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On fixed point theorems for mappings with PPF dependence

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Abstract

In this paper, we consider two families Ψ_1 and Ψ_2 of mappings defined on $[0, +\infty)$ satisfy some certain properties. Using the mentioned properties for Ψ_1 and Ψ_2, we prove the analogous PPF dependent fixed point theorems for mappings as in Drici *et al.* (Nonlinear Anal. 67:641-647, 2007) in partially ordered Banach spaces where mappings satisfy the weaker contractive conditions without assuming the topological closedness with respect to the norm topology for the Razumikhin class $\mathcal{R}_c$.

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1 Introduction and preliminary results

The fixed point theorems for mappings satisfying certain contractive conditions have been continually studied for decade (see [1–9] and references contained therein). Bernfeld *et al.* [10] proved the existence of PPF (past, present and future) dependent fixed points in the Razumikhin class for mappings that have different domains and ranges. After that Dhage [11] extended the existence of PPF dependent fixed points to PPF common dependent fixed points for mappings satisfying the weaker contractive conditions. In 2007, Drici *et al.* [2] proved the fixed point theorems in partially ordered metric spaces for mappings with PPF dependence. In this paper, we consider two families Ψ_1 and Ψ_2 of mappings defined on $[0, +\infty)$ satisfy some certain properties. Moreover, the PPF dependent fixed point theorems for mappings satisfying some generalized contractive conditions in partially ordered Banach spaces are proven using the mentioned properties for Ψ_1 and Ψ_2.

Suppose that E is a real Banach space with the norm $\|\cdot\|_E$ and I is a closed interval $[a, b]$ in $\mathbb{R}$. Let $E_0 = C(I, E)$ be the set of all continuous E-valued mappings on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E, \tag{1.1}$$

for all $\phi \in E_0$. For a fixed element $c \in I$, the Razumikhin class of mappings in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{1.2}$$

Recall that a point $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Example 1.1 Let $T : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by

$$T\phi = \frac{1}{2}\Bigl(\sup_{t\in[0,1]} \bigl|\phi(t)\bigr|\Bigr) \quad \text{for all } \phi \in C\bigl([0,1],\mathbb{R}\bigr).$$

Therefore T is a contraction with a constant $\frac{1}{2}$. Suppose that $\phi(t) = t^2 + 1$ for all $t \in [0,1]$. Since $T\phi = \frac{1}{2}(\sup_{t\in[0,1]} |\phi(t)|) = 1 = \phi(0)$, we find that ϕ is a fixed point with dependence of T.

Definition 1.2 Let A be a subset of E. Then

(i) A is said to be topologically closed with respect to the norm topology if for each sequence $\{x_n\}$ in A with $x_n \to x$ as $n \to \infty$ implies $x \in A$.

(ii) A is said to be algebraically closed with respect to the difference if $x - y \in A$ for all $x, y \in A$.

Recently, Dhage [11] proved the existence of PPF fixed points for mappings satisfying the condition of Cirić type generalized contraction assuming topological closedness with respect to the norm topology for a Razumikhin class.

Definition 1.3 (Dhage, [11]) A mapping $T : E_0 \to E$ is said to satisfy the condition of Cirić type generalized contraction if there exists a real number $\lambda \in [0,1)$ satisfying

$$\begin{aligned} \|T\phi - T\alpha\| \le \lambda \max\Bigl\{ &\|\phi - \alpha\|_{E_0}, \bigl\|\phi(c) - T\phi\bigr\|_E, \bigl\|\alpha(c) - T\alpha\bigr\|_E, \\ &\frac{1}{2}\bigl[\bigl\|\phi(c) - T\alpha\bigr\|_E + \bigl\|\alpha(c) - T\phi\bigr\|_E\bigr]\Bigr\}, \end{aligned} \tag{1.3}$$

for all $\phi, \alpha \in E_0$ and for some $c \in I$.

Theorem 1.4 (Dhage, [11]) *Suppose that $T : E_0 \to E$ satisfies the condition of Cirić type generalized contraction. Assume that $\mathcal{R}_c$ is topologically closed with respect to the norm topology and is algebraically closed with respect to the difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

It is a natural question that the result of the previous theorem is still valid by omitting the topological closedness of $\mathcal{R}_c$. In the next result, we will answer the question.

Proposition 1.5 *The Razumikhin class $\mathcal{R}_c$ is topologically closed with respect to the norm topology.*

Proof Let $\{\phi_n\}$ be a sequence in $\mathcal{R}_c$ converging to ϕ. This implies that

$$\lim_{n\to\infty} \|\phi_n - \phi\|_{E_0} = 0 \quad \text{where } \|\phi_n - \phi\|_{E_0} = \sup_{t\in I}\bigl\|\phi_n(t) - \phi(t)\bigr\|_E.$$

Therefore

$$\lim_{n\to\infty}\|\phi_n\|_{E_0} = \|\phi\|_{E_0} \quad \text{and} \quad \lim_{n\to\infty}\bigl\|\phi_n(t)\bigr\|_E = \bigl\|\phi(t)\bigr\|_E \quad \text{for all } t \in I.$$

Since $\phi_n \in \mathcal{R}_c$ for all $n \in \mathbb{N}$, we obtain $\|\phi_n\|_{E_0} = \|\phi_n(c)\|_E$. Therefore

$$\lim_{n\to\infty}\bigl\|\phi_n(c)\bigr\|_E = \|\phi\|_{E_0}.$$

By the uniqueness of the limit, we have $\|\phi\|_{E_0} = \|\phi(c)\|_E$. Hence $\phi \in \mathcal{R}_c$ and thus $\mathcal{R}_c$ is topologically closed with respect to the norm topology. □

Hence, using Proposition 1.5, we can drop the topological closedness with respect to the norm topology for $\mathcal{R}_c$ in Theorem 1.4.

