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An explicit algorithm for solving the optimize hierarchical problems

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Abstract

In this paper, we consider the variational inequality problem over the generalized mixed equilibrium problem which has a hierarchical structure. Strong convergence of the algorithm to the unique solution is guaranteed under some assumptions.

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Keywords: nonexpansive; strong convergence; variational inequality; fixed point; hierarchical problem

1 Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. We denote weak convergence and strong convergence by the notations $\rightharpoonup$ and $\rightarrow$, respectively. Let $A : C \rightarrow H$ be a nonlinear mapping and let F be a bifunction of $C \times C$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Consider the *generalized mixed equilibrium problem* which is to find $x \in C$ such that

$$F(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $GMEP(F,\varphi,A)$. See [1–4].

If $\varphi \equiv 0$, the problem (1.1) is reduced to the *generalized equilibrium problem* which is to find $x \in C$ such that

$$F(x,y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $GEP(F,A)$.

If $A \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced to the *equilibrium problem* [5] which is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of (1.3) is denoted by $EP(F)$.

If $F \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced to the *Hartmann-Stampacchia variational inequality* [6] which is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The solution set of (1.4) is denoted by $VI(C,A)$.

A mapping $T: C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty [7]. A point $x \in H$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in H : Tx = x\}$.

We discuss the following variational inequality problem over the generalized mixed equilibrium problem, which is called the *hierarchical problem over the generalized mixed equilibrium problem*, which is to find a point $x \in GMEP(F, \varphi, B)$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in GMEP(F, \varphi, B),$$

where A, B are two monotone operators. See [8, 9].

A mapping $A: C \to C$ is called α*-strongly monotone* if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2$ for all $x, y \in C$. A mapping $A: C \to C$ is called L*-Lipschitz continuous* if there exists a positive real number L such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$. A linear bounded operator A is called *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ with the property $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$ for all $x \in H$. A mapping $f: C \to H$ is called a ρ*-contraction* if there exists a constant $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in C$.

In 2010, Yao *et al.* [10] considered the hierarchical problem over the generalized equilibrium problem, $x_{s,t}$ being defined by implicit algorithms:

$$x_{s,t} = s\big[tf(x_{s,t}) + (1 - t)(x_{s,t} - \lambda Ax_{s,t})\big] + (1 - s)T_r(x_{s,t} - rBx_{s,t}), \quad s, t \in (0, 1), \tag{1.5}$$

for each $(s, t) \in (0, 1)^2$. The net $x_{s,t}$ hierarchically converges to the unique solution x^* of the problem of the variational inequality which is to find a point $x^* \in GEP(F, B)$ such that

$$\big\langle Ax^*, x - x^*\big\rangle \geq 0, \quad \forall x \in GEP(F, B), \tag{1.6}$$

where A, B are two monotone operators. The solution set of (1.6) is denoted by Ω. Furthermore, x^* also solves the following variational inequality:

$$x^* \in \Omega, \quad \big\langle (I - f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in \Omega.$$

In 2011, Yao *et al.* [11] studied the hierarchical problem over the fixed point set. Let the sequence $\{x_n\}$ be generated by two algorithms as follows.

Implicit Algorithm: $x_t = TP_C[I - t(A - \gamma f)]x_t$, $\forall t \in (0, 1)$ and

Explicit Algorithm: $x_{n+1} = \beta_n x_n + (1 - \beta_n)TP_C[I - \alpha_n(A - \gamma f)]x_n$, $\forall n \geq 0$.

They showed that these two algorithms converge strongly to the unique solution of the problem of the variational inequality which is to find $x^* \in F(T)$ such that

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in F(T),$$

where $A: C \to H$ is a strongly positive linear bounded operator, $f: C \to H$ is a ρ-contraction, and $T: C \to C$ is a nonexpansive mapping.

In this paper, we construct an algorithm and introduce the *hierarchical problem over the generalized mixed equilibrium problem.* The sequence $\{x_n\}$ is generated by the algorithm for $x_0 \in C$,

$$x_{n+1} = \alpha_n\big(\beta_n x_n + (1-\beta_n)P_C\big[I - \lambda_n(A - \gamma f)\big]x_n\big) + (1-\alpha_n)T_{r_n}(I - r_n B)x_n, \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subset [0,1]$, and $r_n \in (0, 2\beta)$ satisfy some conditions. Then $\{x_n\}$ converges strongly to $x^* \in GMEP(F, \varphi, B)$, which is the unique solution of the variational inequality:

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in GMEP(F, \varphi, B). \tag{1.8}$$

Our results improve the results of Yao *et al.* [10], Yao *et al.* [11] and some other authors.

