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Best proximity points of implicit relation type modified α^3-proximal contractions

Farzaneh Zabihi and Abdolrahman Razani*

*Correspondence: razani@ipm.ir
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract

In this paper, we introduce the concept of an α^3-proximal admissible mappings and establish the existence of best proximity point theorems for implicit relation type modified α^3-proximal contractions. As applications of our theorems, we derive some new best proximity point results for implicit relation type contractions whenever the range space is endowed with a graph or with a partial order. The obtained results generalize, extend, and modify some best proximity point results in the literature. Several interesting consequences of our theorems are also provided.

MSC: 46N40; 47H10; 54H25; 46T99

Keywords: fixed point; best proximity point; α^3-proximal admissible mapping; implicit relation type α^3-proximal contractions; metric space endowed with graph

1 Introduction

In nonlinear functional analysis, one of the most significant research areas is fixed point theory. On the other hand, fixed point theory has an application in distinct branches of mathematics and also in different sciences, such as engineering, computer science, economics, *etc.* In 1922, Banach proved that every contraction in a complete metric space has a unique fixed point. Following this celebrated result, many authors have generalized, improved, and extended this result in the context of different abstract spaces for various operators.

On the other hand, several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation (see Popa [1, 2]). Following Popa's approach, many results on fixed point, common fixed points, and coincidence points have been obtained, in various ambient spaces (see [3–8], and references therein). On the other hand, Samet *et al.* [9] introduced and studied α-ψ-contractive mappings in complete metric spaces and provided applications of the results to ordinary differential equations. More recently, Salimi *et al.* [10] modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point theorems to modify the results in [9]. For more details and applications of this line of research, we refer the reader to some related papers [11–13] and references therein. In this paper, we introduce the concept of an α^3-proximal admissible mappings and establish the existence of best proximity point theorems for implicit relation type modified α^3-proximal contractions. As applications of our theorems, we derive some new best proximity point results for implicit relation type contractions whenever the range space is endowed with a graph or with a partial order. The obtained results generalize, extend, and modify some best proximity point results in the literature.

2 Main results

Let A and B be two nonempty subsets of metric space (X,d) and $T:A\to B$ be a nonself mapping. We say that x^* is a best proximity of T if

$$d\bigl(x^*,Tx^*\bigr)=d(A,B),$$

where

$$d(A,B)=\inf\bigl\{d(x,y):x\in A,y\in B\bigr\}.$$

We define A_0 and B_0 as follows:

$$A_0=\bigl\{x\in A:d(x,y)=d(A,B)\text{ for some }y\in B\bigr\}$$

and

$$B_0=\bigl\{y\in B:d(x,y)=d(A,B)\text{ for some }x\in A\bigr\}.$$

We denote by Ψ the set of all nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for all $t>0$, where ψ^n is the nth iterate of ψ.

Let $\mathcal{F}$ be the set of all continuous functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfying the following assertions:

(F1) if $F(u,v,v,u,u+v,0)\le 0$, where $u,v>0$, then $u\le\psi(v)$;
(F2) $F(t_1,\dots,t_6)$ is decreasing in t_5;
(F3) if $F(u,v,0,u+v,u,v)\le 0$, where $u,v\ge 0$, then $u\le\psi(v)$;
(F4) $F(u,u,0,0,u,u)>0$ for all $u>0$.

Example 1 Let

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\psi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)-L\min\{t_3,t_4,t_5,t_6\},$$

where $L\ge 0$ and $\psi\in\Psi$. Then $F\in\mathcal{F}$.

Example 2 Let

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-at_2-\frac{b[1+t_3]t_4}{1+t_2}-c[t_3+t_4]-d[t_5+t_6],$$

where $a+b+2c+2d<1$. Then $F\in\mathcal{F}$.

Definition 1 Let A, B be two nonempty subsets of a metric space (X,d) and $\alpha:A\times A\to[0,+\infty)$ be a function. We say that a nonself mapping $T:A\to B$ is α^3-proximal admissible if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{cases} \alpha(x_1, x_1) \geq 1, \\ \alpha(x_2, x_2) \geq 1, \\ \alpha(x_1, x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \quad \Longrightarrow \quad \begin{cases} \alpha(u_1, u_2) \geq 1, \\ \alpha(u_1, u_1) \geq 1, \\ \alpha(u_2, u_2) \geq 1. \end{cases}$$

Definition 2 Let A and B be nonempty subsets of a metric space (X, d) and $\alpha : A \times A \to [0, \infty)$ be a function. Then $T : A \to B$ is said to be an implicit relation type modified α^3-proximal contraction, if for all $x, y, u, v \in A$,

$$\begin{cases} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\Longrightarrow \quad F\bigl(d(u,v), d(x,y), d(x,u), d(y,v), d(x,v), d(y,u)\bigr) \leq L\bigl[1 - \alpha(x,x)\alpha(y,y)\bigr], \tag{2.1}$$

where $L \geq 0$ and $F \in \mathcal{F}$.

