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Integral inequalities with 'maxima' and their applications to Hadamard type fractional differential equations

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Abstract

In this paper, some new integral inequalities with 'maxima' are established involving Hadamard integral. Applications to Hadamard fractional differential equations with 'maxima' are also presented.

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Keywords: integral inequality; differential equations with 'maxima'; fractional differential equations

1 Introduction

It is well known that integral inequalities play a dominant role in the study of quantitative properties of solutions of differential and integral equations [1–5]. Fractional inequalities are important in studying the existence, uniqueness, and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivatives; see [6–9] and the references therein. In [10, 11], the authors established some weakly singular integral inequalities of Gronwall-Bellman type and also applied them in the qualitative analysis of solutions to certain fractional differential equations of the Caputo type.

Another kind of fractional derivative that appears in the literature is the fractional derivative due to Hadamard, introduced in 1892 [12], which differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral contains a logarithmic function of an arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [13–18]. Recently in the literature there appeared some results on fractional integral inequalities using the Hadamard fractional integral; see [19–22].

Let us recall here the definitions of Hadamard's fractional integral and derivative [23].

Definition 1.1 The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$, is defined as

$$ {}_{\mathrm{H}}J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0^+}^{t} \left(\log \frac{t}{s} \right)^{\alpha-1} f(s) \frac{ds}{s}, \tag{1.1} $$

where Γ is the standard gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1}\, ds$, provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 1.2 The Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$, of a function $f(t)$ is given by

$$ {}_{\mathrm{H}}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_{0^+}^{t} \left(\log \frac{t}{s} \right)^{n-\alpha-1} f(s) \frac{ds}{s}. \tag{1.2} $$

Differential equations with 'maxima' are a special type of differential equations that contain the maximum of the unknown function over a previous interval. Several integral inequalities have been established in the case when the maxima of the unknown scalar function are involved in the integral; see [24, 25] and references cited therein.

Recently in [26] some new types of integral inequalities on time scales with 'maxima' have been established, which can be used as a handy tool in the investigation of making estimates for bounds of solutions of dynamic equations on time scales with 'maxima'. In this paper we establish some new integral inequalities with 'maxima' involving Hadamard's integral. The significance of our work lies in the fact that 'maxima' are taken on intervals $[\beta t, t]$ which have non-constant lengths, where $0 < \beta < 1$. Most papers take the 'maxima' on $[t-h, t]$, where $h > 0$ is a given constant.

The paper is organized as follows: in Section 2 we recall some results from [26] in the special case $\mathbb{T} = \mathbb{R}$, used to prove our main results, which are presented in Section 3. In Section 4 we give applications of our results for a Hadamard fractional differential equation with 'maxima'.

2 Preliminaries

For convenience we let $t_0 > 0$ throughout. The following results in Lemmas 2.1 and 2.2 are obtained by reducing the time scale $\mathbb{T} = \mathbb{R}$, $f(t) = g(t) \equiv 1$, and $a(t) = b(t) \equiv 0$ for all $t \in (t_0, T)$ in Theorems 3.3 and 3.2 ([26], p.8 and p.6), respectively.

Lemma 2.1 ([26]) *Let the following conditions be satisfied*:

(H_1) *The functions p and $q \in C((t_0, T), \mathbb{R}_+)$.*
(H_2) *The function $\phi \in C([\beta t_0, T), \mathbb{R}_+)$ with $\max_{s \in [\beta t_0, t_0]} \phi(s) > 0$, where $0 < \beta < 1$.*
(H_3) *The function $u \in C([\beta t_0, T), \mathbb{R}_+)$ and satisfies the inequalities*

$$ u(t) \le \phi(t) + \int_{t_0}^{t} \left\{ p(s) u(s) + q(s) \max_{\xi \in [\beta s, s]} u(\xi) \right\} ds, \quad t \in (t_0, T), $$
$$ u(t) \le \phi(t), \quad t \in [\beta t_0, t_0]. $$

Then

$$ u(t) \le \phi(t) + h(t) \exp\left(\int_{t_0}^{t} \left\{ p(s) + q(s) \right\} ds \right), \quad t \in (t_0, T), $$

holds, *where*

$$ h(t) = \max_{s \in [\beta t_0, t_0]} \phi(s) + \int_{t_0}^{t} \left\{ p(s)\phi(s) + q(s) \max_{\xi \in [\beta s, s]} \phi(\xi) \right\} ds, \quad t \in (t_0, T). $$

By splitting the initial function ϕ into two functions, we deduce the following corollary.

Corollary 2.1 *Let the following conditions be satisfied*:

(H_4) *The functions* p, q, *and* $v \in C((t_0, T), \mathbb{R}_+)$.

(H_5) *The function* $w \in C([\beta t_0, t_0], \mathbb{R}_+)$ *with* $\max_{s\in[\beta t_0,t_0]} w(s) > 0$ *and* $w(t_0) = v(t_0)$, *where* $0 < \beta < 1$.

