Superimposed algorithms for the split equilibrium problems and fixed point problems

Li-Jun Zhu[1], Zhangsong Yao[2], Yeong-Cheng Liou[3,4*] and Yonghong Yao[5]

*Correspondence:
simplex_liou@hotmail.com
[3]Department of Information
Management, Cheng Shiu
University, Kaohsiung, 833, Taiwan
[4]Center for Fundamental Science,
Kaohsiung Medical University,
Kaohsiung, 807, Taiwan
Full list of author information is
available at the end of the article

Abstract

The split common problems for finding the equilibrium points and fixed points have been studied. A parallel superimposed algorithm is introduced to solve this split common problem. Strong convergence theorems are shown with some analysis techniques.

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1 Introduction

In the present article, our main purpose is to study the split problem. First, we recall some relevant background in the literature.

Problem 1: the split feasibility problem

Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The problem of finding a point x^* such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q \tag{1.1}$$

is called the split feasibility; it was first introduced by Censor and Elfving [1] in finite dimensional Hilbert spaces. Such problems arise in the field of intensity-modulated radiation therapy when one attempts to describe physical dose constraints and equivalent uniform dose constraints within a single model. When $C \in \mathbb{R}^N$ and $Q \in \mathbb{R}^M$ are a single pair of sets, Censor and Elfving [1] introduced the simultaneous multi-projections algorithm:

$$x_{n+1} = A^{-1}\left(\lambda_1 I + \lambda_2 A A^*\right)^{-1}\left(\lambda_1 v_1^{n+1} + \lambda_2 A A^* v_2^{n+1}\right), \quad n \geq 0, \tag{1.2}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, $v_i^{n+1} = P_{C_i}^{f_i}(b_n)$ $(i = 1, 2)$, and b_n is the solution of the equation

$$\sum_{i=1}^{2} \lambda_i \nabla f_i(b_n) = \sum_{i=1}^{2} \nabla f_i(v_i^n).$$

Note that the simultaneous multi-projections algorithm (1.2) involve a matrix inversion A^{-1} at each iterative step. This is very time-consuming, particularly if the dimensions are large. In order to solve this problem, Byrne [2] derived a new algorithm, called the CQ-algorithm:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \quad n \geq 0,$$

where $\tau \in (0, \frac{2}{L})$ with L being the largest eigenvalue of the matrix A^*A, I is the unit matrix or operator, and P_C and P_Q denote the orthogonal projections onto C and Q, respectively. The CQ-algorithm and its variant forms have now been studied for the split feasibility problem; see, for instance [3–13].

Problem 2: the split common fixed point problem

If every closed convex subset of a Hilbert space is the fixed point set of its associating projection, then the split feasibility problem becomes a special case of the split common fixed point problem of finding a point x^* with the property:

$$x^* \in \text{Fix}(U) \quad \text{and} \quad Ax^* \in \text{Fix}(T). \tag{1.3}$$

This problem was first introduced by Censor and Segal [14] who invented an algorithm which generates a sequence $\{x_n\}$ according to the iterative procedure:

$$x_{n+1} = U(x_n - \gamma A^*(I - T)Ax_n), \quad n \in \mathbb{N}. \tag{1.4}$$

Moudafi [15] extended (1.4) to the following relaxed algorithm:

$$x_{n+1} = U_{\alpha_n}(x_n + \gamma A^*(T_\beta - I)Ax_n), \quad n \in \mathbb{N},$$

where $\beta \in (0,1)$, $\alpha_n \in (0,1)$ are relaxation parameters. Consequently, Wang and Xu [16] considered a general cyclic algorithm. Very recently, the split problem has also been extended to solve other problems, such as the split monotone variational inclusions and the split variational inequalities, please refer to [15, 17–22] and [23–25].

Problem 3: the equilibrium problem

Consider the following equilibrium problem: Finding $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.5}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. We will denote by EP(F) the set of solutions of (1.5).

The equilibrium problems, in its various forms, found application in optimization problems, fixed point problems, and convex minimization problems; in other words, equilibrium problems are a unified model for problems arising in physics, engineering, economics, and so on (see [26–29]).