Definition 3 Let (X, d) be a metric space and A and B be two nonempty subsets of X. Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $d(x, y_n) \to d(x, B)$ for some x in A, has a convergent subsequence.

Theorem 1 *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete and A_0 is nonempty. Assume that $T : A \to B$ is a continuous implicit relation type modified α^3-proximal contraction such that the following conditions hold*:

(i) *T is an α^3-proximal admissible mapping and*

$$T(A_0) \subseteq B_0,$$

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B), \qquad \alpha(x_0, x_1) \geq 1, \qquad \alpha(x_0, x_0) \geq 1 \quad \textit{and} \quad \alpha(x_1, x_1) \geq 1.$$

Then T has a best proximity point. Further, the best proximity point is unique if

(iii) *for every $x, y \in A$ with $d(x, Tx) = d(A,B) = d(y, Ty)$, we have $\alpha(x, y) \geq 1$, $\alpha(x, x) \geq 1$, and $\alpha(y, y) \geq 1$.*

Proof By (ii) there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A,B), \qquad \alpha(x_0, x_1) \geq 1, \qquad \alpha(x_0, x_0) \geq 1 \quad \text{and} \quad \alpha(x_1, x_1) \geq 1.$$

On the other hand, $T(A_0) \subseteq B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B).$$

Now, since T is α^3-proximal admissible, we have

$$\alpha(x_1,x_2) \geq 1, \qquad \alpha(x_1,x_1) \geq 1 \quad \text{and} \quad \alpha(x_2,x_2) \geq 1.$$

Hence,

$$d(x_2, Tx_1) = d(A,B), \qquad \alpha(x_1,x_2) \geq 1, \qquad \alpha(x_1,x_1) \geq 1 \quad \text{and} \quad \alpha(x_2,x_2) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A,B).$$

Then we have

$$\begin{aligned} &d(x_2, Tx_1) = d(A,B), \qquad d(x_3, Tx_2) = d(A,B), \qquad \alpha(x_1,x_2) \geq 1, \\ &\alpha(x_1,x_1) \geq 1 \quad \text{and} \quad \alpha(x_2,x_2) \geq 1. \end{aligned}$$

Again, since T is α^3-proximal admissible, we obtain

$$\alpha(x_2,x_3) \geq 1, \qquad \alpha(x_2,x_2) \geq 1 \quad \text{and} \quad \alpha(x_3,x_3) \geq 1.$$

Also, there exists $x_4 \in A_0$ such that

$$d(x_4, Tx_3) = d(A,B),$$

and hence

$$\begin{aligned} &d(x_3, Tx_2) = d(A,B), \qquad d(x_4, Tx_3) = d(A,B), \qquad \alpha(x_2,x_3) \geq 1, \\ &\alpha(x_2,x_2) \geq 1 \quad \text{and} \quad \alpha(x_3,x_3) \geq 1. \end{aligned}$$

By continuing this process, we construct a sequence $\{x_n\}$ such that

$$\alpha(x_n,x_n) \geq 1, \qquad \alpha(x_{n-1},x_{n-1}) \geq 1 \quad \text{and} \quad \begin{cases} \alpha(x_{n-1},x_n) \geq 1, \\ d(x_n, Tx_{n-1}) = d(A,B), \\ d(x_{n+1}, Tx_n) = d(A,B) \end{cases} \tag{2.2}$$

for all $n \in \mathbb{N}$. Now, from (4.2) with $u = x_n$, $v = x_{n+1}$, $x = x_{n-1}$, and $y = x_n$, we get

$$\begin{aligned} &F\big(d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), d(x_n,x_n)\big) \\ &\quad \leq L\big[1 - \alpha(x_{n-1},x_{n-1})\alpha(x_n,x_n)\big]. \end{aligned}$$

On the other hand from (2.2) we obtain

$$\alpha(x_{n-1},x_{n-1})\alpha(x_n,x_n) \geq 1.$$

That is, $1 - \alpha(x_{n-1}, x_{n-1})\alpha(x_n, x_n) \le 0$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}&F\bigl(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\bigr)\\ &\quad \le L\bigl[1 - \alpha(x_{n-1}, x_{n-1})\alpha(x_n, x_n)\bigr] \le 0.\end{aligned}$$