The following example shows that the algebraic closedness with respect to the difference of Razumikhin class $\mathcal{R}_c$ may fail.

Example 1.6 Let $E_0 = C([0,1],\mathbb{R})$ and $c = 1$. If we take $\phi(t) = t^2$ and $\alpha(t) = t$ for all $t \in [0,1]$, then $\phi, \alpha \in \mathcal{R}_c$ while $\phi - \alpha \notin \mathcal{R}_c$.

Proposition 1.7 *If the Razumikhin class $\mathcal{R}_c$ is algebraically closed with respect to the difference, then $\mathcal{R}_c$ is a convex set.*

Proof Since $\mathcal{R}_c$ is algebraically closed with respect to the difference, we have $\mathcal{R}_c - \mathcal{R}_c \subseteq \mathcal{R}_c$. Using the fact that $-\mathcal{R}_c = \mathcal{R}_c$, we obtain $\mathcal{R}_c + \mathcal{R}_c \subseteq \mathcal{R}_c$. Since $\lambda\mathcal{R}_c \subseteq \mathcal{R}_c$ for all $\lambda \in [0,1]$, we get $\lambda\mathcal{R}_c + (1-\lambda)\mathcal{R}_c \subseteq \mathcal{R}_c$. Hence $\mathcal{R}_c$ is a convex set. □

One can verify that Razumikhin class $\mathcal{R}_c$ is a cone (*i.e.*, $\lambda\phi \in \mathcal{R}_c$, for each $\phi \in \mathcal{R}_c$ and $\lambda \geq 0$). Then by applying the previous theorem Razumikhin class $\mathcal{R}_c$ is a convex cone (also closed).

In 2007, Drici *et al.* [2] proved the following the fixed point theorems in partially ordered complete metric spaces for mappings with PPF dependence.

Theorem 1.8 ([2]) *Let $(E, d, \leq)$ be a partially ordered complete metric space and $T : E_0 \to E$ where $E_0 = C(I, E)$ and $I = [a, b]$. Assume that*

(i) *T is a nondecreasing mapping;*
(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, $d(T\phi, T\alpha) \leq kd_0(\phi, \alpha)$ where $d_0(\phi, \alpha) = \max_{s\in I} d(\phi(s), \alpha(s))$ and $k \in [0,1)$;*
(iii) *there exists a lower solution ϕ_0 such that $\phi_0(c) \leq T\phi_0$;*
(iv) *T is a continuous mapping or if $\{\phi_n\}$ is a nondecreasing sequence in E_0 converging to $\phi \in E_0$, then $\phi_n \leq \phi$ for all $n \in \mathbb{N}$.*

Then T has a PPF dependent fixed point in E_0.

It is a natural question if one can obtain the result of the aforementioned theorem for a generalized contraction (that is, condition (ii)). One of the aims of this paper is to answer the question by considering a general case of Ćirić type generalized contractions.

In this paper, we consider two families Ψ_1 and Ψ_2 of mappings defined on $[0, +\infty)$ satisfying some certain properties. Using the mentioned properties for Ψ_1 and Ψ_2, we prove the analogous PPF dependent fixed point theorems for mappings as in [2] in partially ordered real Banach spaces where mappings satisfy the weaker contractive conditions.

2 PPF dependent fixed points in partially ordered Banach spaces

We begin this part with the consideration of the example of a partially ordered real Banach space. Recall that the set $B(X,\mathbb{R})$ of all bounded linear operators from a normed space X into $\mathbb{R}$ is a real Banach space with a norm defined by

$$\|f\| = \sup_{x\in X,\|x\|=1} \bigl|f(x)\bigr| \quad \text{for all } f \in B(X,\mathbb{R}).$$

We know that $B(X,\mathbb{R})$ is a partially ordered real Banach space with a partial order defined as follows:

$$f \le g \quad \text{if and only if} \quad f(x) \le g(x) \quad \text{for all } x \in X.$$

From now on, let $(E,\le)$ be a partially ordered real Banach space. In this paper, we use the following notations:

$$\begin{aligned}
\Psi_1 = \Biggl\{ &\psi : \psi : [0,+\infty) \to [0,+\infty) \text{ is nondecreasing with} \\
&\sum_{n=1}^{\infty} \psi^n(t) < \infty \text{ for all } t \in (0,+\infty) \Biggr\} \quad \text{and} \\
\Psi_2 = \bigl\{ &\psi : \psi : [0,+\infty) \to [0,+\infty) \text{ is continuous nondecreasing with} \\
&\psi(t) < t \text{ for all } t \in (0,+\infty) \bigr\}.
\end{aligned}$$

Lemma 2.1 ([12]) *Suppose that* $\psi : [0,+\infty) \to [0,+\infty)$. *If* ψ *is nondecreasing, then for each* $t \in (0,+\infty)$, $\lim_{n\to\infty} \psi^n(t) = 0$ *implies* $\psi(t) < t$.

Hence the difference between an element of Ψ_1 and an element of Ψ_2 is continuity.