Motivated by the split common fixed point problem and the equilibrium problem, He and Du [30] presented the following split equilibrium problem and fixed point problem:

$$\text{Find a point } x^* \in \text{Fix}(T) \cap \text{EP}(F) \text{ such that } Ax^* \in \text{Fix}(S) \cap \text{EP}(G), \tag{1.6}$$

where Fix(S) and Fix(T) are the sets of fixed points of two nonlinear mappings S and T, respectively, EP(F) and EP(G) are the solution sets of two equilibrium problems with bifunctions F and G, respectively, and A is a bounded linear mapping. Denote the solution set of (1.6) by

$$\Gamma = \big\{ x \in \text{Fix}(T) \cap \text{EP}(F) : Ax \in \text{Fix}(S) \cap \text{EP}(G) \big\}.$$

Special cases

1. If $F = 0$ and $G = 0$, then (1.6) is reduced to the following split common fixed point problem, which has been considered by many authors, for example, [14, 15, 17] and [21]:

$$\text{Find a point } x^* \in \text{Fix}(T) \text{ such that } Ax^* \in \text{Fix}(S). \tag{1.7}$$

2. If $S = P_Q$ and $T = P_C$, then (1.7) is reduced to the split feasibility problem (1.1).
3. If S and T are all identity operators, then (1.6) is reduced to the split equilibrium problem which has been considered in [18]:

$$\text{Find a point } x^* \in \text{EP}(F) \text{ such that } Ax^* \in \text{EP}(G).$$

Based on the work in this direction, in this paper we will develop new algorithms to solve the split equilibrium problem and the fixed point problem (1.6). We first introduce a parallel superimposed algorithm. Consequently, strong convergence theorems are shown with some analysis techniques.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H.

Definition 2.1 A mapping $T : C \to C$ is called *nonexpansive* if

$$\| Tx - Ty \| \leq \| x - y \|$$

for all $x, y \in C$.

We will use Fix(T) to denote the set of fixed points of T, that is, Fix(T) = $\{ x \in C : x = Tx \}$.

Definition 2.2 A mapping $f : C \to C$ is called *contractive* if

$$\big\| f(x) - f(y) \big\| \leq \rho \| x - y \|$$

for all $x, y \in C$ and for some constant $\rho \in (0, 1)$. In this case, we call f is a ρ-contraction.

Definition 2.3 A linear bounded operator $B : H \to H$ is called *strongly positive* if there exists a constant $\gamma > 0$ such that

$$\langle Bx, x \rangle \geq \gamma \| x \|^2$$

for all $x, y \in H$.

Definition 2.4 We call $P_C : H \to C$ the metric projection if for each $x \in H$

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that the metric projection $P_C : H \to C$ is characterized by

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0$$

for all $x \in H$, $y \in C$. From this, we can deduce that P_C is firmly nonexpansive, that is,

$$\|P_C(x) - P_C(y)\|^2 \le \langle x - y, P_C(x) - P_C(y)\rangle \tag{2.1}$$

for all $x, y \in H$. Hence P_C is also nonexpansive.

It is well known that in a real Hilbert space H, the following two equalities hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $t \in [0, 1]$, and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \tag{2.3}$$

for all $x, y \in H$. It follows that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle \tag{2.4}$$

for all $x, y \in H$.

Throughout this paper, we assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;

(H2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x, y)$;

(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.5 ([31]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (H1)-(H4). *Let $r > 0$ and $x \in C$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, if $U_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$, then the following hold:

(i) U_r^F *is single-valued and U_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,*
$$\|U_r^F x - U_r^F y\|^2 \le \langle U_r^F x - U_r^F y, x - y\rangle;$$

(ii) $\mathrm{EP}(F)$ *is closed and convex and $\mathrm{EP}(F) = \mathrm{Fix}(U_r^F)$.*

Lemma 2.6 ([32]) *Let the mapping U_r^F be defined as in Lemma* 2.5. *Then, for $r, s > 0$ and $x, y \in H$,*

$$\|U_r^F(x) - U_s^F(y)\| \le \|x - y\| + \frac{|s - r|}{s}\|U_s^F(y) - y\|.$$

Lemma 2.7 ([33]) *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.8 ([34]) *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.9 ([35]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce our algorithm and prove our main results.

Let H_1 and H_2 be two real Hilbert spaces and let C and D be two nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*, B be a strongly positive bounded linear operator on H_1 with coefficient $\gamma > 0$. Let $f : C \to C$ be a ρ-contraction and $F : C \times C \to \mathbb{R}$ and $G : D \times D \to \mathbb{R}$ be two bifunctions satisfying the conditions (H1)-(H4). Let $S : D \to D$ and $T : C \to C$ be two nonexpansive mappings.