Now, since F is decreasing in t_5

$$F\bigl(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}) + d(x_{n-1}, x_n), 0\bigr) \le 0,$$

and so from (F1) we get

$$d(x_n, x_{n+1}) \le \psi\bigl(d(x_{n-1}, x_n)\bigr).$$

By induction, we have

$$d(x_n, x_{n+1}) \le \psi^n\bigl(d(x_0, x_1)\bigr).$$

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n \ge N} \psi^n\bigl(d(x_0, x_1)\bigr) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let $m, n \in \mathbb{N}$ with $m > n \ge N$. Then by the triangular inequality, we get

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{n \ge N} \psi^n\bigl(d(x_0, x_1)\bigr) < \epsilon.$$

Consequently $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since A is complete, there is $z \in A$ such that $x_n \to z$. Since T is continuous, $Tx_n \to Tz$ as $n \to \infty$. Hence,

$$d(A, B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz).$$

Thus z is the desired best proximity point of T.

Let $x, y \in A$ be two best proximity point of T such that $x \ne y$. That is, $d(x, Tx) = d(A, B) = d(y, Ty)$. From (iii), we get $\alpha(x, y) \ge 1$, $\alpha(x, x) \ge 1$, and $\alpha(y, y) \ge 1$. So by (4.2) we derive

$$F\bigl(d(x, y), d(x, y), d(x, x), d(y, y), d(y, x), d(x, y)\bigr) \le L\bigl[1 - \alpha(x, x)\alpha(y, y)\bigr] \le 0,$$

which implies

$$F\bigl(d(x, y), d(x, y), 0, 0, d(y, x), d(x, y)\bigr) \le 0,$$

which is a contradiction to (F4). Hence, T has a unique best proximity point. □

Theorem 2 *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T : A \to B$ is an implicit relation type modified α^3-proximal contraction such that the following conditions hold*:

(i) *T is an α^3-proximal admissible mapping and* $T(A_0) \subseteq B_0$,

(ii) *there exist* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = d(A,B), \qquad \alpha(x_0,x_0) \geq 1, \qquad \alpha(x_1,x_1) \geq 1 \quad \text{and} \quad \alpha(x_0,x_1) \geq 1,$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *with* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n,x) \geq 1$ *and* $\alpha(x,x) \geq 1$.

Then T has a best proximity point. Further, the best proximity point is unique if

(iv) *for every* $x, y \in A$, *where* $d(x,Tx) = d(A,B) = d(y,Ty)$, *we have* $\alpha(x,y) \geq 1$, $\alpha(x,x) \geq 1$, *and* $\alpha(y,y) \geq 1$.

Proof Following the proof of Theorem 1, there exist a Cauchy sequence $\{x_n\} \subseteq A$ and $z \in A$ such that (4.2) holds and $x_n \to z$ as $n \to +\infty$. On the other hand, for all $n \in \mathbb{N}$, we can write

$$\begin{aligned} d(z,B) &\leq d(z,Tx_n) \\ &\leq d(z,x_{n+1}) + d(x_{n+1},Tx_n) \\ &= d(z,x_{n+1}) + d(A,B). \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we get

$$\lim_{n\to+\infty} d(z,Tx_n) = d(z,B) = d(A,B). \tag{2.3}$$

Since B is approximatively compact with respect to A, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y^* \in B$. Hence,

$$d\big(z,y^*\big) = \lim_{n\to\infty} d(x_{n_k+1}, Tx_{n_k}) = d(A,B)$$

and so $z \in A_0$. Now, since $T(A_0) \subseteq B_0$, we have $d(w,Tz) = d(A,B)$ for some $w \in A$. By (iii) and (2.2), we have $\alpha(x_n,z) \geq 1$, $\alpha(z,z) \geq 1$, and $d(x_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$. Also, since T is an implicit relation type α^3-proximal contraction, we get

$$F\big(d(x_{n+1},w), d(x_n,z), d(x_n,x_{n+1}), d(z,w), d(x_n,w), d(z,x_{n+1})\big) \leq 0.$$

Taking the limit as $n \to +\infty$ in the above inequality and applying continuity of F, we have

$$F\big(d(z,w), 0, 0, d(z,w), d(z,w), 0\big) \leq 0.$$

Now, if we take $u = d(z,w)$ and $v = 0$, then we have

$$F(u,v,0,u+v,u,v) \leq 0$$

and so from (F3) we get $u \leq \psi(v)$. That is, $d(z,w) \leq \psi(0) = 0$. Thus, $z = w$. Hence z is a best proximity point of T. Uniqueness follows similarly to the proof of Theorem 1. □

Using Example 2 and Theorem 2 we obtain the following corollary.