Remark 2.2
(i) It is easily seen that if $\psi : [0,+\infty) \to [0,+\infty)$ is nondecreasing and $\psi(t) < t$ for all $t \in (0,+\infty)$, then $\psi(0) = 0$.
(ii) We can see that if $\psi : [0,+\infty) \to [0,+\infty)$, $\psi(t) < t$ for all $t \in (0,+\infty)$ and $\psi(0) = 0$, then ψ is continuous at 0.

We now prove the PPF dependent fixed point theorems for mappings satisfying the generalized contractive conditions concerning with $\psi \in \Psi_1$ without assuming the topological closedness with respect to the norm topology for the Razumikhin class $\mathcal{R}_c$.

Theorem 2.3 *Suppose that* $\psi \in \Psi_1$, $c \in I$ *and* $T : E_0 \to E$ *satisfies the following conditions*:
(i) T *is a nondecreasing mapping*;

(ii) *for all* $\phi, \alpha \in E_0$ *with* $\phi \preceq \alpha$, *we have*

$$\|T\phi - T\alpha\|_E \leq \psi\left(\max\left\{\|\phi - \alpha\|_{E_0}, \left\|\phi(c) - T\phi\right\|_E, \left\|\alpha(c) - T\alpha\right\|_E, \frac{1}{2}\left[\left\|\phi(c) - T\alpha\right\|_E + \left\|\alpha(c) - T\phi\right\|_E\right]\right\}\right);$$

(iii) *there exists a lower solution* $\phi_0 \in \mathcal{R}_c$ *such that* $\phi_0(c) \preceq T\phi_0$;
(iv) T *is a continuous mapping.*
Assume that $\mathcal{R}_c$ *is algebraically closed with respect to the difference. Then* T *has a PPF dependent fixed point in* $\mathcal{R}_c$.

Proof Since $\phi_0 \in \mathcal{R}_c$ and $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that $x_1 = \phi_1(c)$. Since ϕ_0 is a lower solution in $\mathcal{R}_c$ such that $\phi_0(c) \preceq T\phi_0$, it follows that $\phi_0 \preceq \phi_1$. Using the algebraic closedness with respect to the difference of $\mathcal{R}_c$, this yields

$$\|\phi_0 - \phi_1\|_{E_0} = \left\|\phi_0(c) - \phi_1(c)\right\|_E.$$

By the fact that T is nondecreasing, we obtain

$$\phi_1(c) = T\phi_0 \preceq T\phi_1.$$

By induction, we can construct the sequence $\{\phi_n\}$ such that

$$\begin{aligned}
&T\phi_0 = \phi_1(c) \preceq T\phi_1 = \phi_2(c) \preceq \cdots \preceq T\phi_n = \phi_{n+1}(c) \preceq T\phi_{n+1} \cdots, \\
&\phi_n \preceq \phi_{n+1} \quad \text{and} \quad \|\phi_n - \phi_{n+1}\|_{E_0} = \left\|\phi_n(c) - \phi_{n+1}(c)\right\|_E,
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Assume that $\phi_{n-1} = \phi_n$ for some $n \in \mathbb{N}$. It follows that $\phi_{n-1}(c) = \phi_n(c) = T\phi_{n-1}$. Therefore T has a fixed point in $\mathcal{R}_c$. Suppose that $\phi_{n-1} \neq \phi_n$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \left\|\phi_n(c) - \phi_{n+1}(c)\right\|_E \\
&= \|T\phi_n - T\phi_{n-1}\|_E \\
&\leq \psi\left(\max\left\{\|\phi_n - \phi_{n-1}\|_{E_0}, \left\|\phi_n(c) - T\phi_n\right\|_E, \left\|\phi_{n-1}(c) - T\phi_{n-1}\right\|_E,\right.\right. \\
&\quad \left.\left.\frac{1}{2}\left[\left\|\phi_n(c) - T\phi_{n-1}\right\|_E + \left\|\phi_{n-1}(c) - T\phi_n\right\|_E\right]\right\}\right) \\
&= \psi\left(\max\left\{\|\phi_n - \phi_{n-1}\|_{E_0}, \left\|\phi_n(c) - \phi_{n+1}(c)\right\|_E, \left\|\phi_{n-1}(c) - \phi_n(c)\right\|_E,\right.\right. \\
&\quad \left.\left.\frac{1}{2}\left[\left\|\phi_n(c) - \phi_n(c)\right\|_E + \left\|\phi_{n-1}(c) - \phi_{n+1}(c)\right\|_E\right]\right\}\right) \\
&\leq \psi\left(\max\left\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \frac{1}{2}\|\phi_{n-1} - \phi_{n+1}\|_{E_0}\right\}\right) \\
&\leq \psi\left(\max\left\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0},\right.\right.
\end{aligned}$$

$$\left.\left.\frac{1}{2}\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_n-\phi_{n+1}\|_{E_0}\right\}\right)$$
$$\leq \psi\left(\max\left\{\|\phi_n-\phi_{n-1}\|_{E_0},\|\phi_n-\phi_{n+1}\|_{E_0}\right\}\right).$$