Algorithm 3.1 Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$x_{n+1} = \alpha_n \sigma f(x_n) + \beta_n x_n + \big((1 - \beta_n)I - \alpha_n B\big) T U_{\lambda_n}^F \big(x_n + \delta A^*\big(S U_{\gamma_n}^G - I\big)A x_n\big) \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are two real number sequences in $(0,\infty)$, $\delta \in (0, \frac{1}{\|A\|^2})$ and $\sigma > 0$ are two constants and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Theorem 3.2 *Suppose $\Gamma \neq \emptyset$ and suppose the following conditions hold*:

(C1): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2): $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C3): $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$;
(C4): $\liminf_{n\to\infty} \gamma_n > 0$ *and* $\lim_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = 1$;
(C5): $\sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ generated by algorithm (3.1) *converges strongly to* $p = \mathrm{Proj}_\Gamma (\sigma f + I - B)p$, *which solves the following VI:*

$$\langle (\sigma f - B)x, y - x \rangle \le 0, \quad \forall y \in \Gamma. \tag{3.2}$$

Proof First, we know that the solution of (3.2) is unique. We denote the unique solution by p. That is, $p = \mathrm{Proj}_\Gamma (\sigma f + I - B)p$. Then we have $p \in \mathrm{Fix}(T) \cap \mathrm{EP}(F)$ and $Ap \in \mathrm{Fix}(S) \cap \mathrm{EP}(G)$. Set $z_n = U^G_{\gamma_n} Ax_n$, $y_n = x_n + \delta A^*(SU^G_{\gamma_n} - I)Ax_n$ and $u_n = U^F_{\lambda_n}(x_n + \delta A^*(SU^G_{\gamma_n} - I)Ax_n)$ for all $n \in \mathbb{N}$. Then $u_n = U^F_{\lambda_n} y_n$. From Lemma 2.5, we know that $U^F_{\lambda_n}$ and $U^G_{\gamma_n}$ are firmly nonexpansive. By these facts, we have the following conclusions:

$$\|z_n - Ap\| = \left\| U^G_{\gamma_n} Ax_n - Ap \right\| \le \|Ax_n - Ap\|, \tag{3.3}$$

$$\|u_n - p\| = \left\| U^F_{\lambda_n} y_n - p \right\| \le \|y_n - p\| \tag{3.4}$$

and

$$\left\| SU^G_{\gamma_n} Ax_n - Ap \right\|^2 \le \left\| U^G_{\gamma_n} Ax_n - Ap \right\|^2$$
$$\le \|Ax_n - Ap\|^2 - \left\| U^G_{\gamma_n} Ax_n - Ax_n \right\|^2. \tag{3.5}$$

Applying Lemma 2.6, we deduce

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\| U^F_{\lambda_{n+1}} y_{n+1} - U^F_{\lambda_n} y_n \right\| \\
&\le \|y_{n+1} - y_n\| + \left| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \right| \|u_{n+1} - y_{n+1}\|
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| U^G_{\gamma_{n+1}} Ax_{n+1} - U^G_{\gamma_n} Ax_n \right\| \\
&\le \|Ax_{n+1} - Ax_n\| + \left| \frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}} \right| \|z_{n+1} - Ax_n\|.
\end{aligned} \tag{3.7}$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \alpha_n (\sigma f(x_n) - Bp) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n B)(Tu_n - p) \right\| \\
&\le \alpha_n \sigma \|f(x_n) - f(p)\| + \alpha_n \|\sigma f(p) - Bp\| + \beta_n \|x_n - p\| \\
&\quad + (1 - \beta_n - \alpha_n \gamma) \|u_n - p\|.
\end{aligned} \tag{3.8}$$

Using (2.3), we get

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| x_n - p + \delta A^*(Sz_n - Ax_n) \right\|^2 \\
&= \|x_n - p\|^2 + \delta^2 \left\| A^*(Sz_n - Ax_n) \right\|^2 \\
&\quad + 2\delta \langle x_n - p, A^*(Sz_n - Ax_n) \rangle.
\end{aligned} \tag{3.9}$$

Since A is a linear operator with its adjoint A^*, we have

$$
\begin{aligned}
\langle x_n - p, A^*(Sz_n - Ax_n) \rangle &= \langle A(x_n - p), Sz_n - Ax_n \rangle \\
&= \langle Ax_n - Ap + Sz_n - Ax_n - (Sz_n - Ax_n), Sz_n - Ax_n \rangle \\
&= \langle Sz_n - Ap, Sz_n - Ax_n \rangle - \|z_n - Ax_n\|^2.
\end{aligned}
\tag{3.10}
$$