(H_6) *The function* $u \in C([\beta t_0, T), \mathbb{R}_+)$ *and satisfies the inequalities*

$$u(t) \leq v(t) + \int_{t_0}^{t} \Big\{ p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi) \Big\}\, ds, \quad t \in (t_0, T),$$

$$u(t) \leq w(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \leq v(t) + h(t) \exp\left(\int_{t_0}^{t} \big\{ p(s) + q(s) \big\}\, ds \right), \quad t \in (t_0, T),$$

holds, where

$$h(t) = \max_{s\in[\beta t_0,t_0]} w(s) + \int_{t_0}^{t} \Big\{ p(s)v(s) + q(s) \max_{\xi\in[\beta s,s]} m(\xi) \Big\}\, ds, \quad t \in (t_0, T),$$

with

$$m(t) = \begin{cases} v(t), & t \in (t_0, T), \\ w(t), & t \in [\beta t_0, t_0]. \end{cases}$$

Lemma 2.2 ([26]) *Let the condition* (H_1) *of Lemma* 2.1 *is satisfied. In addition, assume that*:

(H_7) *The function* $k \in C((t_0, T), (0, \infty))$ *is nondecreasing.*

(H_8) *The function* $\phi \in C([\beta t_0, t_0), \mathbb{R}_+)$, *where* $0 < \beta < 1$.

(H_9) *The function* $u \in C([\beta t_0, T), \mathbb{R}_+)$ *and satisfies the inequalities*

$$u(t) \leq k(t) + \int_{t_0}^{t} \Big\{ p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi) \Big\}\, ds, \quad t \in (t_0, T),$$

$$u(t) \leq \phi(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \leq Nk(t) \exp\left(\int_{t_0}^{t} \big\{ p(s) + q(s) \big\}\, ds \right), \quad t \in (t_0, T),$$

holds, where

$$N = \max\left\{ 1, \frac{\max_{s\in[\beta t_0,t_0]} \phi(s)}{k(t_0)} \right\}.$$

The following lemma is a consequence of Jensen's inequality, which can be found in [27].

Lemma 2.3 ([27]) *Let $n \in N$, and let $x_1, \ldots, x_n$ be non-negative real numbers. Then for $\sigma > 1$,*

$$\left(\sum_{i=1}^{n} x_i\right)^{\sigma} \leq n^{\sigma-1} \sum_{i=1}^{n} x_i^{\sigma}.$$

3 Main results

Theorem 3.1 *Suppose that the following conditions are satisfied*:

(A_1) *The functions p and $r \in C((t_0, T), \mathbb{R}_+)$.*
(A_2) *The function $\phi \in C([\beta t_0, t_0], \mathbb{R}_+)$ with $\max_{s \in [\beta t_0, t_0]} \phi(s) > 0$, where $0 < \beta < 1$.*
(A_3) *The function $u \in C([\beta t_0, T), \mathbb{R}_+)$ with*

$$u(t) \leq r(t) + \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} p(s) \max_{\xi \in [\beta s, s]} u(\xi) \frac{ds}{s}, \quad t \in (t_0, T), \tag{3.1}$$

$$u(t) \leq \phi(t), \quad t \in [\beta t_0, t_0], \tag{3.2}$$

where $\alpha > 0$.

Then the following assertions hold:

(i) *Suppose $\alpha > \frac{1}{2}$, then*

$$u(t) \leq t\left[c_1 r^2(t) + h_1(t) \exp\left(\frac{2\Gamma(2\alpha-1)}{t} \int_{t_0}^{t} p^2(s)\, ds\right)\right]^{\frac{1}{2}}, \quad t \in (t_0, T), \tag{3.3}$$

where

$$c_1 = \max\left\{2t_0^{-2}, (\beta t_0)^{-2}\right\} \tag{3.4}$$

and

$$\begin{aligned} h_1(t) = c_1 \max_{s \in [\beta t_0, t_0]} \phi^2(s) + \frac{2c_1 \Gamma(2\alpha-1)}{t} \\ \times \int_{t_0}^{t} p^2(s) \max_{\xi \in [\beta s, s]} m_1^2(\xi)\, ds, \quad t \in (t_0, T), \end{aligned} \tag{3.5}$$

with

$$m_1(t) = \begin{cases} r(t), & t \in (t_0, T), \\ \phi(t), & t \in [\beta t_0, t_0]. \end{cases} \tag{3.6}$$

In addition, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then

$$u(t) \leq \sqrt{c_1 N_1}\, t r(t) \exp\left(\frac{\Gamma(2\alpha-1)}{t} \int_{t_0}^{t} p^2(s)\, ds\right), \quad t \in (t_0, T), \tag{3.7}$$

where

$$N_1 = \max\left\{1, \frac{\max_{s \in [\beta t_0, t_0]} \phi^2(s)}{r^2(t_0)}\right\}. \tag{3.8}$$