If $\max\{\|\phi_n-\phi_{n-1}\|_{E_0},\|\phi_n-\phi_{n+1}\|_{E_0}\}=\|\phi_n-\phi_{n+1}\|_{E_0}$, then

$$\|\phi_n-\phi_{n+1}\|_{E_0}\leq\psi\left(\|\phi_n-\phi_{n+1}\|_{E_0}\right)<\|\phi_n-\phi_{n+1}\|_{E_0},$$

which leads to a contradiction. Therefore, for each $n\in\mathbb{N}$, we have

$$\|\phi_n-\phi_{n+1}\|_{E_0}\leq\psi\left(\|\phi_n-\phi_{n-1}\|_{E_0}\right).$$

By induction, we obtain

$$\|\phi_n-\phi_{n+1}\|_{E_0}\leq\psi^n\left(\|\phi_0-\phi_1\|_{E_0}\right).$$

Fix $\varepsilon>0$. This implies that there exists $N\in\mathbb{N}$ such that

$$\sum_{n\geq N}\psi^n\left(\|\phi_0-\phi_1\|_{E_0}\right)<\varepsilon.$$

For each $m,n\in\mathbb{N}$ with $m>n>N$, we obtain

$$\|\phi_n-\phi_m\|_{E_0}\leq\sum_{k=n}^{m-1}\|\phi_k-\phi_{k+1}\|_{E_0}\leq\sum_{k=n}^{m-1}\psi^k\left(\|\phi_0-\phi_1\|_{E_0}\right)\leq\sum_{n\geq N}\psi^n\left(\|\phi_0-\phi_1\|_{E_0}\right)<\varepsilon.$$

This implies that $\{\phi_n\}$ is a Cauchy sequence. By the completeness of E_0, we have $\lim_{n\to\infty}\phi_n=\phi$ for some $\phi\in E_0$ and

$$\lim_{n\to\infty}T\phi_n=\lim_{n\to\infty}\phi_{n+1}(c)=\phi(c).$$

Since $\mathcal{R}_c$ is algebraically closed with respect to the norm topology, we have $\phi\in\mathcal{R}_c$. We next prove that ϕ is a PPF dependent fixed point of T. Using the continuity of T, we obtain $\lim_{n\to\infty}T\phi_n=T\phi$. By the uniqueness of the limit, we have $T\phi=\phi(c)$. □

Remark 2.4

(i) From the proof of Theorem 2.3, we assume that the Razumikhin class $\mathcal{R}_c$ is algebraically closed with respect to difference, that is, $\phi-\alpha\in\mathcal{R}_c$ for all $\phi,\alpha\in\mathcal{R}_c$, in order to construct the sequence $\{\phi_n\}$ satisfying

$$\|\phi_n-\phi_{n+1}\|_{E_0}=\left\|\phi_n(c)-\phi_{n+1}(c)\right\|_E\quad\text{for all } n\in\mathbb{N}\cup\{0\}.$$

(ii) In the proof of Theorem 2.3, if we choose $\phi_n\in\mathcal{R}_c$ to be a constant mapping for each $n\in\mathbb{N}\cup\{0\}$, then

$$\|\phi_n-\phi_{n+1}\|_{E_0}=\left\|\phi_n(c)-\phi_{n+1}(c)\right\|_E\quad\text{for all } n\in\mathbb{N}\cup\{0\}.$$

Therefore the algebraic closedness with respect to the difference of $\mathcal{R}_c$ can be dropped.

Example 2.5 Let $E = \mathbb{R}^2$ with respect to the norm $\|(x,y)\| = |x| + |y|$ and $E_0 = C([0,1],\mathbb{R}^2)$. Define a mapping $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{t}{3} \quad \text{for all } t \in [0,\infty).$$

We see that ψ is a nondecreasing mapping with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \in (0,+\infty)$. Let $\phi \in E_0$. Therefore, for each $t \in [0,1]$, we obtain $\phi(t) \in \mathbb{R}^2$. Thus we can define mappings $f_\phi, g_\phi : [0,1] \to \mathbb{R}$ such that

$$\phi(t) = \bigl(f_\phi(t), g_\phi(t)\bigr) \quad \text{for all } t \in [0,1].$$

Define a mapping $T : E_0 \to E$ by

$$T\phi = \left(\frac{1}{4}\|f_\phi\|, \frac{1}{4}\|g_\phi\|\right) \quad \text{for all } \phi \in E_0,$$

where $\|f_\phi\| = \sup_{t\in[0,1]} |f_\phi(t)|$ and $\|g_\phi\| = \sup_{t\in[0,1]} |g_\phi(t)|$. Suppose that $\phi, \alpha \in E_0$ with $\phi \preceq \alpha$. For each $c \in [0,1]$, we obtain