2 Preliminaries

Let C be a nonempty closed convex subset of H. We have the following inequality in an inner product space: $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in H$. For every point $x \in H$, there exists a unique *nearest point* in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, y - P_C x\rangle \leq 0, \\ &\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \end{aligned} \tag{2.1}$$

for all $x \in H$, $y \in C$. Let B be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.1) implies the following:

$$u \in VI(C, B) \quad \Leftrightarrow \quad u = P_C(u - \lambda Bu), \quad \lambda > 0.$$

It is also well known that H satisfies the Opial condition [12], *i.e.*, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, holds for every $y \in H$ with $x \neq y$.

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ, and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;
(A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.2}$$

(B2) C is a bounded set.

Lemma 2.1 [13] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)-(A5) *and let $\varphi : C \to \mathcal{R}$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{2.3}$$

for all $x \in H$. Assume that either (B1) *or* (B2) *holds. Then the following results hold*:
(1) *for each $x \in H$, $T_r(x) \neq \emptyset$*;
(2) *T_r is single-valued*;
(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$*;
(4) *$F(T_r) = MEP(F, \varphi)$*;
(5) *$MEP(F, \varphi)$ is closed and convex.*

Lemma 2.2 [14] *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.3 [15] *Assume A is a self adjoint and strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4 [16] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ are sequences in $\mathcal{R}$ such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Strong convergence theorems

In this section, we introduce an explicit algorithm for solving some hierarchical problem over the set of fixed points of a nonexpansive and the generalized mixed equilibrium problem.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : H \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $B : C \to H$ be β-inverse-strongly monotone and F be a bifunction from $C \times C \to \mathcal{R}$ satisfying* (A1)-(A5) *and let $\varphi : C \to \mathcal{R}$ be convex* *and lower semicontinuous with either* (B1) *or* (B2). *Let* $\{x_n\}$ *be a sequence generated by the following algorithm for arbitrary* $x_0 \in C$:

$$x_{n+1} = \alpha_n\big(\beta_n x_n + (1-\beta_n)P_C\big[I - \lambda_n(A-\gamma f)\big]x_n\big) + (1-\alpha_n)T_{r_n}(I - r_n B)x_n, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subset [0,1]$, $\alpha_n \le \lambda_n$, *and* $r_n \in (0, 2\beta)$ *satisfy the following conditions*:

(C1) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(C2) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;
(C3) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$;
(C4) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *converges strongly to* $x^* \in GMEP(F,\varphi,B)$, *which is the unique solution of the variational inequality*:

$$\big\langle (A-\gamma f)x^*, x - x^*\big\rangle \ge 0, \quad \forall x \in GMEP(F,\varphi,B). \tag{3.2}$$

Proof We will divide the proof into five steps.

Step 1. We will show $\{x_n\}$ is bounded. For any $q \in GMEP(F,\varphi,B)$ and taking $y_n = P_C[I - \lambda_n(A-\gamma f)]x_n$, we note that

$$\begin{aligned} \|y_n - q\| &= \big\|P_C\big[I - \lambda_n(A-\gamma f)\big]x_n - P_C q\big\| \\ &\le \big\|\big[I - \lambda_n(A-\gamma f)\big]x_n - q\big\| \\ &\le \lambda_n\big\|\gamma f(x_n) - \gamma f(q)\big\| + \lambda_n\big\|\gamma f(q) - Aq\big\| + |I - \lambda_n A|\|x_n - q\| \\ &\le \lambda_n\gamma\rho\|x_n - q\| + \lambda_n\big\|\gamma f(q) - Aq\big\| + (1 - \lambda_n\bar{\gamma})\|x_n - q\| \\ &= \big[1 - (\bar{\gamma} - \gamma\rho)\lambda_n\big]\|x_n - q\| + \lambda_n\big\|\gamma f(q) - Aq\big\|. \end{aligned} \tag{3.3}$$

From (3.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \big\|\alpha_n\big\{\beta_n x_n + (1-\beta_n)y_n\big\} + (1-\alpha_n)T_{r_n}(I - r_n B)x_n - q\big\| \\ &\le \alpha_n\big\|\beta_n x_n + (1-\beta_n)y_n - q\big\| + (1-\alpha_n)\big\|T_{r_n}(I - r_n B)x_n - q\big\| \\ &\le \alpha_n\beta_n\|x_n - q\| + \alpha_n(1-\beta_n)\|y_n - q\| \\ &\quad + (1-\alpha_n)\big\|T_{r_n}(I - r_n B)x_n - T_{r_n}(I - r_n B)q\big\| \\ &\le \alpha_n\beta_n\|x_n - q\| + \alpha_n(1-\beta_n)\big\{\big[1 - (\bar{\gamma} - \gamma\rho)\lambda_n\big]\|x_n - q\| + \lambda_n\big\|\gamma f(q) - Aq\big\|\big\} \\ &\quad + (1-\alpha_n)\|x_n - q\| \\ &= \alpha_n\beta_n\|x_n - q\| + \alpha_n(1-\beta_n)\big[1 - (\bar{\gamma} - \gamma\rho)\lambda_n\big]\|x_n - q\| \\ &\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(q) - Aq\big\| + (1-\alpha_n)\|x_n - q\| \\ &= \big[1 - \alpha_n(1-\beta_n)\big]\|x_n - q\| + \alpha_n(1-\beta_n)\big[1 - (\bar{\gamma} - \gamma\rho)\lambda_n\big]\|x_n - q\| \\ &\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(q) - Aq\big\| \\ &= \big\{1 - \alpha_n(1-\beta_n)\big(1 - \big[1 - (\bar{\gamma} - \gamma\rho)\lambda_n\big]\big)\big\}\|x_n - q\| \\ &\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(q) - Aq\big\| \end{aligned}$$