(ii) *Suppose* $0 < \alpha \leq \frac{1}{2}$, *then*

$$u(t) \leq t\left[c_2 r^b(t) + h_2(t)\exp\left(\frac{(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t}\int_{t_0}^t p^b(s)\,ds\right)\right]^{\frac{1}{b}}, \quad t \in (t_0, T), \tag{3.9}$$

where $b = 1 + \frac{1}{\alpha}$,

$$c_2 = \max\left\{2^{\frac{1}{\alpha}} t_0^{-b}, (\beta t_0)^{-b}\right\} \tag{3.10}$$

and

$$h_2(t) = c_2 \max_{s\in[\beta t_0, t_0]} \phi^b(s) + \frac{c_2(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \times \int_{t_0}^t p^b(s) \max_{\xi\in[\beta s, s]} m_1^b(\xi)\,ds, \quad t \in (t_0, T). \tag{3.11}$$

Moreover, if $r \in C((t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \leq (c_2 N_2)^{\frac{1}{b}} t r(t) \exp\left(\frac{(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{bt}\int_{t_0}^t p^b(s)\,ds\right), \quad t \in (t_0, T), \tag{3.12}$$

where

$$N_2 = \max\left\{1, \frac{\max_{s\in[\beta t_0, t_0]} \phi^b(s)}{r^b(t_0)}\right\}. \tag{3.13}$$

Proof (i) $\alpha > \frac{1}{2}$. For $t \in (t_0, T)$, by using the Cauchy-Schwarz inequality in (3.1), we get

$$u(t) \leq r(t) + \left\{\int_{t_0}^t \left(\log\frac{t}{s}\right)^{2\alpha-2} ds\right\}^{\frac{1}{2}} \times \left\{\int_{t_0}^t p^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2 \frac{ds}{s^2}\right\}^{\frac{1}{2}}. \tag{3.14}$$

It is easy to observe that

$$\int_{t_0}^t \left(\log\frac{t}{s}\right)^{2\alpha-2} ds = t\int_0^{\log\frac{t}{t_0}} \tau^{2\alpha-2}e^{-\tau}\,d\tau < \Gamma(2\alpha-1)t. \tag{3.15}$$

Substituting (3.15) in (3.14), we obtain

$$u(t) \leq r(t) + \left(\Gamma(2\alpha-1)t\right)^{\frac{1}{2}}\left\{\int_{t_0}^t p^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2\frac{ds}{s^2}\right\}^{\frac{1}{2}}.$$

Applying Lemma 2.3 with $n = 2$, $\sigma = 2$, we get the estimate

$$u^2(t) \leq 2r^2(t) + 2\Gamma(2\alpha-1)t\int_{t_0}^t p^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2\frac{ds}{s^2}, \quad t \in (t_0, T).$$

Setting $v(t) = t^{-2}u^2(t)$, we have, for $t \in (t_0, T)$,

$$\begin{aligned} v(t) &\le 2t^{-2}r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^2 \frac{ds}{s^2} \\ &\le 2t_0^{-2}r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]}\big(\xi^{-2}u^2(\xi)\big)\,ds \\ &\le c_1 r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]} v(\xi)\,ds, \end{aligned} \tag{3.16}$$

and for $t \in [\beta t_0, t_0]$,

$$v(t) \le t^{-2}\phi^2(t) \le (\beta t_0)^{-2}\phi^2(t) \le c_1\phi^2(t). \tag{3.17}$$

A suitable application of Corollary 2.1 for (3.16) and (3.17) leads to

$$v(t) \le c_1 r^2(t) + h_1(t)\exp\bigg(\frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\,ds\bigg), \quad t \in (t_0, T),$$

where c_1 and h_1 are defined by (3.4) and (3.5), respectively. Therefore, we obtain the desired bound in (3.3).

Now, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then, by Lemma 2.2 with (3.16) and (3.17), it follows that

$$v(t) \le c_1 N_1 r^2(t)\exp\bigg(\frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\,ds\bigg), \quad t \in (t_0, T),$$

where N_1 is defined by (3.8). Thus, we get the inequality in (3.7). This completes the proof of the first part.

(ii) $0 < \alpha \le \frac{1}{2}$. Let $a = 1 + \alpha$ and $b = 1 + \frac{1}{\alpha}$. It is obvious that $\frac{1}{a} + \frac{1}{b} = 1$. Using the Hölder inequality in (3.1), for $t \in (t_0, T)$, we obtain

$$u(t) \le r(t) + \bigg\{\int_{t_0}^{t}\Big(\log\frac{t}{s}\Big)^{a(\alpha-1)} ds\bigg\}^{\frac{1}{a}}\bigg\{\int_{t_0}^{t} p^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b \frac{ds}{s^b}\bigg\}^{\frac{1}{b}}. \tag{3.18}$$

For the first integral in (3.18), repeating the process to get (3.15), we have

$$\int_{t_0}^{t}\Big(\log\frac{t}{s}\Big)^{a(\alpha-1)} ds < \Gamma\big(1 - a(1-\alpha)\big)t. \tag{3.19}$$

Obviously, $1 - a(1-\alpha) = \alpha^2 > 0$ and $\Gamma(1 - a(1-\alpha)) \in \mathbb{R}$. Substituting (3.19) in (3.18), we get

$$u(t) \le r(t) + \big(\Gamma(\alpha^2)t\big)^{\frac{1}{a}}\bigg\{\int_{t_0}^{t} p^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b \frac{ds}{s^b}\bigg\}^{\frac{1}{b}}.$$