$$\begin{aligned}
\|T\phi - T\alpha\|_E &= \left\| \left(\frac{1}{4}\|f_\phi\|, \frac{1}{4}\|g_\phi\|\right) - \left(\frac{1}{4}\|f_\alpha\|, \frac{1}{4}\|g_\alpha\|\right) \right\|_E \\
&= \left| \frac{1}{4}\|f_\phi\| - \frac{1}{4}\|f_\alpha\| \right| + \left| \frac{1}{4}\|g_\phi\| - \frac{1}{4}\|g_\alpha\| \right| \\
&\leq \frac{1}{4}\bigl(\|f_\phi - f_\alpha\| + \|g_\phi - g_\alpha\|\bigr) \\
&= \frac{1}{4}\Bigl(\sup_{t\in[0,1]} \bigl|f_\phi(t) - f_\alpha(t)\bigr| + \sup_{t\in[0,1]} \bigl|g_\phi(t) - g_\alpha(t)\bigr|\Bigr) \\
&= \frac{1}{4}\Bigl(\sup_{t\in[0,1]} \bigl(\bigl|f_\phi(t) - f_\alpha(t)\bigr| + \bigl|g_\phi(t) - g_\alpha(t)\bigr|\bigr)\Bigr) \\
&= \frac{1}{4}\Bigl(\sup_{t\in[0,1]} \bigl\|\bigl(f_\phi(t) - f_\alpha(t), g_\phi(t) - g_\alpha(t)\bigr)\bigr\|_E\Bigr) \\
&= \frac{1}{4}\Bigl(\sup_{t\in[0,1]} \bigl\|\bigl(f_\phi(t), g_\phi(t)\bigr) - \bigl(f_\alpha(t), g_\alpha(t)\bigr)\bigr\|_E\Bigr) \\
&= \frac{1}{4}\Bigl(\sup_{t\in[0,1]} \bigl\|\phi(t) - \alpha(t)\bigr\|_E\Bigr) \\
&= \frac{1}{4}\bigl(\|\phi - \alpha\|_{E_0}\bigr) \\
&\leq \psi\Bigl(\max\Bigl\{\|\phi - \alpha\|_{E_0}, \bigl\|\phi(c) - T\phi\bigr\|_E, \bigl\|\alpha(c) - T\alpha\bigr\|_E, \\
&\qquad \frac{1}{2}\bigl[\bigl\|\phi(c) - T\alpha\bigr\|_E + \bigl\|\alpha(c) - T\phi\bigr\|_E\bigr]\Bigr\}\Bigr).
\end{aligned}$$

Suppose that $\phi_0 = 0$. We see that all assumptions in Theorem 2.3 are now satisfied and 0 is the PPF dependent fixed point of T in $\mathcal{R}_c$.

By applying Theorem 2.3, we obtain the following corollary.

Corollary 2.6 *Suppose that $c \in I$ and $T : E_0 \to E$ satisfies the following conditions*:

(i) *T is a nondecreasing mapping*;

(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, we have*

$$\|T\phi - T\alpha\|_E \leq k\biggl(\max\biggl\{\|\phi - \alpha\|_{E_0}, \bigl\|\phi(c) - T\phi\bigr\|_E, \bigl\|\alpha(c) - T\alpha\bigr\|_E, \frac{1}{2}\bigl[\bigl\|\phi(c) - T\alpha\bigr\|_E + \bigl\|\alpha(c) - T\phi\bigr\|_E\bigr]\biggr\}\biggr), \quad \textit{where } k \in [0,1);$$

(iii) *there exists a lower solution $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \leq T\phi_0$*;

(iv) *T is a continuous mapping.*

Assume that $\mathcal{R}_c$ is algebraically closed with respect to the difference. Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof Define a function $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = kt$ for all $t \in [0,+\infty)$. Therefore ψ is a nondecreasing mapping and

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty \quad \text{for all } t \in (0,+\infty).$$

This implies that all assumptions in Theorem 2.3 are satisfied. Hence we obtain the desired result. □

Theorem 2.7 *Suppose that $\psi \in \Psi_1$, $c \in I$ and $T : E_0 \to E$ satisfies the following conditions*:

(i) *T is a nondecreasing mapping*;

(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, $\|T\phi - T\alpha\|_E \leq \psi(\|\phi - \alpha\|_{E_0})$*;

(iii) *there exists a lower solution $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \leq T\phi_0$*;

(iv) *if $\{\phi_n\}$ is a nondecreasing sequence in E_0 converging to $\phi \in E_0$, then $\phi_n \leq \phi$ for all $n \in \mathbb{N}$.*

Assume that $\mathcal{R}_c$ is algebraically closed with respect to the difference. Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof By the analogous proof as in Theorem 2.3, we can construct a nondecreasing sequence $\{\phi_n\}$ in $\mathcal{R}_c$ converging to $\phi \in \mathcal{R}_c$. This implies that $\phi_n \leq \phi$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\bigl\|T\phi - \phi(c)\bigr\|_E &\leq \bigl\|T\phi - \phi_{n+1}(c)\bigr\|_E + \bigl\|\phi_{n+1}(c) - \phi(c)\bigr\|_E \\
&\leq \|T\phi - T\phi_n\|_E + \|\phi_{n+1} - \phi\|_{E_0} \\
&\leq \psi\bigl(\|\phi - \phi_n\|_{E_0}\bigr) + \|\phi_{n+1} - \phi\|_{E_0}.
\end{aligned}$$

Since ψ is continuous at 0, we get $\lim_{n\to\infty} \psi(\|\phi - \phi_n\|_{E_0}) = \psi(0) = 0$. Taking the limit of the above inequality, this yields $\|T\phi - \phi(c)\|_E = 0$ and so ϕ is a PPF dependent fixed point of T in $\mathcal{R}_c$. □

We next ensure the result on PPF dependent fixed points for mappings concerning with $\psi \in \Psi_2$.