$$
\begin{aligned}
&= \big[1-\alpha_n(1-\beta_n)\lambda_n(\bar{\gamma}-\gamma\rho)\big]\|x_n-q\| + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(q)-Aq\big\| \\
&= \big[1-\alpha_n(1-\beta_n)\lambda_n(\bar{\gamma}-\gamma\rho)\big]\|x_n-q\| \\
&\quad + \alpha_n(1-\beta_n)\lambda_n(\bar{\gamma}-\gamma\rho)\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho}.
\end{aligned}
$$

It follows by induction that

$$\|x_n-q\| \le \max\left\{\|x_0-q\|, \frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho}\right\}, \quad \forall n\ge 0.$$

Therefore $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{Ax_n\}$, and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Setting $v_n = [I-\lambda_n(A-\gamma f)]x_n$ and we observe that

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|P_C v_{n+1} - P_C v_n\| \\
&\le \big\|\big[I-\lambda_{n+1}(A-\gamma f)\big]x_{n+1} - \big[I-\lambda_n(A-\gamma f)\big]x_n\big\| \\
&= \big\|\lambda_{n+1}\gamma\big[f(x_{n+1})-f(x_n)\big] + (\lambda_{n+1}-\lambda_n)\gamma f(x_n) + (I-\lambda_{n+1}A)(x_{n+1}-x_n) \\
&\quad +(\lambda_{n+1}-\lambda_n)Ax_n\big\| \\
&\le \lambda_{n+1}\gamma\big\|f(x_{n+1})-f(x_n)\big\| + (1-\lambda_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big) \\
&\le \lambda_{n+1}\gamma\rho\|x_{n+1}-x_n\| + (1-\lambda_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\big(\big\|\gamma f(x_n)\big\| + \|Ax_n\|\big) \\
&= \big[1-(\bar{\gamma}-\gamma\rho)\lambda_{n+1}\big]\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|M_1,
\end{aligned} \tag{3.4}
$$

where $M_1 = \sup\{\|\gamma f(x_n)\| + \|Ax_n\| : n\in\mathbb{N}\}$. Setting $z_n = \beta_n x_n + (1-\beta_n)y_n$ for all $n\ge 0$. We observes that

$$
\begin{aligned}
\|z_{n+1}-z_n\| &= \big\|\beta_{n+1}x_{n+1} + (1-\beta_{n+1})y_{n+1} - \big(\beta_n x_n + (1-\beta_n)y_n\big)\big\| \\
&\le \beta_{n+1}\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_n-y_n\| + |1-\beta_{n+1}|\|y_{n+1}-y_n\|.
\end{aligned} \tag{3.5}
$$

Substituting (3.4) into (3.5) it follows that

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\le \beta_{n+1}\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_n-y_n\| \\
&\quad + |1-\beta_{n+1}|\big\{\big[1-(\bar{\gamma}-\gamma\rho)\lambda_{n+1}\big]\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|M_1\big\} \\
&\le \beta_{n+1}\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_n-y_n\| \\
&\quad + \big[1-\beta_{n+1}-(1-\beta_{n+1})(\bar{\gamma}-\gamma\rho)\lambda_{n+1}\big]\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|M_1 \\
&= \big[1-(1-\beta_{n+1})(\bar{\gamma}-\gamma\rho)\lambda_{n+1}\big]\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|M_2 \\
&\quad +|\lambda_{n+1}-\lambda_n|M_1,
\end{aligned} \tag{3.6}
$$

where $M_2 = \sup\{\|x_n - y_n\| : n \in \mathbb{N}\}$. On the other hand, from $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1}Bx_{n-1})$ and $u_n = T_{r_n}(x_n - r_nBx_n)$ it follows that

$$F(u_{n-1}, y) + \langle Bx_{n-1}, y - u_{n-1}\rangle + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \tag{3.7}$$

and

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$

Substituting $y = u_n$ into (3.7) and $y = u_{n-1}$ into (3.8), we have

$$F(u_{n-1}, u_n) + \langle Bx_{n-1}, u_n - u_{n-1}\rangle + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0$$

and

$$F(u_n, u_{n-1}) + \langle Bx_n, u_{n-1} - u_n\rangle + \varphi(u_{n-1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0.$$