Applying Lemma 2.3 with $n = 2$, $\sigma = b$, we get the following estimate:

$$\begin{aligned} u^b(t) &\le 2^{b-1}r^b(t) + 2^{b-1}\big(\Gamma(\alpha^2)t\big)^{\frac{b}{a}}\int_{t_0}^{t} p^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b \frac{ds}{s^b} \\ &= 2^{\frac{1}{\alpha}}r^b(t) + \big(2\Gamma(\alpha^2)t\big)^{\frac{1}{\alpha}}\int_{t_0}^{t} p^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b \frac{ds}{s^b}, \quad t \in (t_0, T). \end{aligned}$$

By taking $v(t) = t^{-b}u^b(t)$, we have

$$v(t) \le c_2 r^b(t) + \frac{(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \int_{t_0}^t p^b(s) \max_{\xi\in[\beta s,s]} v(\xi)\,ds, \quad t \in (t_0, T) \tag{3.20}$$

and

$$v(t) \le c_2 \phi^b(t), \quad t \in [\beta t_0, t_0]. \tag{3.21}$$

An application of Corollary 2.1 to (3.20) and (3.21) yields

$$v(t) \le c_2 r^b(t) + h_2(t) \exp\left(\frac{(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \int_{t_0}^t p^b(s)\,ds\right), \quad t \in (t_0, T),$$

where c_2 and h_2 are defined by (3.10) and (3.11), respectively. Thus, we get the required inequality in (3.9).

Furthermore, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then, by applying Lemma 2.2 to (3.21) and (3.22), we get

$$v(t) \le c_2 N_2 r^b(t) \exp\left(\frac{(2\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \int_{t_0}^t p^b(s)\,ds\right), \quad t \in (t_0, T),$$

where N_2 is defined by (3.13). Therefore, the desired inequality (3.12) is proved. This completes the proof. □

Theorem 3.2 *Suppose that the conditions* (A_1) *and* (A_2) *of Theorem* 3.1 *are satisfied. In addition we assume that*:

(A_4) *The function* $q \in C((t_0, T), \mathbb{R}_+)$.
(A_5) *The function* $u \in C([\beta t_0, T), \mathbb{R}_+)$ *with*

$$u(t) \le r(t) + \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left\{p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi)\right\} \frac{ds}{s}, \quad t \in (t_0, T), \tag{3.22}$$

$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0], \tag{3.23}$$

where $\alpha > 0$.

Then the following assertions hold:

(a) *Suppose* $\alpha > \frac{1}{2}$, *then*

$$u(t) \le t\left\{c_3 r^2(t) + h_3(t) \exp\left(\frac{3\Gamma(2\alpha-1)}{t} \int_{t_0}^t \left\{p^2(s) + q^2(s)\right\} ds\right)\right\}^{\frac{1}{2}},$$
$$t \in (t_0, T), \tag{3.24}$$

where

$$c_3 = \max\left\{3t_0^{-2}, (\beta t_0)^{-2}\right\} \tag{3.25}$$

and

$$h_3(t) = c_3 \max_{s\in[\beta t_0,t_0]} \phi^2(s) + \frac{3c_3\Gamma(2\alpha-1)}{t} \times \int_{t_0}^{t} \Bigl\{p^2(s)r^2(s) + q^2(s) \max_{\xi\in[\beta s,s]} m_1^2(\xi)\Bigr\}\,ds, \quad t\in(t_0,T), \tag{3.26}$$

with m_1 *is defined by* (3.6).

Furthermore, if $r\in C((t_0,T),(0,\infty))$ *is a nondecreasing function, then*

$$u(t) \le \sqrt{c_3N_1}\,tr(t)\exp\biggl(\frac{3\Gamma(2\alpha-1)}{2t}\int_{t_0}^{t}\bigl\{p^2(s)+q^2(s)\bigr\}\,ds\biggr), \quad t\in(t_0,T), \tag{3.27}$$

where N_1 *is defined by* (3.8).

(b) *Suppose* $0<\alpha\le\frac{1}{2}$, *then*

$$u(t) \le t\biggl\{c_4r^b(t) + h_4(t)\exp\biggl(\frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t}\int_{t_0}^{t}\bigl\{p^b(s)+q^b(s)\bigr\}\,ds\biggr)\biggr\}^{\frac{1}{b}}, \quad t\in(t_0,T), \tag{3.28}$$

where $b = 1+\frac{1}{\alpha}$,

$$c_4 = \max\bigl\{3^{\frac{1}{\alpha}}t_0^{-b},(\beta t_0)^{-b}\bigr\} \tag{3.29}$$

and

$$h_4(t) = c_4 \max_{s\in[\beta t_0,t_0]} \phi^b(s) + \frac{c_4(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \times \int_{t_0}^{t} \Bigl\{p^b(s)r^b(s) + q^b(s) \max_{\xi\in[\beta s,s]} m_1^b(\xi)\Bigr\}\,ds, \quad t\in(t_0,T). \tag{3.30}$$

In addition, if $r\in C((t_0,T),(0,\infty))$ *is a nondecreasing function, then*

$$u(t) \le (c_4N_2)^{\frac{1}{b}}\,tr(t)\exp\biggl(\frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{bt}\int_{t_0}^{t}\bigl\{p^b(s)+q^b(s)\bigr\}\,ds\biggr), \quad t\in(t_0,T), \tag{3.31}$$

where N_2 *is defined by* (3.13).