Theorem 2.8 *Suppose that $\psi \in \Psi_2$, $c \in I$ and $T : E_0 \to E$ satisfies the following conditions*:

(i) *T is a nondecreasing mapping*;

(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, we have*

$$\begin{aligned}\|T\phi - T\alpha\|_E \leq \psi\Bigg(\max\bigg\{&\|\phi - \alpha\|_{E_0}, \big\|\phi(c) - T\phi\big\|_E, \big\|\alpha(c) - T\alpha\big\|_E, \\ &\frac{1}{2}\big[\big\|\phi(c) - T\alpha\big\|_E + \big\|\alpha(c) - T\phi\big\|_E\big]\bigg\}\Bigg);\end{aligned}$$

(iii) *there exists a lower solution $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \leq T\phi_0$*;

(iv) *T is a continuous mapping.*

Assume that $\mathcal{R}_c$ is algebraically closed with respect to the difference. Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof Since $\phi_0 \in \mathcal{R}_c$ and $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. As in the proof of Theorem 2.3, we can construct the sequence $\{\phi_n\}$ such that

$$\begin{aligned}&T\phi_0 = \phi_1(c) \leq T\phi_1 = \phi_2(c) \leq \cdots \leq T\phi_n = \phi_{n+1}(c) \leq T\phi_{n+1}\cdots, \\ &\phi_n \leq \phi_{n+1} \quad \text{and} \quad \|\phi_n - \phi_{n+1}\|_{E_0} = \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E,\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Assume that $\phi_{n-1} = \phi_n$ for some $n \in \mathbb{N}$. It follows that $\phi_{n-1}(c) = \phi_n(c) = T\phi_{n-1}$. Therefore T has a fixed point in $\mathcal{R}_c$. Suppose that $\phi_{n-1} \neq \phi_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E \\ &= \|T\phi_n - T\phi_{n-1}\|_E \\ &\leq \psi\Bigg(\max\bigg\{\|\phi_n - \phi_{n-1}\|_{E_0}, \big\|\phi_n(c) - T\phi_n\big\|_E, \big\|\phi_{n-1}(c) - T\phi_{n-1}\big\|_E, \\ &\quad \frac{1}{2}\big[\big\|\phi_n(c) - T\phi_{n-1}\big\|_E + \big\|\phi_{n-1}(c) - T\phi_n\big\|_E\big]\bigg\}\Bigg) \\ &= \psi\Bigg(\max\bigg\{\|\phi_n - \phi_{n-1}\|_{E_0}, \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E, \big\|\phi_{n-1}(c) - \phi_n(c)\big\|_E, \\ &\quad \frac{1}{2}\big[\big\|\phi_n(c) - \phi_n(c)\big\|_E + \big\|\phi_{n-1}(c) - \phi_{n+1}(c)\big\|_E\big]\bigg\}\Bigg) \\ &\leq \psi\bigg(\max\bigg\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \frac{1}{2}\|\phi_{n-1} - \phi_{n+1}\|_{E_0}\bigg\}\bigg) \\ &\leq \psi\Bigg(\max\bigg\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}, \\ &\quad \frac{1}{2}\|\phi_{n-1} - \phi_n\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}\bigg\}\Bigg) \\ &\leq \psi\big(\max\big\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\big\}\big).\end{aligned}$$

If $\max\{\|\phi_n - \phi_{n-1}\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\} = \|\phi_n - \phi_{n+1}\|_{E_0}$, then

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \psi\big(\|\phi_n - \phi_{n+1}\|_{E_0}\big) < \|\phi_n - \phi_{n+1}\|_{E_0}.$$

This leads to a contradiction. Therefore

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &\leq \psi\big(\|\phi_n - \phi_{n-1}\|_{E_0}\big)\\ &< \|\phi_n - \phi_{n-1}\|_{E_0}.\end{aligned}$$

It follows that $\|\phi_n - \phi_{n+1}\|_{E_0} \leq \|\phi_{n-1} - \phi_n\|_{E_0}$ for all $n \in \mathbb{N}$. Since the sequence $\{\|\phi_n - \phi_{n+1}\|_{E_0}\}$ is a nonincreasing sequence of nonnegative real numbers, we see that it is a convergent sequence. Suppose that

$$\lim_{n\to\infty} \|\phi_n - \phi_{n+1}\|_{E_0} = \alpha,$$

for some nonnegative real number α. We will prove that $\alpha = 0$. Suppose that $\alpha > 0$. Since

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \psi\big(\|\phi_n - \phi_{n-1}\|_{E_0}\big),$$

for all $n \in \mathbb{N}$ and the continuity of ψ, we have $\alpha \leq \psi(\alpha) < \alpha$ which leads to a contradiction. This implies that $\alpha = 0$. We next prove that the sequence $\{\phi_n\}$ is a Cauchy sequence in E_0. Assume that $\{\phi_n\}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ satisfying $m_k > n_k > k$ for each $k \in \mathbb{N}$ and

$$\|\phi_{m_k} - \phi_{n_k}\|_{E_0} \geq \varepsilon. \tag{2.1}$$

Let $\{m_k\}$ be the sequence of the least positive integers exceeding $\{n_k\}$ which satisfies (2.1) and

$$\|\phi_{m_k-1} - \phi_{n_k}\|_{E_0} < \varepsilon. \tag{2.2}$$

We will prove that $\lim_{k\to\infty} \|\phi_{m_k} - \phi_{n_k}\|_{E_0} = \varepsilon$. Since $\|\phi_{m_k} - \phi_{n_k}\|_{E_0} \geq \varepsilon$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \|\phi_{m_k} - \phi_{n_k}\|_{E_0} \geq \varepsilon.$$

For each $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\|\phi_{m_k} - \phi_{n_k}\|_{E_0} &\leq \|\phi_{m_k} - \phi_{m_k-1}\|_{E_0} + \|\phi_{m_k-1} - \phi_{n_k}\|_{E_0}\\ &\leq \|\phi_{m_k} - \phi_{m_k-1}\|_{E_0} + \varepsilon.\end{aligned}$$

