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Some convexity inequalities in noncommutative L_p-spaces

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Abstract
In this paper, we prove some convexity inequalities in noncommutative L_p spaces generalizing the previous result of Hiai and Zhan. Moreover, we generalize a variational inequality for positive definite matrices due to Hansen to the case of noncommutative L_p spaces.

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1 Introduction

Let $\|\cdot\|$ be a unitarily invariant norm on matrices. For matrices A, B, X in $M_n(\mathbb{C})$ with A, B being positive semidefinite and X arbitrary, Bhatia and Davis [1] presented the matrix Cauchy-Schwarz inequality $\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \le \||AX|^r\| \cdot \||XB|^r\|$. In 2002, Hiai and Zhan [2] proved that under the same assumption for matrices A, B, and X, the function $t \mapsto \||A^tXB^{1-t}|^r\| \cdot \||A^{1-t}XB^t|^r\|$ is convex on $[0,1]$ for each $r > 0$. Among other things, this convexity result interpolated the above matrix Cauchy-Schwarz inequality showed by Bhatia and Davis. Another result of interest is that, in 2014, Hansen [3], among other things, gave new and simplified proofs of the Carlen-Lieb theorem concerning concavity of certain trace functions by applying the theory of operator monotone functions. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Equipped with the usual adjoint as involution $\mathcal{B}(\mathcal{H})$ becomes a unital C^*-algebra. We say a function $|||\cdot||| : \mathcal{B}(\mathcal{H}) \to \mathcal{R}$ is a unitary invariant norm (or symmetric norm) if it is a norm satisfying the invariance property $|||uxv||| = |||x|||$ for all x and all unitary operators u and v in $\mathcal{B}(\mathcal{H})$.

In this paper, we consider the noncommutative L_p spaces of τ-measurable operators affiliated with a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace τ. On one hand, we use the method of Hiai and Zhan, via the notion of generalized singular value studied by Fack and Kosaki [4], to prove some convexity inequalities in noncommutative L_p spaces generalizing the previous result of Hiai and Zhan. On the other hand, by making use of the joint convexity and concavity of trace functions obtained by Bekjan [5] we generalize a variational inequality for positive definite matrices due to Hansen to the case of noncommutative L_p spaces.

2 Preliminaries

Throughout the paper, unless specified, we always denote by $\mathcal{M}$ a semi-finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful semi-finite trace τ. We denote the identity in $\mathcal{M}$ by 1 and let $\mathcal{P}$ denote the projection lattice of $\mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be τ-measurable if for every $\epsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1-e) < \epsilon$. The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M}, \tau)$, or simply $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. A closed densely defined linear operator x admits a unique polar decomposition $x = u|x|$, where u is a partial isometry such that $u^*u = (\ker x)^{\perp}$ and $uu^* = \overline{\operatorname{im} x}$ (with $\operatorname{im} x = x(D(x))$). We call $r(x) = (\ker x)^{\perp}$ and $l(x) = \overline{\operatorname{im} x}$ the left and right supports of x, respectively. Thus $l(x) \sim r(x)$. Moreover, if x is self-adjoint, we let $s(x) = r(x)$, the support of x.

Let $\mathcal{M}_+$ be the positive part of $\mathcal{M}$. Set $\mathcal{S}_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$ and let $\mathcal{S}(\mathcal{M})$ be the linear span of $\mathcal{S}_+(\mathcal{M})$, we will often abbreviate $\mathcal{S}_+(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$, respectively, as $\mathcal{S}_+$ and $\mathcal{S}$. Let $0 < p < \infty$, the noncommutative L_p-space $L_p(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}, \|\cdot\|_p)$, where $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty$, $\forall x \in L_p(\mathcal{M}, \tau)$. In addition, we put $L^{\infty}(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\|\cdot\|_{\infty}$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L_p(\mathcal{M}, \tau)$ are Banach spaces under $\|\cdot\|_p$ for $1 \leq p < \infty$ and they have a lot of expected properties of classical L_p-spaces (see [6] or [7]).