Again from (2.3), we obtain

$$
\langle Sz_n - Ap, Sz_n - Ax_n \rangle = \frac{1}{2} \left(\|Sz_n - Ap\|^2 + \|Sz_n - Ax_n\|^2 - \|Ax_n - Ap\|^2 \right).
\tag{3.11}
$$

From (3.5), (3.10), and (3.11), we have

$$
\begin{aligned}
\langle x_n - p, A^*(Sz_n - Ax_n) \rangle &= \frac{1}{2} \left(\|Sz_n - Ap\|^2 + \|Sz_n - Ax_n\|^2 - \|Ax_n - Ap\|^2 \right) \\
&\quad - \|Sz_n - Ax_n\|^2 \\
&\leq \frac{1}{2} \left(\|Ax_n - Ap\|^2 - \|z_n - Ax_n\|^2 + \|Sz_n - Ax_n\|^2 \right. \\
&\quad \left. - \|Ax_n - Ap\|^2 \right) - \|Sz_n - Ax_n\|^2 \\
&= -\frac{1}{2} \|z_n - Ax_n\|^2 - \frac{1}{2} \|Sz_n - Ax_n\|^2.
\end{aligned}
\tag{3.12}
$$

Substituting (3.12) into (3.9) to deduce

$$
\begin{aligned}
\|y_n - p\|^2 &\leq \|x_n - p\|^2 + \delta^2 \|A\|^2 \|Sz_n - Ax_n\|^2 + 2\delta \left(-\frac{1}{2} \|z_n - Ax_n\|^2 - \frac{1}{2} \|Sz_n - Ax_n\|^2 \right) \\
&= \|x_n - p\|^2 + \left(\delta^2 \|A\|^2 - \delta \right) \|Sz_n - Ax_n\|^2 - \delta \|z_n - Ax_n\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned}
$$

It follows that

$$
\|y_n - p\| \leq \|x_n - p\|.
$$

Thus, from (3.8), we get

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \sigma \rho \|x_n - p\| + \alpha_n \|\sigma f(p) - Bp\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \gamma) \|x_n - p\| \\
&= \left[1 - (\gamma - \sigma \rho) \alpha_n \right] \|x_n - p\| + \alpha_n \|\sigma f(p) - Bp\| \\
&\leq \max \left\{ \|x_n - p\|, \frac{\|\sigma f(p) - Bp\|}{\gamma - \sigma \rho} \right\}.
\end{aligned}
$$

The boundedness of the sequence $\{x_n\}$ follows.

Next, we estimate $\|u_{n+1} - u_n\|$. Observe that

$$
\begin{aligned}
&\|y_{n+1} - y_n\|^2 \\
&= \left\| x_{n+1} - x_n + \delta \left[A^*(Sz_{n+1} - Ax_{n+1}) - A^*(Sz_n - Ax_n) \right] \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|x_{n+1} - x_n\|^2 + \delta^2 \|A^*(Sz_{n+1} - Ax_{n+1}) - A^*(Sz_n - Ax_n)\|^2 \\
&\quad + 2\delta\langle x_{n+1} - x_n, A^*[(Sz_{n+1} - Ax_{n+1}) - (Sz_n - Ax_n)]\rangle \\
&\leq \|x_{n+1} - x_n\|^2 + \delta^2\|A\|^2 \|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad + 2\delta\langle Ax_{n+1} - Ax_n, Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\rangle \\
&= \|x_{n+1} - x_n\|^2 + \delta^2\|A\|^2 \|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad + 2\delta\langle Sz_{n+1} - Sz_n, Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\rangle \\
&\quad - 2\delta\|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \delta^2\|A\|^2 \|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad + \delta\big(\|Sz_{n+1} - Sz_n\|^2 + \|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad - \|Ax_{n+1} - Ax_n\|^2\big) - 2\delta\|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \big(\delta^2\|A\|^2 - \delta\big)\|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad + \delta\big(\|Sz_{n+1} - Sz_n\|^2 - \|Ax_{n+1} - Ax_n\|^2\big) \\
&\leq \|x_{n+1} - x_n\|^2 + \big(\delta^2\|A\|^2 - \delta\big)\|Sz_{n+1} - Sz_n - (Ax_{n+1} - Ax_n)\|^2 \\
&\quad + \delta\big(\|z_{n+1} - z_n\|^2 - \|Ax_{n+1} - Ax_n\|^2\big).
\end{aligned}
\tag{3.13}
$$