Corollary 1 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T:A\to B$ is a nonself mapping satisfying the following conditions*:

(i) *T is an α^3-proximal admissible mapping and $T(A_0)\subseteq B_0$,*

(ii) *there exist $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B),\qquad \alpha(x_0,x_0)\ge 1,\qquad \alpha(x_1,x_1)\ge 1\quad \text{and}\quad \alpha(x_0,x_1)\ge 1,$$

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$ with $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x)\ge 1$ and $\alpha(x,x)\ge 1$,*

(iv) *there exist nonnegative real numbers a, b, c, d with $a+b+2c+2d<1$, such that for all $x_1,x_2,u_1,u_2\in A$,*

$$\begin{cases}\alpha(x_1,x_2)\ge 1,\\ d(u_1,Tx_1)=d(A,B),\\ d(u_2,Tx_2)=d(A,B)\end{cases}$$
$$\begin{aligned}\Longrightarrow\quad d(u_1,u_2)+L\alpha(x_1,x_1)\alpha(x_2,x_2)\le{}& ad(x_1,x_2)+b\frac{[1+d(x_1,u_1)]d(x_2,u_2)}{1+d(x_1,x_2)}\\ &+c\bigl[d(x_1,u_1)+d(x_2,u_2)\bigr]\\ &+d\bigl[d(x_1,u_2)+d(x_2,u_1)\bigr]+L,\end{aligned}$$

where $L\ge 0$.

Then T has a best proximity point. Further, the best proximity point is unique if

(v) *for every $x,y\in A$, where $d(x,Tx)=d(A,B)=d(y,Ty)$, we have $\alpha(x,y)\ge 1$, $\alpha(x,x)\ge 1$, and $\alpha(y,y)\ge 1$.*

If in Corollary 1 we take $b=c=d=0$, then we have the following corollary.

Corollary 2 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T:A\to B$ is a nonself mapping satisfying the following conditions*:

(i) *T is an α^3-proximal admissible mapping and $T(A_0)\subseteq B_0$,*

(ii) *there exist $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B),\qquad \alpha(x_0,x_0)\ge 1,\qquad \alpha(x_1,x_1)\ge 1\quad \text{and}\quad \alpha(x_0,x_1)\ge 1,$$

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$ with $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x)\ge 1$ and $\alpha(x,x)\ge 1$,*

(iv) *there exists a nonnegative real number a with $a<1$, such that for all $x_1,x_2,u_1,u_2\in A$,*

$$\begin{cases}\alpha(x_1,x_2)\ge 1,\\ d(u_1,Tx_1)=d(A,B),\\ d(u_2,Tx_2)=d(A,B)\end{cases}\quad\Longrightarrow\quad d(u_1,u_2)+L\alpha(x_1,x_1)\alpha(x_2,x_2)\le ad(x_1,x_2)+L,$$

where $L\ge 0$.

Then T has a best proximity point. Further, the best proximity point is unique if

(v) *for every* $x, y \in A$, *where* $d(x, Tx) = d(A, B) = d(y, Ty)$, *we have* $\alpha(x, y) \geq 1$, $\alpha(x, x) \geq 1$, *and* $\alpha(y, y) \geq 1$.

Example 3 Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. Consider $A = (-\infty, -1]$, $B = [1, +\infty)$ and define $T : A \to B$ by

$$Tx = \begin{cases} 11, & \text{if } x \in (-\infty, -14), \\ 7, & \text{if } x \in [-14, -12), \\ 5, & \text{if } x \in [-12, -10), \\ 2, & \text{if } x \in [-10, -8), \\ 10, & \text{if } x \in [-8, -6), \\ 17, & \text{if } x \in [-6, -4), \\ 14, & \text{if } x \in [-4, -2), \\ 1, & \text{if } x \in [-2, -1]. \end{cases}$$

Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [-2, -1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Clearly, B is approximatively compact with respect to A and $d(A, B) = 2$. Then $A_0 = \{-1\}$ and $B_0 = \{1\}$. Clearly, $T(A_0) \subseteq B_0$, $d(-1, T(-1)) = d(A, B) = 2$, and $\alpha(-1, -1) \geq 1$.