Proof (a) $\alpha>\frac{1}{2}$. By using the Cauchy-Schwarz inequality in (3.22), for $t\in(t_0,T)$, we have

$$\begin{aligned} u(t) &\le r(t) + \biggl\{\int_{t_0}^{t}\Bigl(\log\frac{t}{s}\Bigr)^{2\alpha-2}ds\biggr\}^{\frac{1}{2}}\biggl\{\int_{t_0}^{t}p^2(s)u^2(s)\frac{ds}{s^2}\biggr\}^{\frac{1}{2}} \\ &\quad + \biggl\{\int_{t_0}^{t}\Bigl(\log\frac{t}{s}\Bigr)^{2\alpha-2}ds\biggr\}^{\frac{1}{2}}\biggl\{\int_{t_0}^{t}q^2(s)\Bigl(\max_{\xi\in[\beta s,s]}u(\xi)\Bigr)^2\frac{ds}{s^2}\biggr\}^{\frac{1}{2}} \\ &\le r(t) + \bigl(\Gamma(2\alpha-1)t\bigr)^{\frac{1}{2}}\biggl\{\biggl(\int_{t_0}^{t}p^2(s)u^2(s)\frac{ds}{s^2}\biggr)^{\frac{1}{2}} \\ &\quad + \biggl(\int_{t_0}^{t}q^2(s)\Bigl(\max_{\xi\in[\beta s,s]}u(\xi)\Bigr)^2\frac{ds}{s^2}\biggr)^{\frac{1}{2}}\biggr\}. \end{aligned}$$

By applying Lemma 2.3 with $n = 3$, $\sigma = 2$, we get

$$u^2(t) \le 3r^2(t) + 3\Gamma(2\alpha-1)t\left\{\int_{t_0}^t p^2(s)u^2(s)\frac{ds}{s^2} + \int_{t_0}^t q^2(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^2\frac{ds}{s^2}\right\},\quad t\in(t_0,T).$$

Setting $v(t) = t^{-2}u^2(t)$, we obtain

$$v(t) \le c_3 r^2(t) + \frac{3\Gamma(2\alpha-1)}{t}\left\{\int_{t_0}^t p^2(s)v(s)\,ds + \int_{t_0}^t q^2(s)\max_{\xi\in[\beta s,s]} v(\xi)\,ds\right\},\quad t\in(t_0,T) \tag{3.32}$$

and

$$v(t) \le c_3\phi^2(t),\quad t\in[\beta t_0,t_0]. \tag{3.33}$$

Using Corollary 2.1 for (3.32) and (3.33), it follows that

$$v(t) \le c_3 r^2(t) + h_3(t)\exp\left(\frac{3\Gamma(2\alpha-1)}{t}\int_{t_0}^t \{p^2(s)+q^2(s)\}\,ds\right),\quad t\in(t_0,T),$$

where c_3 and h_3 are defined by (3.25) and (3.26), respectively. Therefore, we get the desired inequality in (3.24).

As a special case, if $r \in C((t_0,T),(0,\infty))$ is a nondecreasing function, then by applying Lemma 2.2 with (3.32) and (3.33), we have

$$v(t) \le c_3 N_1 r^2(t)\exp\left(\frac{3\Gamma(2\alpha-1)}{t}\int_{t_0}^t \{p^2(s)+q^2(s)\}\,ds\right),\quad t\in(t_0,T),$$

where N_1 is defined by (3.8). Thus, we get the required inequality in (3.27). This completes the proof of the first part.

(b) $0 < \alpha \le \frac{1}{2}$. Let $a = 1+\alpha$ and $b = 1+\frac{1}{\alpha}$. Using the Hölder inequality in (3.22), for $t\in(t_0,T)$, we obtain

$$\begin{aligned} u(t) &\le r(t) + \left\{\int_{t_0}^t \left(\log\frac{t}{s}\right)^{a(\alpha-1)} ds\right\}^{\frac{1}{a}}\left\{\left(\int_{t_0}^t p^b(s)u^b(s)\frac{ds}{s^b}\right)^{\frac{1}{b}} + \left(\int_{t_0}^t q^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b\frac{ds}{s^b}\right)^{\frac{1}{b}}\right\} \\ &\le r(t) + \left(\Gamma\left(\alpha^2\right)t\right)^{\frac{1}{a}}\left\{\left(\int_{t_0}^t p^b(s)u^b(s)\frac{ds}{s^b}\right)^{\frac{1}{b}} + \left(\int_{t_0}^t q^b(s)\Big(\max_{\xi\in[\beta s,s]} u(\xi)\Big)^b\frac{ds}{s^b}\right)^{\frac{1}{b}}\right\}. \end{aligned}$$