This implies that $\lim_{k\to\infty} \|\phi_{m_k} - \phi_{n_k}\|_{E_0} \leq \varepsilon$. Therefore

$$\lim_{k\to\infty} \|\phi_{m_k} - \phi_{n_k}\|_{E_0} = \varepsilon.$$

Similarly, we can prove that

$$\lim_{k\to\infty} \|\phi_{m_k+1} - \phi_{n_k}\|_{E_0} = \varepsilon, \qquad \lim_{k\to\infty} \|\phi_{m_k} - \phi_{n_k-1}\|_{E_0} = \varepsilon,$$

and

$$\lim_{k\to\infty} \|\phi_{m_k+1} - \phi_{n_k-1}\|_{E_0} = \varepsilon.$$

Since $\mathcal{R}_c$ is algebraically closed with respect to the difference, for each $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|\phi_{n_k} - \phi_{m_k+1}\|_{E_0} &= \bigl\|\phi_{n_k}(c) - \phi_{m_k+1}(c)\bigr\|_E \\
&= \|T\phi_{m_k} - T\phi_{n_k-1}\|_E \\
&\leq \psi\Bigl(\max\Bigl\{\|\phi_{m_k} - \phi_{n_k-1}\|_{E_0}, \bigl\|\phi_{m_k}(c) - T\phi_{m_k}\bigr\|_E, \\
&\quad \bigl\|\phi_{n_k-1}(c) - T\phi_{n_k-1}\bigr\|_E, \\
&\quad \frac{1}{2}\bigl[\bigl\|\phi_{m_k}(c) - T\phi_{n_k-1}\bigr\|_E + \bigl\|\phi_{n_k-1}(c) - T\phi_{n_k}\bigr\|_E\bigr]\Bigr\}\Bigr) \\
&= \psi\Bigl(\max\Bigl\{\|\phi_{m_k} - \phi_{n_k-1}\|_{E_0}, \bigl\|\phi_{m_k}(c) - \phi_{m_k+1}(c)\bigr\|_E, \\
&\quad \bigl\|\phi_{n_k-1}(c) - \phi_{n_k}(c)\bigr\|_E, \\
&\quad \frac{1}{2}\bigl[\bigl\|\phi_{m_k}(c) - \phi_{n_k}(c)\bigr\|_E + \bigl\|\phi_{n_k-1}(c) - \phi_{n_k+1}(c)\bigr\|_E\bigr]\Bigr\}\Bigr) \\
&\leq \psi\Bigl(\max\Bigl\{\|\phi_{m_k} - \phi_{n_k-1}\|_{E_0}, \|\phi_{m_k} - \phi_{m_k+1}\|_{E_0}, \|\phi_{n_k-1} - \phi_{n_k}\|_{E_0}, \\
&\quad \frac{1}{2}\bigl[\|\phi_{m_k} - \phi_{n_k}\|_{E_0} + \|\phi_{n_k-1} - \phi_{n_k+1}\|_{E_0}\bigr]\Bigr\}\Bigr) \\
&\leq \psi\Bigl(\max\Bigl\{\|\phi_{m_k} - \phi_{n_k-1}\|_{E_0}, \|\phi_{m_k} - \phi_{m_k+1}\|_{E_0}, \|\phi_{n_k-1} - \phi_{n_k}\|_{E_0}, \\
&\quad \frac{1}{2}\bigl[\|\phi_{m_k} - \phi_{n_k}\|_{E_0} + \|\phi_{n_k-1} - \phi_{n_k}\|_{E_0} + \|\phi_{n_k} - \phi_{n_k+1}\|_{E_0}\bigr]\Bigr\}\Bigr).
\end{aligned}$$

By taking the limit of both sides, we have

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon.$$

This leads to a contradiction. It follows that the sequence $\{\phi_n\}$ is a Cauchy sequence. By the completeness of E_0, we have $\lim_{n\to\infty} \phi_n = \phi$ for some $\phi \in E_0$ and

$$\lim_{n\to\infty} T\phi_n = \lim_{n\to\infty} \phi_{n+1}(c) = \phi(c).$$

Since $\mathcal{R}_c$ is algebraically closed with respect to the norm topology, we have $\phi \in \mathcal{R}_c$. We will prove that ϕ is a PPF dependent fixed point of T. Using the continuity of T, we obtain $\lim_{n\to\infty} T\phi_n = T\phi$. By the uniqueness of the limit, we can conclude that $T\phi = \phi(c)$. □

Example 2.9 Assume that $E = \mathbb{R}$ and $E_0 = C([0,1],\mathbb{R})$. Define a mapping $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{t}{2} \quad \text{for all } t \in [0,\infty).$$

We see that ψ is a continuous nondecreasing mapping with $\psi(t) < t$. Define a mapping $T : E_0 \to E$ by

$$T\phi = \frac{1}{3}\phi\left(\frac{1}{4}\right) \quad \text{for all } \phi \in E_0.$$

Suppose that $\phi, \alpha \in E_0$ with $\psi \leq \alpha$ and $c \in [0,1]$. Therefore

$$\begin{aligned} \|T\phi - T\alpha\|_E &= \frac{1}{3}\left|\phi\left(\frac{1}{4}\right) - \alpha\left(\frac{1}{4}\right)\right| \\ &\leq \frac{1}{3}\|\phi - \alpha\|_{E_0} \\ &\leq \psi\Bigg(\max\bigg\{\|\phi - \alpha\|_{E_0}, \big\|\phi(c) - T\phi\big\|_E, \big\|\alpha(c) - T\alpha\big\|_E, \\ &\quad \frac{1}{2}\big[\big\|\phi(c) - T\alpha\big\|_E + \big\|\alpha(c) - T\phi\big\|_E\big]\bigg\}\Bigg). \end{aligned}$$

Suppose that $\phi_0 = 0$. We find that all assumptions in Theorem 2.8 are now satisfied and 0 is the PPF dependent fixed point of T in $\mathcal{R}_c$.