Assume

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ d(u_1, Tx_1) = d(A, B) = 2, \\ d(u_2, Tx_2) = d(A, B) = 2, \end{cases}$$

then

$$\begin{cases} x_1, x_2 \in [-2, -1], \\ d(u_1, Tx_1) = 2, \\ d(u_2, Tx_2) = 2. \end{cases}$$

Therefore, $u_1 = u_2 = -1$, that is, $\alpha(u_1, u_2) \geq 1$, $\alpha(u_1, u_1) \geq 1$, and $\alpha(u_2, u_2) \geq 1$. Further,

$$\begin{aligned} d(u_1, u_2) \leq{} & ad(x_1, x_2) + b\frac{[1 + d(x_1, u_1)]d(x_2, u_2)}{1 + d(x_1, x_2)} \\ & + c\bigl[d(x_1, u_1) + d(x_2, u_2)\bigr] \\ & + d\bigl[d(x_1, u_2) + d(x_2, u_1)\bigr] \\ & + L\bigl[1 - \alpha(x_1, x_1)\alpha(x_2, x_2)\bigr], \end{aligned}$$

that is, T is an α^3-proximal admissible mapping and condition (iv) of Corollary 1 holds true. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq [-2, -1]$ and hence $x \in [-2, -1]$. Consequently, $\alpha(x, x) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore all the conditions of Corollary 1 hold for this example and T has a best proximity point. Here $z = -1$ is the best proximity point of T.

If in Corollary 1 we take $\alpha(x,y) = 1$, then we have the following corollary.

Corollary 3 (Theorem 3.1 of [14]) *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that B is approximatively compact with respect to A. Assume that $a+b+2c+2d<1$. Let A_0 and B_0 be nonempty and $T: A \to B$ be a nonself mapping satisfying the following assertions*:

(i) $T(A_0) \subseteq B_0$,

(ii)

$$\begin{cases} d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases}$$
$$\Longrightarrow \quad d(u_1,u_2) \le ad(x_1,x_2) + b\frac{[1+d(x_1,u_1)]d(x_2,u_2)}{1+d(x_1,x_2)} + c\bigl[d(x_1,u_1)+d(x_2,u_2)\bigr] + d\bigl[d(x_1,u_2)+d(x_2,u_1)\bigr].$$

Then there exists $z \in A$ such that

$$d(z,Tz) = d(A,B).$$

By taking $\alpha(x,y) = 1$ in Theorem 2, we deduce the following corollary.

Corollary 4 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T: A \to B$ is a nonself mapping such that $TA_0 \subseteq B_0$ and for all $x,y,u,v \in A$,*

$$\begin{cases} d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\Longrightarrow \quad F\bigl(d(u,v), d(x,y), d(x,u), d(y,v), d(y,u), d(x,v)\bigr) \le 0,$$

where $F \in \mathcal{F}$. Then T has a unique best proximity point.

Using Example 1 and Corollary 4, we deduce the following result.

Corollary 5 *Let A, B be two nonempty subsets of a metric space (X,d) such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T: A \to B$ is a nonself mapping such that $TA_0 \subseteq B_0$ and, for all $x,y,u,v \in A$,*

$$\begin{cases} d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases}$$
$$\Longrightarrow \quad d(u,v) \le \psi\left(\max\left\{d(x,y), d(x,u), d(y,v), \frac{d(y,u)+d(x,v)}{2}\right\}\right) + L\min\bigl\{d(x,u), d(y,v), d(y,u), d(x,v)\bigr\},$$

where $\psi \in \Psi$. Then T has a unique best proximity point.

3 Some results in metric spaces endowed with a graph

Consistent with Jachymski [15], let (X,d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, *i.e.*, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [15]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for details [12, 15, 16]).

In 2006, Espínola and Kirk [17] established an important combination of fixed point theory and graph theory.

Definition 4 Let A, B be two nonempty closed subsets of a metric space (X,d) endowed with a graph G. Then $T: A \to B$ is said to be an implicit relation type G-proximal contraction, if, for all $x, y, u, v \in A$,

$$\begin{cases} (x,y) \in E(G), \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{cases} \quad \Longrightarrow \quad (u,v) \in E(G)$$

and

$$\begin{cases} (x,y) \in E(G), \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{cases}$$
$$\Longrightarrow \quad F\bigl(d(u,v), d(x,y), d(x,u), d(y,v), d(y,u), d(x,v)\bigr) \le 0,$$

where $F \in \mathcal{F}$.

Theorem 3 *Let A, B be two nonempty closed subsets of a metric space (X,d) endowed with a graph G. Assume that A is complete, A_0 is nonempty, and $T: A \to B$ is a continuous implicit relation type G-proximal contraction such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0, x_1) \in E(G).$$

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x,Tx) = d(A,B) = d(y,Ty)$, we have $(x,y) \in E(G)$.