By applying Lemma 2.3 with $n = 3$, $\sigma = b$, we get

$$u^b(t) \le 3^{\frac{1}{\alpha}} r^b(t) + \big(3\Gamma(\alpha^2)t\big)^{\frac{1}{\alpha}} \left\{ \int_{t_0}^{t} p^b(s) u^b(s) \frac{ds}{s^b} + \int_{t_0}^{t} q^b(s) \Big(\max_{\xi \in [\beta s, s]} u(\xi) \Big)^b \frac{ds}{s^b} \right\}, \quad t \in (t_0, T).$$

Taking $v(t) = t^{-b} u^b(t)$, it follows that

$$v(t) \le c_4 r^b(t) + \frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \left\{ \int_{t_0}^{t} p^b(s) v(s)\, ds + \int_{t_0}^{t} q^b(s) \max_{\xi \in [\beta s, s]} v(\xi)\, ds \right\}, \quad t \in (t_0, T) \tag{3.34}$$

and

$$v(t) \le c_4 \phi^b(t), \quad t \in [\beta t_0, t_0]. \tag{3.35}$$

Applying Corollary 2.1 for (3.34) and (3.35), we have the following estimate:

$$v(t) \le c_4 r^b(t) + h_4(t) \exp\left(\frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \int_{t_0}^{t} \big\{ p^b(s) d + q^b(s) \big\}\, ds \right), \quad t \in (t_0, T),$$

where c_4 and h_4 are defined by (3.29) and (3.30), respectively. Hence, the result (3.28) is proved.

As a special case, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then by using Lemma 2.2 with (3.34) and (3.35), we get

$$v(t) \le c_4 N_2 r^b(t) \exp\left(\frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{t} \int_{t_0}^{t} \big\{ p^b(s) d + q^b(s) \big\}\, ds \right), \quad t \in (t_0, T),$$

where N_2 is defined by (3.13). Thus, the required inequality in (3.31) is proved. This completes the proof. □

4 Applications to Hadamard fractional differential equations with 'maxima'

In this section, the dependence of solutions on the orders with initial conditions and the bound of solutions for the Hadamard fractional differential equations, are investigated. We consider the following fractional differential equation with 'maxima':

$${}_{\mathrm{H}}D^{\alpha} y(t) = f\Big(t, y(t), \max_{s \in [\beta t, t]} y(s)\Big), \quad t \in I = (t_0, T), \tag{4.1}$$

$${}_{\mathrm{H}}D^{\alpha - k} y(t)|_{t = t_0^+} = \eta_k, \quad k = 1, 2, \ldots, n, n = -[-\alpha], \tag{4.2}$$

and the initial function

$$y(t) = \phi(t), \quad t \in [\beta t_0, t_0], \tag{4.3}$$

where ${}_{\mathrm{H}}D^{\alpha}$ represents the Hadamard fractional derivative of order α ($\alpha > 0$), $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, ϕ is a given continuous function on $[\beta t_0, t_0]$, $0 < \beta < 1$ and η_k are constants.

The problem (4.1)-(4.3) describes a model of a fractional problem in real world phenomena in which often some parameters are involved. The values of these parameters can be measured only up to certain degree of accuracy. Hence, the orders of fractional differential equation α in (4.1) and the initial conditions $\alpha - k$ in (4.2) may be subject to some errors either by necessity or for convenience. Thus, it is important to know how the solution of (4.1)-(4.3) changes when the values of α and $\alpha - k$ are slightly altered.

Theorem 4.1 *Let $\alpha > 0$ and $\delta > 0$ such that $0 \le n - 1 < \alpha - \delta < \alpha \le n$. Also let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption*:

(A_6) *There exist constants $L_1, L_2 > 0$ such that $|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L_1|u_1 - v_1| + L_2|u_2 - v_2|$, for each $t \in I$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.*

If y and z are the solutions of the initial value problems (4.1)-(4.3) *and*

$$ {}_{\mathrm{H}}D^{\alpha-\delta} z(t) = f\Big(t, z(t), \max_{s\in[\beta t, t]} z(s)\Big), \quad t \in I, \tag{4.4}$$

$$ {}_{\mathrm{H}}D^{\alpha-\delta-k} z(t)|_{t=t_0^+} = \overline{\eta}_k, \quad k = 1, 2, \ldots, n, n = -\big[-(\alpha - \delta)\big], \tag{4.5}$$

with initial function

$$ z(t) = \overline{\phi}(t), \quad t \in [\beta t_0, t_0], \tag{4.6}$$

respectively, where $\overline{\eta}_k$ are constants and $\overline{\phi}$ is a given continuous function on $[\beta t_0, t_0]$ such that $\phi(t) \not\equiv \overline{\phi}(t)$ for all $t \in [\beta t_0, t_0]$, then the following estimates hold for $t_0 < t \le h < T$:

(I) *Suppose $\alpha - \delta > \frac{1}{2}$. Then for $t \in I$*

$$ \big|z(t) - y(t)\big| \le t\Bigg\{ c_5 A^2(t) + h_5(t) \times \exp\left(\frac{3\Gamma(2\alpha - 2\delta - 1)(L_1^2 + L_2^2)(t - t_0)}{\Gamma^2(\alpha) t}\right)\Bigg\}^{\frac{1}{2}}. \tag{4.7}$$