For the next result, we drop the continuity of T.

Theorem 2.10 *Suppose that $\psi \in \Psi_2$, $c \in I$, and that $T : E_0 \to E$ satisfies the following conditions*:

(i) *T is a nondecreasing mapping*;
(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, we have*

$$\begin{aligned} \|T\phi - T\alpha\|_E \leq \psi\Bigg(&\max\bigg\{\|\phi - \alpha\|_{E_0}, \big\|\phi(c) - T\phi\big\|_E, \big\|\alpha(c) - T\alpha\big\|_E, \\ &\frac{1}{2}\big[\big\|\phi(c) - T\alpha\big\|_E + \big\|\alpha(c) - T\phi\big\|_E\big]\bigg\}\Bigg); \end{aligned}$$

(iii) *there exists a lower solution $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \leq T\phi_0$*;
(iv) *if $\{\phi_n\}$ is a nondecreasing sequence in E_0 converging to $\phi \in E_0$, then $\phi_n \leq \phi$ for all $n \in \mathbb{N}$.*

Assume that $\mathcal{R}_c$ is algebraically closed with respect to the difference. Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof As in the proof of Theorem 2.8, we can construct a nondecreasing sequence $\{\phi_n\}$ converging to $\phi \in \mathcal{R}_c$ and this yields

$$\lim_{n\to\infty} T\phi_n = \lim_{n\to\infty} \phi_{n+1}(c) = \phi(c).$$

Using (iv), we have $\phi_n \leq \phi$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|T\phi - \phi(c)\|_E &\leq \|T\phi - \phi_{n+1}(c)\|_E + \|\phi_{n+1}(c) - \phi(c)\|_E \\
&\leq \|T\phi - T\phi_n\|_E + \|\phi_{n+1} - \phi\|_{E_0} \\
&\leq \psi\Biggl(\max\Biggl\{\|\phi - \phi_n\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\phi_n(c) - T\phi_n\|_E, \\
&\quad \frac{1}{2}\bigl[\|\phi(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi\|_E\bigr]\Biggr\}\Biggr) + \|\phi_{n+1} - \phi\|_{E_0} \\
&= \psi\Biggl(\max\Biggl\{\|\phi - \phi_n\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \\
&\quad \frac{1}{2}\bigl[\|\phi(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - T\phi\|_E\bigr]\Biggr\}\Biggr) + \|\phi_{n+1} - \phi\|_{E_0} \\
&\leq \psi\Biggl(\max\Biggl\{\|\phi - \phi_n\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\phi_n - \phi_{n+1}\|_{E_0} \\
&\quad \frac{1}{2}\bigl[\|\phi - \phi_{n+1}\|_{E_0} + \|\phi_n(c) - T\phi\|_E\bigr]\Biggr\}\Biggr) + \|\phi_{n+1} - \phi\|_{E_0}.
\end{aligned}$$

Letting $n \to \infty$, we obtain $\|T\phi - \phi(c)\|_E \leq \psi(\|\phi(c) - T\phi\|_E)$. If $\phi(c) \neq T\phi$, then

$$\|T\phi - \phi(c)\|_E \leq \psi(\|\phi(c) - T\phi\|_E) < \|\phi(c) - T\phi\|_E.$$

This leads to a contradiction. Therefore $T\phi = \phi(c)$. This implies that ϕ is a PPF dependent fixed point of T. □

By applying Theorem 2.8 and Theorem 2.10, we obtain the following corollary.

Corollary 2.11 *Suppose that $c \in I$ and that $T : E_0 \to E$ satisfies the following conditions*:

(i) *T is a nondecreasing mapping*;

(ii) *for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha$, we have*

$$\begin{aligned}
\|T\phi - T\alpha\|_E \leq k\Biggl(&\max\Biggl\{\|\phi - \alpha\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\alpha(c) - T\alpha\|_E, \\
&\frac{1}{2}\bigl[\|\phi(c) - T\alpha\|_E + \|\alpha(c) - T\phi\|_E\bigr]\Biggr\}\Biggr), \quad \textit{where } k \in [0,1);
\end{aligned}$$

(iii) *there exists a lower solution $\phi_0 \in \mathcal{R}_c$ such that $\phi_0(c) \leq T\phi_0$*;

(iv) *T is a continuous mapping or if $\{\phi_n\}$ is a nondecreasing sequence in E_0 converging to $\phi \in E_0$, then $\phi_n \leq \phi$ for all $n \in \mathbb{N}$.*

Assume that $\mathcal{R}_c$ is algebraically closed with respect to the difference. Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof Define a function $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = kt$ for all $t \in [0, +\infty)$. Therefore ψ is a continuous nondecreasing mapping and

$$\psi(t) < t \quad \text{for all } t \in (0, +\infty).$$

This implies that all assumptions in Theorem 2.8 or Theorem 2.10 are satisfied. Hence the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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