Proof Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Firstly, we prove that T is an α^3-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Therefore, we have

$$\begin{cases} (x,y) \in E(G), \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Since T is an implicit relation type G-proximal contraction, we get $(u,v) \in E(G)$. Also, since $\Delta \subseteq E(G)$, $(u,u),(v,v) \in E(G)$. That is, $\alpha(u,v) \geq 1$, $\alpha(u,u) \geq 1$, $\alpha(v,v) \geq 1$, and

$$F\bigl(d(u,v),d(x,y),d(x,u),d(y,v),d(y,u),d(x,v)\bigr) \leq 0 = L\bigl[1 - \alpha(x,x)\alpha(y,y)\bigr]$$

when $L = 0$. Thus T is an α^3-proximal admissible mapping with $T(A_0) \subseteq B_0$ and continuous implicit relation type G-proximal contraction. From (ii) there exist $x_0, x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $(x_0,x_1) \in E(G)$, that is, $d(x_1,Tx_0) = d(A,B)$, $\alpha(x_0,x_1) \geq 1$, $\alpha(x_0,x_0) \geq 1$, and $\alpha(x_1,x_1) \geq 1$. Hence, all the conditions of Theorem 1 are satisfied and T has a best proximity point. □

Similarly, by using Theorem 2, we can prove the following theorem.

Theorem 4 *Let A, B be two nonempty closed subsets of a metric space (X,d) endowed with a graph G. Assume that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Also suppose that $T : A \to B$ is an implicit relation type G-proximal contraction mapping such that the following conditions hold*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1,Tx_0) = d(A,B) \quad \textit{and} \quad (x_0,x_1) \in E(G),$$

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n,x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x,y \in A$ such that $d(x,Tx) = d(A,B) = d(y,Ty)$, we have $(x,y) \in E(G)$.

Corollary 6 *Let A, B be two nonempty closed subsets of a metric space (X,d) endowed with a graph G. Assume that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume $a + b + 2c + 2d < 1$. Also, suppose that $T : A \to B$ satisfies the following conditions*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1,Tx_0) = d(A,B) \quad \textit{and} \quad (x_0,x_1) \in E(G),$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $(x_n, x) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$,

(iv) *for* $x_1, x_2, u_1, u_2 \in A_0$,

$$\begin{cases} (x_1, x_2) \in E(G), \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \quad d(u_1,u_2) &\le ad(x_1,x_2) + b\frac{[1 + d(x_1,u_1)]d(x_2,u_2)}{1 + d(x_1,x_2)} \\ &\quad + c\bigl[d(x_1,u_1) + d(x_2,u_2)\bigr] \\ &\quad + d\bigl[d(x_1,u_2) + d(x_2,u_1)\bigr]. \end{aligned}$$

Then T *has a best proximity point. Further, the best proximity point is unique if, for every* $x, y \in A$ *such that* $d(x,Tx) = d(A,B) = d(y,Ty)$, *we have* $(x,y) \in E(G)$.

Corollary 7 *Let* A, B *be two nonempty closed subsets of a metric space* (X,d) *endowed with a graph* G. *Assume that* A *is complete*, B *is approximatively compact with respect to* A, *and* A_0 *is nonempty. Also, suppose that* $T : A \to B$ *satisfies the following conditions*:

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0, x_1) \in E(G),$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *then* $(x_n, x) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$,

(iv) *for* $x_1, x_2, u_1, u_2 \in A_0$,

$$\begin{cases} (x_1, x_2) \in E(G), \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \quad d(u_1,u_2) &\le \psi\left(\max\left\{d(x_1,x_2), d(x_1,u_1), d(x_2,u_2), \right.\right. \\ &\qquad \left.\left.\frac{d(x_2,u_1) + d(x_1,u_2)}{2}\right\}\right) \\ &\quad + L\min\bigl\{d(x_1,u_1), d(x_2,u_2), d(x_2,u_1), d(x_1,u_2)\bigr\}, \end{aligned}$$

where $\psi \in \Psi$.

Then T *has a best proximity point. Further, the best proximity point is unique if, for every* $x, y \in A$ *such that* $d(x,Tx) = d(A,B) = d(y,Ty)$, *we have* $(x,y) \in E(G)$.

4 Some results in metric spaces endowed with a partially ordered

The study of existence of fixed points in partially ordered sets has been established by Ran and Reurings [18] with applications to matrix equations. Agarwal *et al.* [19], Ćirić *et al.* [20], and Hussain *et al.* [12, 21] obtained some new fixed point results for nonlinear contractions in partially ordered Banach and metric spaces with some applications. In this section, as an application of our results we derive some new best proximity point results whenever the range space is endowed with a partial order.