(II) *Suppose $0 < \alpha - \delta \le \frac{1}{2}$. Then for $t \in I$*

$$ \big|z(t) - y(t)\big| \le t\Bigg\{ c_6 A^b(t) + h_6(t) \times \exp\left(\frac{(3\Gamma((\alpha - \delta)^2))^{\frac{1}{\alpha-\delta}}(L_1^b + L_2^b)(t - t_0)}{\Gamma^b(\alpha) t}\right)\Bigg\}^{\frac{1}{b}}, \tag{4.8}$$

where

$$ A(t) = \left|\sum_{j=1}^{n} \frac{\overline{\eta}_j}{\Gamma(\alpha - \delta - j + 1)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j} - \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha - j + 1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}\right| + \left|\left(\log\frac{t}{t_0}\right)^{\alpha-\delta}\left(\frac{1}{\Gamma(\alpha - \delta + 1)} - \frac{1}{(\alpha - \delta)\Gamma(\alpha)}\right)\right| \|f\| $$

$$+\left|\frac{1}{(\alpha-\delta)\Gamma(\alpha)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta}-\frac{1}{\Gamma(\alpha+1)}\left(\log\frac{t}{t_0}\right)^{\alpha}\right|\|f\|, \tag{4.9}$$

$$\|f\| = \sup_{t_0\le t\le h}\left|f\left(t,y(t),\max_{s\in[\beta t,t]}y(s)\right)\right|,$$

$$b = 1+\frac{1}{\alpha-\delta},$$

$$c_5 = \max\left\{3t_0^{-2},(\beta t_0)^{-2}\right\},$$

$$c_6 = \max\left\{3^{\frac{1}{\alpha-\delta}}t_0^{-b},(\beta t_0)^{-b}\right\},$$

$$h_5(t) = c_5\max_{s\in[\beta t_0,t_0]}\left|\overline{\phi}(s)-\phi(s)\right|^2+\frac{3c_5\Gamma(2\alpha-2\delta-1)}{\Gamma^2(\alpha)t}\int_{t_0}^t\left(L_1^2A^2(s)+L_2^2\max_{\xi\in[\beta s,s]}m_2^2(\xi)\right)ds$$

and

$$h_6(t) = c_6\max_{s\in[\beta t_0,t_0]}\left|\overline{\phi}(s)-\phi(s)\right|^b$$
$$+\frac{c_6(3\Gamma((\alpha-\delta)^2))^{\frac{1}{\alpha-\delta}}}{\Gamma^b(\alpha)t}\int_{t_0}^t\left(L_1^bA^b(s)+L_2^b\max_{\xi\in[\beta s,s]}m_2^b(\xi)\right)ds,$$

with a continuous function $m_2(t)$ *is defined by*

$$m_2(t) = \begin{cases} A(t), & t\in I,\\ |\overline{\phi}(t)-\phi(t)|, & t\in[\beta t_0,t_0].\end{cases}$$

Proof The solutions y and z of the initial value problems (4.1)-(4.3) and (4.4)-(4.6) satisfy the following equations:

$$y(t) = \sum_{j=1}^n\frac{\eta_j}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha-1}f\left(s,y(s),\max_{\xi\in[\beta s,s]}y(\xi)\right)\frac{ds}{s}$$

and

$$z(t) = \sum_{j=1}^n\frac{\overline{\eta}_j}{\Gamma(\alpha-\delta-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j}$$
$$+\frac{1}{\Gamma(\alpha-\delta)}\int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}f\left(s,z(s),\max_{\xi\in[\beta s,s]}z(\xi)\right)\frac{ds}{s},$$

respectively. So using the assumption (A_6), it follows that

$$\left|z(t)-y(t)\right| \le \left|\sum_{j=1}^n\frac{\overline{\eta}_j}{\Gamma(\alpha-\delta-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j}-\sum_{j=1}^n\frac{\eta_j}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}\right|$$
$$+\left|\frac{1}{\Gamma(\alpha-\delta)}\int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}f\left(s,z(s),\max_{\xi\in[\beta s,s]}z(\xi)\right)\frac{ds}{s}\right.$$
$$\left.-\frac{1}{\Gamma(\alpha)}\int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}f\left(s,z(s),\max_{\xi\in[\beta s,s]}z(\xi)\right)\frac{ds}{s}\right|$$
$$+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}f\left(s,z(s),\max_{\xi\in[\beta s,s]}z(\xi)\right)\frac{ds}{s}\right.$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-\delta-1} f\Bigl(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\Bigr) \frac{ds}{s} \Biggr| \\
&+ \Biggl| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-\delta-1} f\Bigl(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\Bigr) \frac{ds}{s} \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f\Bigl(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\Bigr) \frac{ds}{s} \Biggr| \\
\le{}& A(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-\delta-1} \\
&\times \Bigl(L_1 \bigl|z(s) - y(s)\bigr| + L_2 \Bigl|\max_{\xi\in[\beta s,s]} z(\xi) - \max_{\xi\in[\beta s,s]} y(\xi)\Bigr|\Bigr) \frac{ds}{s} \\
\le{}& A(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-\delta-1} \\
&\times \Bigl(L_1 \bigl|z(s) - y(s)\bigr| + L_2 \max_{\xi\in[\beta s,s]} \bigl|z(\xi) - y(\xi)\bigr|\Bigr) \frac{ds}{s}, \quad t \in I,
\end{aligned}
$$

where $A(t)$ is defined by (4.9), and

$$\bigl|z(t) - y(t)\bigr| = \bigl|\overline{\phi}(t) - \phi(t)\bigr|, \quad t \in [\beta t_0, t_0].$$