Let x be a τ-measurable operator and $t > 0$. The 'tth singular number (or generalized s-number) of x' is defined by

$$\mu_t(x) = \inf\bigl\{\|xe\| : e \in \mathcal{P}, \tau(1-e) \leq t\bigr\}.$$

See [4] for basic properties and detailed information on the generalized s-numbers.

To achieve one of our main results, we state for easy reference the following fact, obtained from [8], which will be applied below.

Lemma 2.1 *If $0 < r < 1$, then the function $f(z, A) = \tau(z^r + A^r)^{\frac{1}{r}}$ is jointly concave in strictly positive operators $(z, A) \in \mathcal{S}_+ \times \mathcal{S}_+$.*

3 Main results

Lemma 3.1 *Let $x, y \in \mathcal{S}$ such that xy is a self-adjoint τ-measurable operator and let $1 \leq p < \infty$, then*

$$\|xy\|_p \leq \|yx\|_p.$$

Proof Notice that xy is a self-adjoint τ-measurable operator; then

$$\begin{aligned}
\mu_s(xy)^{2n} &= \mu_s(xyxy\cdots) = \mu_s\bigl((xy)^*xy\cdots\bigr) \\
&= \mu_s\bigl(|xy|^{2n}\bigr) = \mu_s^{2n}\bigl(|xy|\bigr) \\
&= \mu_s^{2n}(xy).
\end{aligned}$$

Let f be an increasing function on $\mathcal{R}_+$ satisfying $f(0) = 0$ and where $t \to f(e^t)$ is convex, then applying Lemma 2 of [9] we have

$$\begin{aligned}\int_0^\infty f\bigl(\mu_s(xy)^{2n}\bigr)\,ds &= \int_0^\infty f\bigl(\mu_s(xyxy\cdots)\bigr)\,ds \\ &\le \int_0^\infty f\bigl(\mu_s(yxyx\cdots)\|x\|\|y\|\bigr)\,ds \\ &= \int_0^\infty f\bigl(\mu_s\bigl(\|x\|\|y\|yxyx\cdots\bigr)\bigr)\,ds \\ &\le \int_0^\infty f\bigl(\mu_s^{2n-1}(yx)\|x\|\|y\|\bigr)\,ds.\end{aligned}$$

In particular, if $f(s) = s^{\frac{p}{2n-1}}$, then

$$\int_0^\infty \mu_s^{\frac{2n}{2n-1}p}(xy)\,ds \le \int_0^\infty \mu_s^p(yx)\bigl(\|x\|\|y\|\bigr)^{\frac{p}{2n-1}}\,ds.$$

Taking the $\liminf_{n\to\infty}$, by the usual Fatou lemma we get

$$\begin{aligned}\int_0^\infty \mu_s^p(xy)\,ds &= \int_0^\infty \liminf_{n\to\infty} \mu_s^{\frac{2n}{2n-1}p}(xy)\,ds \\ &\le \liminf_{n\to\infty} \int_0^\infty \mu_s^{\frac{2n}{2n-1}p}(xy)\,ds \\ &\le \liminf_{n\to\infty} \int_0^\infty \mu_s^p(yx)\bigl(\|x\|\|y\|\bigr)^{\frac{p}{2n-1}}\,ds \\ &= \liminf_{n\to\infty}\bigl(\|x\|\|y\|\bigr)^{\frac{p}{2n-1}} \int_0^\infty \mu_s^p(yx)\,ds \\ &= \int_0^\infty \mu_s^p(yx)\,ds.\end{aligned}$$

□

Lemma 3.2 *Let $x, y \in \mathcal{S}$ and $z \in \mathcal{M}$. Let $1 \le p < \infty$, then*

$$\bigl\| |x^*zy|^r \bigr\|_p^2 \le \bigl\| |xx^*z|^r \bigr\|_p \cdot \bigl\| |zyy^*|^r \bigr\|_p, \quad r > 0. \tag{1}$$

Proof It suffices to prove that

$$\left(\int_0^\infty \mu_s\bigl(|x^*zy|^r\bigr)^p\,ds\right)^2 \le \left(\int_0^\infty \mu_s\bigl(|xx^*z|^r\bigr)^p\,ds\right)\left(\int_0^\infty \mu_s\bigl(|zyy^*|^r\bigr)^p\,ds\right).$$

