-1}x_{i-1}^n, T_{ii}x_i^n, T_{ii+1}x_{i+1}^n, \ldots, T_{im}x_m^n)] \\
&\qquad - R_{M_i,\rho_i}^{A_i,\eta_i}[A_i(g_i(x_i^*)) \\
&\qquad - \rho_i F_i(T_{i1}x_1^*, \ldots, T_{ii-1}x_{i-1}^*, T_{ii}x_i^*, T_{ii+1}x_{i+1}^*, \ldots, T_{im}x_m^*)]\| + \|w_i^n\| \\
&\quad\leq \|x_i^n - x_i^* - (g_i(x_i^n) - g_i(x_i^*))\| + \|w_i^n\| \\
&\qquad + \frac{\tau_i}{r_i t_i}\|A_i(g_i(x_i^n)) - A_i(g_i(x_i^*)) \\
&\qquad - \rho_i[F_i(T_{i1}x_1^n, \ldots, T_{ii-1}x_{i-1}^n, T_{ii}x_i^n, T_{ii+1}x_{i+1}^n, \ldots, T_{im}x_m^n) \\
&\qquad - F_i(T_{i1}x_1^n, \ldots, T_{ii-1}x_{i-1}^n, T_{ii}x_i^*, T_{ii+1}x_{i+1}^n, \ldots, T_{im}x_m^n)]\| \\
&\qquad + \frac{\tau_i\rho_i}{r_i t_i}\|F_i(T_{i1}x_1^n, \ldots, T_{ii-1}x_{i-1}^n, T_{ii}x_i^*, T_{ii+1}x_{i+1}^n, \ldots, T_{im}x_m^n) \\
&\qquad - F_i(T_{i1}x_1^*, \ldots, T_{ii-1}x_{i-1}^*, T_{ii}x_i^*, T_{ii+1}x_{i+1}^*, \ldots, T_{im}x_m^*)\|.
\end{aligned} \tag{3.17}$$

Following very similar arguments from (3.14)-(3.16), we have

$$\begin{aligned}
&\|x_i^{n+1} - x_i^*\| \\
&\quad\leq \sqrt{1 - 2\delta_i + \xi_i^2}\,\|x_i^n - x_i^*\| \\
&\qquad + \frac{\tau_i}{r_i t_i}\Bigg[\sqrt{\beta_i^2\xi_i^2 - 2\rho_i\mu_i\delta_i^2 + 2\rho_i c_i\zeta_{ii}^2\gamma_{ii}^2 + \rho_i^2\zeta_{ii}^2\gamma_{ii}^2}\,\|x_i^n - x_i^*\| \\
&\qquad + \rho_i\sum_{j=1, j\neq i}^m \zeta_{ij}\gamma_{ij}\|x_j^n - x_j^*\|\Bigg] + \|w_i^n\|,
\end{aligned} \tag{3.18}$$

which implies that

$$\begin{aligned}
\sum_{j=1}^m \|x_j^{n+1} - x_j^*\| &= \sum_{j=1}^m \theta_j\|x_j^n - x_j^*\| + \sum_{j=1}^m \|w_j^n\| \\
&\leq \theta\sum_{j=1}^m \|x_j^n - x_j^*\| + \sum_{j=1}^m \|w_j^n\|,
\end{aligned}$$

where $a_n = \sum_{j=1}^m \|x_j^n - x_j^*\|$, $b_n = \sum_{j=1}^m \|w_j^n\|$. The condition of Algorithm 3.2 yields $\lim_{n\to\infty} b_n = 0$. Now Lemma 2.4 implies that $\lim_{n\to\infty} a_n = 0$, and so $x_j^n \to x_j^*$ as $n \to \infty$ for $j = 1, 2, \ldots, m$. This completes the proof. □

Remark 3.4 If $m = 2$, $g_1 = g_2 = U_{11} = U_{22} \equiv I$ (right now, $\delta_i = \xi_i = \zeta_{ii} = 1$ for $i = 1, 2$), then Theorem 3.1 reduces to Theorem 4.5 based on Algorithm 4.3 of Agarwal and Verma [34]. Our presented results improve and extend some known results in the literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
TX carried out the proof of the corollaries and gave some examples to show the main results. HL conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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