From (A2), we have

$$\left\langle u_n - u_{n-1}, Bx_{n-1} - Bx_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0,$$

and then

$$\left\langle u_n - u_{n-1}, r_{n-1}(Bx_{n-1} - Bx_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0,$$

so

$$\left\langle u_n - u_{n-1}, r_{n-1}Bx_{n-1} - r_{n-1}Bx_n + u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}
&\left\langle u_n - u_{n-1}, (I - r_{n-1}B)x_n - (I - r_{n-1}B)x_{n-1} + u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\right\rangle \geq 0, \\
&\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n)\right\rangle \geq 0.
\end{aligned}$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\|u_n - u_{n-1}\|^2 &\leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n)\right\rangle \\
&\leq \|u_n - u_{n-1}\|\left\{\|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - x_n\|\right\}
\end{aligned}$$

and hence

$$\begin{aligned}\|u_n - u_{n-1}\| &\le \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\le \|x_n - x_{n-1}\| + \frac{M_3}{c}|r_n - r_{n-1}|,\end{aligned} \tag{3.9}$$

where $M_3 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n z_n + (1-\alpha_n)u_n - \alpha_{n-1}z_{n-1} - (1-\alpha_{n-1})u_{n-1}\| \\ &\le \alpha_n\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|z_{n-1} - u_{n-1}\| + |1-\alpha_n|\|u_n - u_{n-1}\| \\ &= \alpha_n\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_4 + |1-\alpha_n|\|u_n - u_{n-1}\|,\end{aligned} \tag{3.10}$$

where $M_4 = \sup\{\|z_n - u_n\| : n \in \mathbb{N}\}$. Substituting (3.6) and (3.9) into (3.10)

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \alpha_n\big\{\big[1 - (1-\beta_n)(\bar{\gamma} - \gamma\rho)\lambda_n\big]\|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|M_2 + |\lambda_n - \lambda_{n-1}|M_1\big\} \\ &\quad + |\alpha_n - \alpha_{n-1}|M_4 + |1-\alpha_n|\left\{\|x_n - x_{n-1}\| + \frac{M_3}{c}|r_n - r_{n-1}|\right\} \\ &\le \big[1 - (1-\beta_n)(\bar{\gamma} - \gamma\rho)\alpha_n\lambda_n\big]\|x_n - x_{n-1}\| + \alpha_n|\beta_n - \beta_{n-1}|M_2 \\ &\quad + \alpha_n|\lambda_n - \lambda_{n-1}|M_1 + |\alpha_n - \alpha_{n-1}|M_4 + \frac{M_3}{c}|r_n - r_{n-1}|,\end{aligned} \tag{3.11}$$

from (C1)-(C4) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, and $\{Ax_n\}$. Applying Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. For each $q \in GMEP(F,\varphi,B)$, note that T_{r_n} is firmly nonexpansive, then we have

$$\begin{aligned}\|u_n - q\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(q - r_nBq)\|^2 \\ &\le \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(q - r_nBq), u_n - q\rangle \\ &= \langle (x_n - r_nBx_n) - (q - r_nBq), u_n - q\rangle \\ &= \frac{1}{2}\big\{\|(x_n - r_nBx_n) - (q - r_nBq)\|^2 + \|u_n - q\|^2 \\ &\quad - \|(x_n - r_nBx_n) - (q - r_nBq) - (u_n - q)\|^2\big\} \\ &\le \frac{1}{2}\big\{\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n - r_n(Bx_n - Bq)\|^2\big\} \\ &\le \frac{1}{2}\big\{\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Bx_n - Bq\rangle - r_n^2\|Bx_n - Bq\|^2\big\},\end{aligned} \tag{3.13}$$

which implies that

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Bx_n - Bq\|. \tag{3.14}$$

From (3.1), we get

$$\begin{aligned}\|y_n - x_n\| &= \big\|P_C\big(I - \lambda_n(A - \gamma f)\big)x_n - P_C x_n\big\| \\ &\le \big\|\big(I - \lambda_n(A - \gamma f)\big)x_n - x_n\big\| \\ &\le \lambda_n\big\|(A - \gamma f)x_n\big\|.\end{aligned}$$