Since $\delta \in (0, \frac{1}{\|A\|^2})$, we derive by virtue of (3.7) and (3.13) that

$$
\begin{aligned}
\|y_{n+1} - y_n\|^2 &\leq \|x_{n+1} - x_n\|^2 \\
&\quad + \delta\left|\frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}}\right|\big(\|z_{n+1} - z_n\| + \|Ax_{n+1} - Ax_n\|\big).
\end{aligned}
\tag{3.14}
$$

According to (3.6) and (3.14), we have

$$
\begin{aligned}
\|u_{n+1} - u_n\|^2 &= \|U_{\lambda_{n+1}}^F y_{n+1} - U_{\lambda_n}^F y_n\|^2 \\
&\leq \left(\|y_{n+1} - y_n\| + \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right|\|u_{n+1} - y_{n+1}\|\right)^2 \\
&\leq \|y_{n+1} - y_n\|^2 + \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right|\bigg(2\|y_{n+1} - y_n\|\|u_{n+1} - y_{n+1}\| \\
&\quad + \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right|\|u_{n+1} - y_{n+1}\|^2\bigg) \\
&\leq \|x_{n+1} - x_n\|^2 + \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right|\bigg(2\|y_{n+1} - y_n\|\|u_{n+1} - y_{n+1}\| \\
&\quad + \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right|\|u_{n+1} - y_{n+1}\|^2\bigg) \\
&\quad + \delta\left|\frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}}\right|\big(\|z_{n+1} - z_n\| + \|Ax_{n+1} - Ax_n\|\big) \\
&\leq \|x_{n+1} - x_n\|^2 + \left(\left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right| + \delta\left|\frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}}\right|\right)M,
\end{aligned}
\tag{3.15}
$$

where $M > 0$ is a constant such that

$$\sup_n \left\{ 2\|y_{n+1} - y_n\| \|u_{n+1} - y_{n+1}\| + \left| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \right| \|u_{n+1} - y_{n+1}\|^2 \right. $$
$$\left. + \delta \big(\|z_{n+1} - z_n\| + \|Ax_{n+1} - Ax_n\| \big) \right\} \leq M.$$

Therefore,

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \sqrt{\left(\left| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \right| + \delta \left| \frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}} \right| \right) M}. \tag{3.16}$$

From (3.1), we write $x_{n+1} = \beta_n x_n + (1 - \beta_n) w_n$ where $w_n = Tu_n + \frac{\alpha_n}{1 - \beta_n}(\sigma f(x_n) - BTu_n)$ for all $n \in \mathbb{N}$. Then we have

$$\|w_{n+1} - w_n\|$$

$$= \left\| Tu_{n+1} - Tu_n + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \big(\sigma f(x_{n+1}) - BTu_{n+1} \big) - \frac{\alpha_n}{1 - \beta_n} \big(\sigma f(x_n) - BTu_n \big) \right\|$$

$$\leq \|Tu_{n+1} - Tu_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \big\| \sigma f(x_{n+1}) - BTu_{n+1} \big\| + \frac{\alpha_n}{1 - \beta_n} \big\| \sigma f(x_n) - BTu_n \big\|$$

$$\leq \|u_{n+1} - u_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \big\| \sigma f(x_{n+1}) - BTu_{n+1} \big\| + \frac{\alpha_n}{1 - \beta_n} \big\| \sigma f(x_n) - BTu_n \big\|$$

$$\leq \|x_{n+1} - x_n\| + \sqrt{\left(\left| \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \right| + \delta \left| \frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1}} \right| \right) M}$$

$$+ \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \big\| \sigma f(x_{n+1}) - BTu_{n+1} \big\| + \frac{\alpha_n}{1 - \beta_n} \big\| \sigma f(x_n) - BTu_n \big\|.$$

Noting the condition (C1) and the boundedness of the sequences $\{u_{n+1}\}$, $\{y_{n+1}\}$, $\{z_{n+1}\}$, $\{Ax_n\}$, $\{f(x_n)\}$, and $\{BTu_n\}$, we have

$$\limsup_{n \to \infty} \big(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \big) \leq 0.$$

By Lemma 2.7, we deduce

$$\lim_{n \to \infty} \|x_n - w_n\| = 0.$$

Hence,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|x_n - w_n\| = 0. \tag{3.17}$$