Since

$$\int_0^\infty \mu_s\bigl(|x^*zy|^r\bigr)^p\,ds = \int_0^\infty \mu_s^p\bigl((x^*zy)^*x^*zy\bigr)^{\frac{r}{2}}\,ds = \int_0^\infty \mu_s^{\frac{r}{2}p}\bigl((x^*zy)^*x^*zy\bigr)\,ds.$$

Applying Lemma 2 of [9] together with the Hölder inequality we obtain

$$\begin{aligned}\int_0^\infty \mu_s^{\frac{r}{2}p}\bigl(y^*z^*xx^*zy\bigr)\,ds &\le \int_0^\infty \mu_s^p\bigl(|yy^*z^*xx^*z|^{\frac{r}{2}}\bigr)\,ds \\ &= \int_0^\infty \mu_s^{\frac{r}{2}p}\bigl(yy^*z^*xx^*z\bigr)\,ds\end{aligned}$$

$$\begin{aligned}&\le \int_0^\infty \mu_s^{\frac{r}{2}p}(zyy^*)\mu_s^{\frac{r}{2}p}(xx^*z)\,ds\\&\le \left(\int_0^\infty \mu_s^{rp}(zyy^*)\,ds\right)^{\frac{1}{2}}\left(\int_0^\infty \mu_s^{rp}(xx^*z)\,ds\right)^{\frac{1}{2}},\end{aligned}$$

which implies the lemma. □

Theorem 3.3 *Let $x, y \in L_p(\mathcal{M})$ be positive operators with $1 \le p < \infty$ and let $z \in \mathcal{M}$, then for every real number $r > 0$, the function*

$$\phi(t) = \big\| |x^t zy^{1-t}|^r \big\|_p \cdot \big\| |x^{1-t}zy^t|^r \big\|_p$$

is convex on the interval $[0,1]$ and attains its minimum at $t = \frac{1}{2}$.

Proof (i) First we assume that τ is finite. By the density of $\mathcal{S}$ in $L_p(\mathcal{M})$, we first consider the case $x, y \in \mathcal{S}$. Since ϕ is continuous and symmetric with respect to $t = \frac{1}{2}$, all the conclusions will follow after we show that

$$\phi(t) \le \frac{1}{2}\big\{\phi(t+s) + \phi(t-s)\big\} \tag{2}$$

for $t \pm s \in [0,1]$. By (1) we have

$$\begin{aligned}\big\| |x^t zy^{1-t}|^r \big\|_p &= \big\| |x^s(x^{t-s}zy^{1-t-s})y^s|^r \big\|_p\\&\le \big\{\big\| |x^{t+s}zy^{1-(t+s)}|^r \big\|_p \cdot \big\| |x^{t-s}zy^{1-(t-s)}|^r \big\|_p\big\}^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\big\| |x^{1-t}zy^t|^r \big\|_p &= \big\| |x^s(x^{1-t-s}zy^{t-s})y^s|^r \big\|_p\\&\le \big\{\big\| |x^{1-(t-s)}zy^{t-s}|^r \big\|_p \cdot \big\| |x^{1-(t+s)}zy^{t+s}|^r \big\|_p\big\}^{\frac{1}{2}}.\end{aligned}$$

Multiplying the above two inequalities we obtain

$$\big\| |x^t zy^{1-t}|^r \big\|_p \cdot \big\| |x^{1-t}zy^t|^r \big\|_p \le \frac{1}{2}\big\{\phi(t+s) + \phi(t-s)\big\}. \tag{3}$$

For the general case, namely, for any $x, y \in L_p(\mathcal{M})$, there exist $x_n, y_n \in \mathcal{M}_+$ such that x_n, y_n are invertible and $x_n \to x$, $y_n \to y$ in $L_p(\mathcal{M})$. Moreover, we have $\phi_n(t) = \||x_n^t zy_n^{1-t}|^r\|_p \cdot \||x_n^{1-t}zy_n^t|^r\|_p$ is convex for all $t \in [0,1]$ and attains its minimum at $t = \frac{1}{2}$. Applying Theorem 3.7 of [4] and using the method to prove Lemma 3.3 in [10], we get $x_n^t zy_n^{1-t} \to x^t zy^{1-t}$, $x_n^{1-t}zy_n^t \to x^{1-t}zy^t$ in $L_p(\mathcal{M})$. Hence, we obtain $\phi_n(t) \to \phi(t)$, $n \to \infty$. Therefore, $\phi(t)$ is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$.