By (C3), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.15}$$

Setting $w_n = [I - \lambda_n(A - \gamma f)]x_n$. It follows that

$$\begin{aligned}\|w_n - x_n\| &= \big\|\big[I - \lambda_n(A - \gamma f)\big]x_n - x_n\big\| \\ &\le \big\|\big[I - \lambda_n(A - \gamma f)\big]x_n - x_n\big\| \\ &\le \lambda_n\big\|(A - \gamma f)x_n\big\|.\end{aligned}$$

By using (C3) again, we get

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.16}$$

From $y_n = P_C[I - \lambda_n(A - \gamma f)]x_n$, we compute

$$\begin{aligned}\|y_n - q\| &= \big\|P_C\big[I - \lambda_n(A - \gamma f)\big]x_n - P_C q\big\| \\ &\le \big\|\big[I - \lambda_n(A - \gamma f)\big]x_n - q\big\| \\ &= \|w_n - q\|.\end{aligned} \tag{3.17}$$

It follows from (3.15) that

$$\|x_n - q\| \le \|w_n - q\|. \tag{3.18}$$

Then we get

$$\begin{aligned}\|w_n - q\|^2 &\le \big\langle\big[I - \lambda_n(A - \gamma f)\big]x_n - q, w_n - q\big\rangle \\ &= \lambda_n\langle\gamma f x_n - Aq, w_n - q\rangle + \big\langle(I - \lambda_n A)(x_n - q), w_n - q\big\rangle \\ &\le \lambda_n\langle\gamma f x_n - Aq, w_n - q\rangle + (1 - \lambda_n\bar{\gamma})\|x_n - q\|\|w_n - q\| \\ &\le (1 - \lambda_n\bar{\gamma})\|w_n - q\|^2 + \lambda_n\langle\gamma f x_n - Aq, w_n - q\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|w_n - q\|^2 &\le \frac{1}{\bar{\gamma}}\langle\gamma f x_n - Aq, w_n - q\rangle \\ &= \frac{1}{\bar{\gamma}}\big[\gamma\langle f x_n - fq, w_n - q\rangle + \langle\gamma fq - Aq, w_n - q\rangle\big] \\ &\le \frac{1}{\bar{\gamma}}\big[\gamma\rho\|w_n - q\|^2 + \big\langle(A - \gamma f)q, q - w_n\big\rangle\big],\end{aligned}$$

that is,

$$\|w_n - q\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle. \tag{3.19}$$

On the other hand, we note that

$$\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(q - r_n Bq)\|^2 \\
&\leq \|(x_n - r_n Bx_n) - (q - r_n Bq)\|^2 \\
&= \|(x_n - q) - r_n(Bx_n - Bq)\|^2 \\
&\leq \|x_n - q\|^2 - 2r_n\langle x_n - q, Bx_n - Bq\rangle + r_n^2\|Bx_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 - 2r_n\beta\|Bx_n - Bq\|^2 + r_n^2\|Bx_n - Bq\|^2.
\end{aligned} \tag{3.20}$$

Using (3.17), (3.18), (3.19), and (3.20), we note that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \alpha_n\beta_n\|x_n - q\|^2 + \alpha_n(1 - \beta_n)\|y_n - q\|^2 + (1 - \alpha_n)\|u_n - q\|^2 \\
&\leq \alpha_n\beta_n\|w_n - q\|^2 + \alpha_n(1 - \beta_n)\|w_n - q\|^2 + (1 - \alpha_n)\|u_n - q\|^2 \\
&= \alpha_n\|w_n - q\|^2 + (1 - \alpha_n)\|u_n - q\|^2 \\
&\leq \frac{\alpha_n}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle \\
&\quad + (1 - \alpha_n)\{\|x_n - q\|^2 - 2r_n\beta\|Bx_n - Bq\|^2 + r_n^2\|Bx_n - Bq\|^2\} \\
&= \frac{\alpha_n}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle \\
&\quad + (1 - \alpha_n)\{\|x_n - q\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bq\|^2\} \\
&\leq \frac{\alpha_n}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle + \|x_n - q\|^2 \\
&\quad + (1 - \alpha_n)r_n(r_n - 2\beta)\|Bx_n - Bq\|^2.
\end{aligned} \tag{3.21}$$

Then we have

$$\begin{aligned}
&(1 - \alpha_n)c(2\beta - d)\|Bx_n - Bq\|^2 \\
&\quad \leq \frac{\alpha_n}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad \leq \frac{\alpha_n}{\bar{\gamma} - \gamma\rho}\langle (A - \gamma f)q, q - w_n\rangle + \|x_n - x_{n+1}\|(\|x_n - q\| + \|x_{n+1} - q\|).
\end{aligned}$$