Since $x_{n+1} - x_n = \alpha_n(\sigma f(x_n) - BTu_n) + (1 - \beta_n)(Tu_n - x_n)$, we obtain

$$\|Tu_n - x_n\| \leq \frac{1}{\beta_n} \big\{ \alpha_n \big\| \sigma f(x_n) - BTu_n \big\| + \|x_{n+1} - x_n\| \big\}.$$

Thus,

$$\lim_{n \to \infty} \|x_n - Tu_n\| = 0. \tag{3.18}$$

Using the firmly nonexpansiveness of $U_{\lambda_n}^F$, we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \left\| U_{\lambda_n}^F y_n - p \right\|^2 \\
&\leq \|y_n - p\|^2 - \left\| U_{\lambda_n}^F y_n - y_n \right\|^2 \\
&= \|y_n - p\|^2 - \|u_n - y_n\|^2 \\
&= \|y_n - p\|^2 - \left\| u_n - x_n - \delta A^* \left(S U_{\gamma_n}^G - I \right) A x_n \right\|^2 \\
&= \|y_n - p\|^2 - \|u_n - x_n\|^2 - \delta^2 \left\| A^* \left(S U_{\gamma_n}^G - I \right) A x_n \right\|^2 \\
&\quad + 2\delta \langle u_n - x_n, A^* \left(S U_{\gamma_n}^G - I \right) A x_n \rangle.
\end{aligned}
\tag{3.19}
$$

Applying (2.4) to (3.1) to deduce

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \left\| \alpha_n \left(\sigma f(x_n) - Bp \right) + \beta_n (x_n - T u_n) + (I - \alpha_n B)(T u_n - p) \right\|^2 \\
&\leq \left\| (I - \alpha_n B)(T u_n - p) + \beta_n (x_n - T u_n) \right\|^2 + 2\alpha_n \langle \sigma f(x_n) - Bp, x_{n+1} - p \rangle \\
&\leq \left[\|I - \alpha_n B\| \|T u_n - p\| + \beta_n \|x_n - T u_n\| \right]^2 + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\| \\
&\leq \left[(1 - \alpha_n \gamma) \|u_n - p\| + \beta_n \|x_n - T u_n\| \right]^2 + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\| \\
&= (1 - \alpha_n \gamma)^2 \|u_n - p\|^2 + \beta_n^2 \|x_n - T u_n\|^2 + 2(1 - \alpha_n \gamma)\beta_n \|u_n - p\| \|x_n - T u_n\| \\
&\quad + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\|.
\end{aligned}
\tag{3.20}
$$

It follows from (3.19) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 - \|u_n - y_n\|^2 + \beta_n^2 \|x_n - T u_n\|^2 \\
&\quad + 2(1 - \alpha_n \gamma)\beta_n \|u_n - p\| \|x_n - T u_n\| \\
&\quad + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\|.
\end{aligned}
\tag{3.21}
$$

Then

$$
\begin{aligned}
\|u_n - y_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2 \|x_n - T u_n\|^2 \\
&\quad + 2(1 - \alpha_n \gamma)\beta_n \|u_n - p\| \|x_n - T u_n\| + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\| \\
&\leq \left(\|x_n - p\| + \|x_{n+1} - p\| \right) \|x_{n+1} - x_n\| + \beta_n^2 \|x_n - T u_n\|^2 \\
&\quad + 2(1 - \alpha_n \gamma)\beta_n \|u_n - p\| \|x_n - T u_n\| + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\|.
\end{aligned}
$$

This together with (C1), (3.17), and (3.18) implies that

$$
\lim_{n \to \infty} \|u_n - y_n\| = 0.
\tag{3.22}
$$

From (3.20), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n \gamma)^2 \|u_n - p\|^2 + \beta_n^2 \|x_n - T u_n\|^2 \\
&\quad + 2(1 - \alpha_n \gamma)\beta_n \|u_n - p\| \|x_n - T u_n\| + 2\alpha_n \|\sigma f(x_n) - Bp\| \|x_{n+1} - p\|
\end{aligned}
$$

$$\leq \|y_n - p\|^2 + \beta_n^2 \|x_n - Tu_n\|^2 + 2(1 - \alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\|$$
$$+ 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\|$$
$$\leq \|x_n - p\|^2 + (\delta^2\|A\|^2 - \delta)\|Sz_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2$$
$$+ \beta_n^2\|x_n - Tu_n\|^2 + 2(1 - \alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\|$$
$$+ 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\|.$$