(ii) In the general case when τ is semi-finite, there exists an increasing family $(e_i)_{i\in\mathcal{I}} \in \mathcal{P}$ such that $\tau(e_i) < \infty$ for every $i \in \mathcal{I}$ and such that e_i converges to 1 in the strong operator topology (see [6] or [7]). Thus, $e_i\mathcal{M}e_i$ is finite for each $i \in \mathcal{I}$. Let $x, y \in L_2(\mathcal{M})_+$, then $e_ixe_i, e_iye_i \in L_2(e_i\mathcal{M}e_i)_+$. Write $x_i = e_ixe_i$, $y_i = e_iye_i$, it follows from the case (i) that the function $f_i(t) = \||x_i^t zy_i^{1-t}|^r\|_p \cdot \||x_i^{1-t}zy_i^t|^r\|_p$ is convex on $[0,1]$ and attains its minimum at

$t = \frac{1}{2}$. In view of the fact that $x_i \to x$, $y_i \to y$ in $L_p(\mathcal{M})$, by a similar computation we derive $\lim_i f_i(t) = \phi(t)$. Therefore, $\phi(t)$ is convex on $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. □

An immediate consequence of Theorem 3.3 interpolates the inequality (1) as follows.

Corollary 3.4 *Let x, y, z be τ-measurable operators as in Theorem* 3.3. *For every* $r > 0$,

$$\begin{aligned} \bigl\| \bigl|x^{\frac{1}{2}} z y^{\frac{1}{2}}\bigr|^r \bigr\|_p^2 &\le \bigl\| \bigl|x^t z y^{1-t}\bigr|^r \bigr\|_p \cdot \bigl\| \bigl|x^{1-t} z y^t\bigr|^r \bigr\|_p \\ &\le \bigl\| |xz|^r \bigr\| \cdot \bigl\| |zy|^r \bigr\|_p \end{aligned}$$

holds for $0 \le t \le 1$.

The following is another example of convex functions involving the noncommutative L_p-norm.

Theorem 3.5 *Let* $x_i \in L_p(\mathcal{M})$ $(i = 1, 2, \dots, k)$ *be positive operators and let* $1 \le p < \infty$. *Then, for every positive real number r, the function* $t \mapsto \|(\sum_{i=1}^k x_i^t)^r\|_p$ *is convex on* $(0, \infty)$.

Proof The same proof as of Theorem 4 of Hiai and Zhan [2] works. □

If $\tau(1) = 1$, that is to say, we consider the von Neumann algebra $\mathcal{M}$ to be equipped with normal faithful finite unital trace, then we can also get the following noncommutative analog of the variational inequality and we refer the readers to Theorem 2.5 of [3] for more details of the Carlen-Lieb theorems concerning the concavity of certain trace functions.

Theorem 3.6 *Let* $x, y \in L_r(\mathcal{M})$ *be positive operators and let* $0 < r < 1$. *Then*

$$\tau\bigl(x^r + y^r\bigr)^{\frac{1}{r}} \le \tau\bigl(z^{\frac{r-1}{r}} x + (1-z)^{\frac{r-1}{r}} y\bigr)$$

for every $z \in \mathcal{S}_+$ *such that z and* $1 - z$ *are invertible. If* $xy = yx$, *take*

$$z = x^r\bigl(x^r + y^r\bigr)^{-1}$$

and we get the equality.

Proof The same proof as of Theorem 2.5 of Hansen [3] works. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by the corresponding author JJS. JJS and YZH prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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