From (C3), $\{r_n\} \subset [c, d] \subset (0, 2\beta)$, and (3.12), we obtain

$$\lim_{n\to\infty}\|Bx_n - Bq\| = 0. \tag{3.22}$$

Substituting (3.13) into (3.21), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \alpha_n\|w_n - q\|^2 + (1 - \alpha_n)\|u_n - q\|^2 \\
&\leq \alpha_n\|w_n - q\|^2 + (1 - \alpha_n)\{\|x_n - q\|^2 - \|x_n - u_n\|^2
\end{aligned}$$

$$\begin{aligned} &+ 2r_n\|x_n - u_n\|\|Bx_n - Bq\|\Big\} \\ &\le \alpha_n\|w_n - q\|^2 + \|x_n - q\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 \\ &\quad + 2r_n(1-\alpha_n)\|x_n - u_n\|\|Bx_n - Bq\|, \end{aligned}$$

and it follows that

$$\begin{aligned} (1-\alpha_n)\|x_n - u_n\|^2 &\le \alpha_n\|w_n - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + 2r_n(1-\alpha_n)\|x_n - u_n\|\|Bx_n - Bq\| \\ &\le \alpha_n\|w_n - q\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - q\| + \|x_{n+1} - q\|\big) \\ &\quad + 2r_n(1-\alpha_n)\|x_n - u_n\|\|Bx_n - Bq\|. \end{aligned}$$

Since we have (C3), (3.12), and (3.22),

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.23}$$

By (C4), we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.24}$$

Step 4. Next, we will show that

$$\limsup_{n\to\infty}\big\langle(\gamma f - A)x^*, x_n - x^*\big\rangle \le 0.$$

Indeed, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle(\gamma f - A)x^*, x_n - x^*\big\rangle = \lim_{i\to\infty}\big\langle(\gamma f - A)x^*, x_{n_i} - x^*\big\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $z \in C$. We notice that $\|w_n - x_n\| \le \lambda_n\|(A - \gamma f)x_n\| \to 0$. Hence, we get $\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle \le 0$. Next, we will show that $z \in GMEP(F,\varphi,B)$. Since $u_n = T_{r_n}(x_n - r_nBx_n)$, we have

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad \forall y \in C,$$

and hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C. \tag{3.25}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$. So, from (3.25), we have

$$\begin{aligned}
\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Bx_{n_i}\rangle \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weak lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have in the limit

$$\langle y_t - w, By_t\rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w) \tag{3.26}$$

as $i \to \infty$. From (A1), (A4) and (3.26), we also get

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tF(y_t, y) + (1-t)F(y_t, z) + t\varphi(y) - (1-t)\varphi(z) - \varphi(y_t) \\
&= t\bigl[F(y_t, y) + \varphi(y) - \varphi(y_t)\bigr] + (1-t)\bigl[F(y_t, z) + \varphi(z) - \varphi(y_t)\bigr] \\
&\leq t\bigl[F(y_t, y) + \varphi(y) - \varphi(y_t)\bigr] + (1-t)\langle y_t - z, By_t\rangle \\
&= t\bigl[F(y_t, y) + \varphi(y) - \varphi(y_t)\bigr] + (1-t)t\langle y - z, By_t\rangle, \\
0 &\leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - z, By_t\rangle.
\end{aligned}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle \geq 0.$$

This implies that $z \in GMEP(F, \varphi, B)$. It is easy to see that $P_{GMEP(F,\varphi,B)}(I - A + \gamma f)(x^*)$ is a contraction of H into itself. Hence H is complete, there exists a unique fixed point $x^* \in H$, such that $x^* = P_{GMEP(F,\varphi,B)}(I - A + \gamma f)(x^*)$.

Step 5. Next, we will prove $x_n \to x^* \in GMEP(F, \varphi, B)$, which solves the variational inequality (1.8). It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \alpha_n\beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n(1-\beta_n)\bigl\langle P_C\bigl[I - \lambda_n(A - \gamma f)\bigr]x_n - P_C\bigl[I - \lambda_n(A - \gamma f)\bigr]x^*, x_{n+1} - x^*\bigr\rangle \\
&\quad + \alpha_n(1-\beta_n)\bigl\langle P_C\bigl[I - \lambda_n(A - \gamma f)\bigr]x^* - x^*, x_{n+1} - x^*\bigr\rangle \\
&\quad + (1-\alpha_n)\bigl\langle T_{r_n}(I - r_nB)x_n - T_{r_n}(I - r_nB)x^*, x_{n+1} - x^*\bigr\rangle \\
&\leq \alpha_n\beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n(1-\beta_n)\bigl\langle\bigl[I - \lambda_n(A - \gamma f)\bigr]x_n - \bigl[I - \lambda_n(A - \gamma f)\bigr]x^*, x_{n+1} - x^*\bigr\rangle
\end{aligned}$$