Definition 5 [22] Let $(X,d,\preceq)$ be a partially ordered metric space. We say that a nonself mapping $T:A\to B$ is proximally ordered-preserving if and only if, for all $x_1,x_2,u_1,u_2\in A$,

$$\begin{cases} x_1\preceq x_2,\\ d(u_1,Tx_1)=d(A,B), \\ d(u_2,Tx_2)=d(A,B)\end{cases} \implies \quad u_1\preceq u_2.$$

Theorem 5 *Let A, B be two nonempty closed subsets of a partially ordered metric space $(X,d,\preceq)$ such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T:A\to B$ satisfies the following conditions*:

(i) *T is continuous and proximally ordered-preserving such that $T(A_0)\subseteq B_0$*,

(ii) *there exist elements $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=d(A,B)\quad \textit{and}\quad x_0\preceq x_1,$$

(iii) *for all $x,y,u,v\in A$*,

$$\begin{cases} x\preceq y,\\ d(u,Tx)=d(A,B),\\ d(y,Ty)=d(A,B)\end{cases}$$
$$\implies \quad F\bigl(d(u,v),d(x,y),d(x,u),d(y,v),d(y,u),d(x,v)\bigr)\le 0. \tag{4.1}$$

Then T has a best proximity point.

Proof Define $\alpha:A\times A\to[0,+\infty)$ by

$$\alpha(x,y)=\begin{cases} 1, & \text{if } x\preceq y,\\ \frac{1}{2}, & \text{otherwise.}\end{cases}$$

Firstly, we prove that T is an α^3-proximal admissible mapping. To this aim, assume

$$\begin{cases} \alpha(x,y)\ge 1,\\ d(u,Tx)=d(A,B),\\ d(v,Ty)=d(A,B).\end{cases}$$

Therefore, we have

$$\begin{cases} x\preceq y,\\ d(u,Tx)=d(A,B),\\ d(v,Ty)=d(A,B).\end{cases}$$

Now, since T is proximally ordered-preserving, then $u\preceq v$, that is, $\alpha(u,v)\ge 1$. Further, by (ii) we have

$$d(x_1,Tx_0)=d(A,B)\quad \text{and}\quad \alpha(x_0,x_1)\ge 1.$$

Moreover, from (iii) we get

$$\begin{cases} \alpha(x,y) \geq 1, \\ d(u,Tx) = d(A,B), \\ d(y,Ty) = d(A,B) \end{cases} \implies F\bigl(d(u,v),d(x,y),d(x,u),d(y,v),d(y,u),d(x,v)\bigr) \leq 0.$$

Thus all the conditions of Theorem 1 hold (when $L = 0$) and T has a best proximity point. □

Theorem 6 *Let A, B be two nonempty closed subsets of a partially ordered metric space $(X,d,\preceq)$ such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume that $T : A \to B$ satisfies the following conditions*:

(i) *T is proximally ordered-preserving such that $T(A_0) \subseteq B_0$*,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad x_0 \preceq x_1,$$

(iii) *for all $x, y, u, v \in A$*,

$$\begin{cases} x \preceq y, \\ d(u,Tx) = d(A,B), \\ d(y,Ty) = d(A,B) \end{cases}$$
$$\implies F\bigl(d(u,v),d(x,y),d(x,u),d(y,v),d(y,u),d(x,v)\bigr) \leq 0, \tag{4.2}$$

(iv) *if $\{x_n\}$ is an increasing sequence in A converging to $x \in A$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*
Then T has a best proximity point.

Corollary 8 *Let A, B be two nonempty closed subsets of a partially ordered metric space $(X,d,\preceq)$ such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Assume $a + b + 2c + 2d < 1$. Also, suppose that $T : A \to B$ satisfies the following conditions*:

(i) *$T(A_0) \subseteq B_0$*,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad x_0 \preceq x_1,$$

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$*,

(iv) *for $x_1, x_2, u_1, u_2 \in A_0$*,

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1,Tx_1) = d(A,B), \\ d(u_2,Tx_2) = d(A,B) \end{cases}$$
$$\implies d(u_1,u_2) \leq ad(x_1,x_2) + b\frac{[1 + d(x_1,u_1)]d(x_2,u_2)}{1 + d(x_1,x_2)}$$
$$+ c\bigl[d(x_1,u_1) + d(x_2,u_2)\bigr] + d\bigl[d(x_1,u_2) + d(x_2,u_1)\bigr].$$

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $x \preceq y$.

Corollary 9 *Let A, B be two nonempty closed subsets of a partially ordered metric space $(X, d, \preceq)$ such that A is complete, B is approximatively compact with respect to A, and A_0 is nonempty. Also, suppose that $T : A \to B$ satisfies the following conditions:*

(i) $T(A_0) \subseteq B_0$,

(ii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad x_0 \preceq x_1,$$

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,*

(iv) *for $x_1, x_2, u_1, u_2 \in A_0$,*

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B) \end{cases}$$
$$\begin{aligned} \Longrightarrow \quad d(u_1, u_2) \leq{} & \psi \biggl(\max \biggl\{ d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), \\ & \frac{d(x_2, u_1) + d(x_1, u_2)}{2} \biggr\} \biggr) \\ & + L \min \bigl\{ d(x_1, u_1), d(x_2, u_2), d(x_2, u_1), d(x_1, u_2) \bigr\}, \end{aligned}$$

where $\psi \in \Psi$.