Hence,

$$(\delta - \delta^2\|A\|^2)\|Sz_n - Ax_n\|^2 + \delta\|z_n - Ax_n\|^2$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2\|x_n - Tu_n\|^2$$
$$+ 2(1 - \alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\| + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\|$$
$$\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \beta_n^2\|x_n - Tu_n\|^2$$
$$+ 2(1 - \alpha_n\gamma)\beta_n\|u_n - p\|\|x_n - Tu_n\| + 2\alpha_n\|\sigma f(x_n) - Bp\|\|x_{n+1} - p\|,$$

which implies that

$$\lim_{n\to\infty} \|Sz_n - Ax_n\| = \lim_{n\to\infty} \|z_n - Ax_n\| = 0.$$

So,

$$\lim_{n\to\infty} \|Sz_n - z_n\| = 0. \tag{3.23}$$

Note that

$$\|y_n - x_n\| = \|\delta A^*(SU_{\gamma_n}^G - I)Ax_n\|$$
$$\leq \delta\|A\|\|Sz_n - Ax_n\|.$$

Therefore,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.24}$$

From (3.18), (3.22), and (3.24), we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.25}$$

Now, we show that $\limsup_{n\to\infty}\langle(\sigma f - B)p, x_n - p\rangle \leq 0$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\sigma f - B)p, x_n - p\rangle = \lim_{i\to\infty}\langle(\sigma f - B)p, x_{n_i} - p\rangle. \tag{3.26}$$

Since the sequence $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $x_{n_i} \rightharpoonup z$.

Consequently, we derive from the above conclusions that

$$y_{n_i} \rightharpoonup z, \qquad u_{n_i} \rightharpoonup z, \qquad Ax_{n_i} \rightharpoonup z \quad \text{and} \quad z_{n_i} \rightharpoonup Az. \tag{3.27}$$

By the demi-closed principle of the nonexpansive mappings S and T (see Lemma 2.8), we deduce $z \in \text{Fix}(T)$ and $Az \in \text{Fix}(S)$ (according to (3.25) and (3.23), respectively).

Next, we show that $z \in \text{EP}(F)$. Since $u_n = U_{\lambda_n}^F y_n$, we have

$$F(u_n, y) + \frac{1}{\lambda_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

It follows from the monotonicity of F that

$$\frac{1}{\lambda_n} \langle y - u_n, u_n - y_n \rangle \geq F(y, u_n), \tag{3.29}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.30}$$

Since $\|u_n - y_n\| \to 0$, $u_{n_i} \rightharpoonup z$, and $\liminf_{n \to \infty} \lambda_n > 0$, we obtain $\frac{u_{n_i} - y_{n_i}}{\lambda_{n_i}} \to 0$. It follows that $0 \geq F(y, z)$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z \in C$. It follows that $F(y_t, z) \leq 0$. So,

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, z) \leq tF(y_t, y). \tag{3.31}$$

Therefore, $0 \leq F(y_t, y)$. Thus $0 \leq F(z, y)$. This implies that $z \in \text{EP}(F)$. Similarly, we can prove that $Az \in \text{EP}(G)$. To this end, we deduce $z \in \text{Fix}(T) \cap \text{EP}(F)$ and $Az \in \text{Fix}(S) \cap \text{EP}(G)$. That is to say, $z \in \Gamma$. Therefore,

$$\begin{aligned}
\limsup_{n \to \infty} \langle (\sigma f - B)p, x_n - p \rangle &= \lim_{i \to \infty} \langle (\sigma f - B)p, x_{n_i} - p \rangle \\
&= \lim_{i \to \infty} \langle (\sigma f - B)p, z - p \rangle \\
&\leq 0. \tag{3.32}
\end{aligned}$$

Finally, we prove $x_n \to p$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \\
&= \langle \alpha_n (\sigma f(x_n) - Bp) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n B)(Tu_n - p), x_{n+1} - p \rangle \\
&= \alpha_n \langle \sigma f(x_n) - Bp, x_{n+1} - p \rangle + \beta_n \langle x_n - p, x_{n+1} - p \rangle \\
&\quad + \langle ((1 - \beta_n)I - \alpha_n B)(Tu_n - p), x_{n+1} - p \rangle \\
&\leq \alpha_n \sigma \langle f(x_n) - f(p), x_{n+1} - p \rangle + \alpha_n \langle \sigma f(p) - Bp, x_{n+1} - p \rangle \\
&\quad + \beta_n \|x_n - p\| \|x_{n+1} - p\| + (1 - \beta_n - \alpha_n \gamma) \|Tu_n - p\| \|x_{n+1} - p\| \\
&\leq [1 - (\gamma - \sigma\rho)\alpha_n] \|x_n - p\| \|x_{n+1} - p\| + \alpha_n \langle \sigma f(p) - Bp, x_{n+1} - p \rangle \\
&\leq \frac{1 - (\gamma - \sigma\rho)\alpha_n}{2} \|x_n - p\|^2 + \frac{1}{2} \|x_{n+1} - p\|^2 + \alpha_n \langle \sigma f(p) - Bp, x_{n+1} - p \rangle. \tag{3.33}
\end{aligned}$$