$$\begin{aligned}
&\quad + \alpha_n(1-\beta_n)\big\langle \big[I - \lambda_n(A-\gamma f)\big]x^* - x^*, x_{n+1}-x^*\big\rangle \\
&\quad + (1-\alpha_n)\big\| T_{r_n}(I-r_nB)x_n - T_{r_n}(I-r_nB)x^*\big\| \|x_{n+1}-x^*\| \\
&\le \alpha_n\beta_n\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\big\{\big[1-(\bar{\gamma}-\gamma\rho)\lambda_n\big]\|x_n-x^*\| \\
&\quad + \lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\big\}\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle + (1-\alpha_n)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&= \alpha_n\beta_n\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\alpha_n)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\big[1-(\bar{\gamma}-\gamma\rho)\lambda_n\big]\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle \\
&= \big[1-\alpha_n(1-\beta_n)\big]\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\big[1-(\bar{\gamma}-\gamma\rho)\lambda_n\big]\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle \\
&= \big[1-\alpha_n(1-\beta_n)\big[1-1+(\bar{\gamma}-\gamma\rho)\lambda_n\big]\big]\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle \\
&= \big[1-\alpha_n(1-\beta_n)(\bar{\gamma}-\gamma\rho)\lambda_n\big]\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle \\
&\le \frac{1-\alpha_n(1-\beta_n)(\bar{\gamma}-\gamma\rho)\lambda_n}{2}\big(\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2\big) \\
&\quad + \alpha_n(1-\beta_n)\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n(1-\beta_n)\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle \\
&\le \frac{1-(1-\beta_n)(\bar{\gamma}-\gamma\rho)\alpha_n\lambda_n}{2}\|x_n-x^*\|^2 + \frac{1}{2}\|x_{n+1}-x^*\|^2 \\
&\quad + (1-\beta_n)\alpha_n\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - (1-\beta_n)\alpha_n\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \big[1-(1-\beta_n)(\bar{\gamma}-\gamma\rho)\alpha_n\lambda_n\big]\|x_n-x^*\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\lambda_n\big\|\gamma f(x^*) - Ax^*\big\|\|x_{n+1}-x^*\| \\
&\quad - 2(1-\beta_n)\alpha_n\lambda_n\big\langle (A-\gamma f)x^*, x_{n+1}-x^*\big\rangle.
\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{Ax_n\}$ are all bounded, we can choose a constant $M > 0$ such that

$$\sup \frac{1}{\bar{\gamma} - \gamma\rho}\left\{2\left\|\gamma f\left(x^*\right) - Ax^*\right\|\left\|x_{n+1} - x^*\right\| + 2\left\langle (A - \gamma f)x^*, x_{n+1} - x^*\right\rangle\right\} \leq M.$$

It follows that

$$\left\|x_{n+1} - x^*\right\|^2 \leq \left[1 - (1 - \beta_n)(\bar{\gamma} - \gamma\rho)\alpha_n\lambda_n\right]\left\|x_n - x^*\right\|^2 + (1 - \beta_n)\alpha_n\lambda_n M.$$

By (C3), we conclude that $x_n \to x^*$, as $n \to \infty$. This completes the proof. □

4 An example

Next, the following example shows that all conditions of Theorem 3.1 are satisfied.

Example 4.1 For instance, let $\alpha_n = \frac{n+1}{n^2+1}$, $\beta_n = \frac{1}{n+1}$, $\lambda_n = \frac{1}{2(n+1)}$, and $r_n = \frac{n}{n+1}$. Then clearly the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following condition:

$$\frac{n+1}{n^2+1} \leq \frac{1}{2(n+1)}.$$

We will show that the condition (C1) is fulfilled. Indeed, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| &= \sum_{n=1}^{\infty}\left|\frac{n+1}{n^2+1} - \frac{n}{(n-1)^2+1}\right| \\
&= \sum_{n=1}^{\infty}\left|\frac{(n+1)(n^2-2n+2) - n(n^2+1)}{(n^2+1)(n^2-2n+2)}\right| \\
&= \sum_{n=1}^{\infty}\left|\frac{2+n-n^2}{n^4-2n^3+3n^2-2n+2}\right|.
\end{aligned}$$

The sequence $\{\alpha_n\}$ satisfies the condition (C1) by a p-series.

Next, we will show that the condition (C2) is fulfilled. We compute

$$\begin{aligned}
\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| &= \sum_{n=1}^{\infty}\left|\frac{1}{n+1} - \frac{1}{n}\right| \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\
&= 1.
\end{aligned}$$

The sequence $\{\beta_n\}$ satisfies the condition (C2).