Applying Theorem 3.2 yields the desired inequalities (4.7) and (4.8). This completes the proof. □

In the following theorem, we give the upper bounds of solution of the Hadamard fractional differential equation with 'maxima' and initial conditions (4.1)-(4.3).

Theorem 4.2 *Assume that*:

(A_7) *There exist functions $\mu, \nu \in C(I, \mathbb{R}_+)$ such that for $t \in I$, $u_1, u_2 \in \mathbb{R}$,*

$$\bigl|f(t, u_1, u_2)\bigr| \le \mu(t)|u_1| + \nu(t)|u_2|. \tag{4.10}$$

If y is solution of the initial value problem (4.1)-(4.3) *such that $\phi(t) \not\equiv 0$ for all $t \in [\beta t_0, t_0]$, then the following estimates hold*:

(III) *Suppose $\alpha > \frac{1}{2}$. Then for $t \in I$*

$$
\begin{aligned}
\bigl|y(t)\bigr| \le t \Biggl\{ & c_3 \left(\sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{t_0}\right)^{\alpha-j} \right)^2 \\
& + h_7(t) \exp\left(\frac{3\Gamma(2\alpha - 1)}{\Gamma^2(\alpha) t} \int_{t_0}^{t} \bigl\{\mu^2(s) + \nu^2(s)\bigr\} \, ds \right) \Biggr\}^{\frac{1}{2}}.
\end{aligned} \tag{4.11}
$$

(IV) *Suppose $0 < \alpha \le \frac{1}{2}$. Then for $t \in I$*

$$
\begin{aligned}
\bigl|y(t)\bigr| \le t \Biggl\{ & \frac{c_4 |\eta_1|^b}{\Gamma^b(\alpha)} \left(\log \frac{t}{t_0}\right)^{b(\alpha-1)} \\
& + h_8(t) \exp\left(\frac{(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{\Gamma^b(\alpha) t} \int_{t_0}^{t} \bigl\{\mu^2(s) + \nu^2(s)\bigr\} \, ds \right) \Biggr\}^{\frac{1}{b}},
\end{aligned} \tag{4.12}
$$

where b, c_3, c_4 are defined as in Theorem 3.2,

$$h_7(t) = c_3 \max_{s\in[\beta t_0,t_0]} \phi^2(s) + \frac{3c_3\Gamma(2\alpha-1)}{\Gamma^2(\alpha)t} \times \int_{t_0}^{t} \left\{ \mu^2(s)\left(\sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma(\alpha-j+1)}\left(\log\frac{s}{t_0}\right)^{\alpha-j}\right)^2 + \nu^2(s)\max_{\xi\in[\beta s,s]} m_3^2(\xi)\right\} ds$$

and

$$h_8(t) = c_4 \max_{s\in[\beta t_0,t_0]} \left|\phi(s)\right|^b + \frac{c_4(3\Gamma(\alpha^2))^{\frac{1}{\alpha}}}{\Gamma^b(\alpha)t} \int_{t_0}^{t} \left\{ \frac{|\eta_1|^b \mu^b(s)}{\Gamma^b(\alpha)}\left(\log\frac{s}{t_0}\right)^{b(\alpha-1)} + \nu^b(s)\max_{\xi\in[\beta s,s]} m_3^b(\xi)\right\} ds,$$

with a continuous function $m_3(t)$ defined by

$$m_3(t) = \begin{cases} \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma(\alpha-j+1)}(\log\frac{t}{t_0})^{\alpha-j}, & t\in I, \\ |\phi(t)|, & t\in[\beta t_0,t_0]. \end{cases}$$

Proof The solution y of the initial value problem (4.1)-(4.3) satisfies the following equations:

$$y(t) = \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right)\frac{ds}{s}, \quad t\in I,$$
$$y(t) = \phi(t), \quad t\in[\beta t_0,t_0].$$

For $\alpha > 0$, by using the assumption (A_7), it follows that

$$\left|y(t)\right| \le \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\mu(s)\left|y(s)\right| + \nu(s)\max_{\xi\in[\beta s,s]}\left|y(\xi)\right|\right)\frac{ds}{s}, \quad t\in I,$$
$$\left|y(t)\right| = \left|\phi(t)\right|, \quad t\in[\beta t_0,t_0].$$

Hence Theorem 3.2 yields the estimate of the inequalities (4.11) and (4.12). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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