It follows that

$$\|x_{n+1} - p\|^2 \leq \left[1 - (\gamma - \sigma\rho)\alpha_n\right]\|x_n - p\|^2 + 2\alpha_n\langle\sigma f(p) - Bp, x_{n+1} - p\rangle. \tag{3.34}$$

Applying Lemma 2.9 and (3.32) to (3.34), we deduce $x_n \to p$. The proof is completed. □

Algorithm 3.3 Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$x_{n+1} = \alpha_n\sigma f(x_n) + \beta_n x_n + \left((1 - \beta_n)I - \alpha_n B\right)T\left(x_n + \delta A^*(S - I)Ax_n\right) \tag{3.35}$$

for all $n \in \mathbb{N}$, where $\delta \in (0, \frac{1}{\|A\|^2})$ and $\sigma > 0$ are two constants and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Corollary 3.4 *Suppose* $\Gamma_1 = \{x \in \mathrm{Fix}(T) : Ax \in \mathrm{Fix}(S)\} \neq \emptyset$ *and suppose the following conditions hold*:

(C1): $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2): $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3): $\sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.35) *converges strongly to* $p = \mathrm{Proj}_{\Gamma_1}(\sigma f + I - B)p$, *which solves the following VI*:

$$\langle(\sigma f - B)x, y - x\rangle \leq 0, \quad \forall y \in \Gamma_1.$$

Algorithm 3.5 Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$x_{n+1} = \alpha_n\sigma f(x_n) + \beta_n x_n + \left((1 - \beta_n)I - \alpha_n B\right)U_{\lambda_n}^F\left(x_n + \delta A^*\left(U_{\gamma_n}^G - I\right)Ax_n\right) \tag{3.36}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are two real number sequences in $(0, \infty)$, $\delta \in (0, \frac{1}{\|A\|^2})$ and $\sigma > 0$ are two constants and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Corollary 3.6 *Suppose* $\Gamma_2 = \{x \in \mathrm{EP}(F) : Ax \in \mathrm{EP}(G)\} \neq \emptyset$ *and suppose the following conditions hold*:

(C1): $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2): $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3): $\liminf_{n\to\infty}\lambda_n > 0$ *and* $\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n} = 1$;

(C4): $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\lim_{n\to\infty}\frac{\gamma_{n+1}}{\gamma_n} = 1$;

(C5): $\sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.36) *converges strongly to* $p = \mathrm{Proj}_{\Gamma_2}(\sigma f + I - B)p$, *which solves the following VI*:

$$\langle(\sigma f - B)x, y - x\rangle \leq 0, \quad \forall y \in \Gamma_2.$$

Algorithm 3.7 Taking $x_0 \in H_1$ arbitrarily, we define a sequence $\{x_n\}$ by the following:

$$x_{n+1} = \alpha_n\sigma f(x_n) + \beta_n x_n + \left((1 - \beta_n)I - \alpha_n B\right)P_C\left(x_n + \delta A^*(P_Q - I)Ax_n\right) \tag{3.37}$$

for all $n \in \mathbb{N}$, where $\delta \in (0, \frac{1}{\|A\|^2})$ and $\sigma > 0$ are two constants and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0, 1)$.

Corollary 3.8 *Suppose* $\Gamma_3 = \{x \in C : Ax \in Q\} \neq \emptyset$ *and suppose the following conditions hold*:

(C1): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2): $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3): $\sigma\rho < \gamma$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.37) *converges strongly to* $p = \mathrm{Proj}_{\Gamma_3}(\sigma f + I - B)p$, *which solves the following VI*:

$$\langle (\sigma f - B)x, y - x \rangle \leq 0, \quad \forall y \in \Gamma_3.$$

Author details
[1]School of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan, 750021, China. [2]School of Mathematics and Information Technology, Nanjing Xiaozhuang University, Nanjing, 211171, China. [3]Department of Information Management, Cheng Shiu University, Kaohsiung, 833, Taiwan. [4]Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung, 807, Taiwan. [5]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China.

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