Next, we will show that the condition (C3) is fulfilled. We compute

$$\begin{aligned}
\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| &= \sum_{n=1}^{\infty}\left|\frac{1}{2(n+1)} - \frac{1}{2n}\right| \\
&= \left(\frac{1}{2\cdot 1} - \frac{1}{2\cdot 2}\right) + \left(\frac{1}{2\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{2\cdot 3} - \frac{1}{2\cdot 4}\right) + \cdots \\
&= \frac{1}{2},
\end{aligned}$$

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{1}{2(n+1)} = 0,$$

and

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \frac{1}{2(n+1)} = \infty.$$

The sequence $\{\lambda_n\}$ satisfies the condition (C3).

Finally, we will show that the condition (C4) is fulfilled. We compute

$$\begin{aligned}\sum_{n=1}^{\infty} |r_n - r_{n-1}| &= \sum_{n=1}^{\infty} \left| \frac{n}{n+1} - \frac{n-1}{(n-1)+1} \right| \\ &= \sum_{n=1}^{\infty} \left| \frac{n(n) - (n-1)(n+1)}{(n+1)n} \right| \\ &= \sum_{n=1}^{\infty} \left| \frac{n^2 - n^2 + 1}{(n+1)n} \right| \\ &= \sum_{n=1}^{\infty} \left| \frac{1}{n(n+1)} \right|\end{aligned}$$

and

$$\liminf_{n\to\infty} r_n = \liminf_{n\to\infty} \frac{n}{n+1} = 1.$$

The sequence $\{r_n\}$ satisfies the condition (C4).

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$ and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm for arbitrary $x_0 \in C$:*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T P_C \bigl[I - \lambda_n (A - \gamma f)\bigr] x_n, \tag{4.1}$$

where $\{\beta_n\}, \{\lambda_n\} \subset [0,1]$ satisfy the following conditions:

(C1) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$.

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of the variational inequality:

$$\bigl\langle (A - \gamma f)x^*, x - x^* \bigr\rangle \geq 0, \quad \forall x \in F(T). \tag{4.2}$$

Proof Setting $\{\alpha_n\} \equiv 1$ and T to be a nonexpansive mapping in Theorem 3.1, we obtain the desired conclusion immediately. □

Remark 4.3 Corollary 4.2 generalizes and improves the results of Yao *et al.* [11].

Corollary 4.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a strongly positive linear bounded operator, $B : C \to H$ be a β-inverse-strongly monotone and F be a bifunction from $C \times C \to \mathcal{R}$ satisfying* (A1)-(A5) *and let $\varphi : C \to \mathcal{R}$ be convex and lower semicontinuous with either* (B1) *or* (B2). *Suppose $GMEP(F,\varphi,B) \neq \emptyset$. Let $\{x_n\}$ be a sequence by the following algorithm for arbitrary $x_0 \in C$:*

$$x_{n+1} = \alpha_n\big(\beta_n x_n + (1-\beta_n)[I-\lambda_n A]x_n\big) + (1-\alpha_n)T_{r_n}(I-r_nB)x_n, \tag{4.3}$$

where $\{\alpha_n\},\{\beta_n\},\{\lambda_n\} \subset [0,1]$, $\alpha_n \leq \lambda_n$, and $r_n \in (0,2\beta)$ satisfy the following conditions:

(C1) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(C2) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;
(C3) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$;
(C4) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in GMEP(F,\varphi,B)$, which is the unique solution of the variational inequality:

$$\big\langle Ax^*, x - x^*\big\rangle \geq 0, \quad \forall x \in GMEP(F,\varphi,B). \tag{4.4}$$

Proof Setting T, P_C to be the identity and $f \equiv 0$ in Theorem 3.1, we obtain the desired conclusion immediately. □

Corollary 4.5 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be ρ-contraction, $B : C \to H$ be β-inverse-strongly monotone and F be a bifunction from $C \times C \to \mathcal{R}$ satisfying* (A1)-(A5) *and let $\varphi : C \to \mathcal{R}$ be convex and lower semicontinuous with either* (B1) *or* (B2). *Let $\{x_n\}$ be a sequence generated by the following algorithm for arbitrary $x_0 \in C$:*

$$x_{n+1} = \alpha_n\big(\lambda_n(1-\beta_n)f(x_n) + \big[I - \lambda_n(1-\beta_n)A\big]x_n\big) + (1-\alpha_n)T_{r_n}(I-r_nB)x_n, \tag{4.5}$$

where $\{\alpha_n\},\{\beta_n\},\{\lambda_n\} \subset [0,1]$, $\lambda_n \leq \beta_n$, and $r_n \in (0,2\beta)$ satisfy the following conditions:

(C1) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C2) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$;
(C4) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in GMEP(F,\varphi,B)$, which is the unique solution of the variational inequality

$$\big\langle (A-f)x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in GMEP(F,\varphi,B). \tag{4.6}$$

Proof Setting T, P_C to be the identity and $\gamma \equiv 1$ in Theorem 3.1, we obtain the desired conclusion immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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