Then T has a best proximity point. Further, the best proximity point is unique if, for every $x, y \in A$ such that $d(x, Tx) = d(A, B) = d(y, Ty)$, we have $x \preceq y$.

5 Application to fixed point theory

5.1 Implicit relation type modified α-contraction

Definition 6 [9] Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Remark 1 Note that every α-admissible mappings are α^3-proximal admissible mappings when $A = B = X$.

Definition 7 Let (X, d) be a metric space and $\alpha : A \times A \to [0, \infty)$ be a function. Then $T : X \to X$ is said to be an implicit relation type α-contraction, if for all $x, y \in X$ with $\alpha(x, y) \geq 1$, we have

$$\begin{aligned} & F\bigl(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\bigr) \\ & \quad \leq L\bigl[1 - \alpha(x, x)\alpha(y, y)\bigr], \end{aligned} \tag{5.1}$$

where $L \geq 0$ and $F \in \mathcal{F}$.

Theorem 7 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *T is α-admissible,*
(ii) *there exists x_0 in X such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is an implicit relation type modified α-contraction.*

Then T has a fixed point.

Theorem 8 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping and the following conditions hold*:

(i) *T is α-admissible,*
(ii) *there exists x_0 in X such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is an implicit relation type modified α-contraction,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, x) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Using Example 2 and Theorem 8, we deduce the following result.

Corollary 10 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping and the following conditions hold*:

(i) *T is α-admissible,*
(ii) *there exists x_0 in X such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *for all $x, y \in X$ with $\alpha(x, y) \geq 1$ we have*

$$\begin{aligned} d(Tx, Ty) + L\alpha(x,x)\alpha(y,y) \leq{} & ad(x,y) + \frac{b[1 + d(x, Tx)]d(y, Ty)}{1 + d(x,y)} \\ & + c\big[d(x, Tx) + d(y, Ty)\big] \\ & + d\big[d(y, Tx) + d(x, Ty)\big] + L, \end{aligned}$$

where $a + b + 2c + 2d < 1$ and $L \geq 0$,
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, x) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 11 *Let (X, d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping and the following conditions hold*:

(i) *T is α-admissible,*
(ii) *there exists x_0 in X such that $\alpha(x_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *for all $x, y \in X$ with $\alpha(x, y) \geq 1$ we have*

$$d(Tx, Ty) + L\alpha(x,x)\alpha(y,y) \leq ad(x,y) + L,$$

where $0 \leq a < 1$ and $L \geq 0$,
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, x) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

5.2 Implicit relation type G-contraction

Definition 8 [15] We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, *i.e.*,

$$\forall x, y \in X \quad \bigl((x,y) \in E(G) \Rightarrow \bigl(T(x), T(y)\bigr) \in E(G)\bigr)$$

and T decreases weights of edges of G in the following way:

$$\exists \alpha \in (0,1), \forall x, y \in X \quad \bigl((x,y) \in E(G) \Rightarrow d\bigl(T(x), T(y)\bigr) \leq \alpha d(x,y)\bigr).$$

Definition 9 [15] A mapping $T : X \to X$ is called G-continuous, if for given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \quad \text{as } n \to \infty \quad \text{and} \quad (x_n, x_{n+1}) \in E(G) \quad \text{for all } n \in \mathbb{N} \quad \text{imply} \quad Tx_n \to Tx.$$

Definition 10 Let (X,d) be a metric space endowed with a graph G. Then $T : X \to X$ is said to be an implicit relation type G-contraction, if, for all $x, y \in X$,

$$(x,y) \in E(G) \quad \Longrightarrow \quad (Tx, Ty) \in E(G)$$

and

$$(x,y) \in E(G) \quad \Longrightarrow \quad F\bigl(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\bigr) \leq 0,$$

where $F \in \mathcal{F}$.

Theorem 9 *Let (X,d) be a complete metric space endowed with a graph G. Assume that $T : X \to X$ is a continuous self-mapping satisfying the following conditions*:

(i) *there exists x_0 in X such that $(x_0, Tx_0) \in E(G)$,*
(ii) *T is an implicit relation type G-contraction.*

Then T has a fixed point.

Theorem 10 *Let (X,d) be a complete metric space endowed with a graph G. Assume that $T : X \to X$ is a self-mapping satisfying the following conditions*:

(i) *there exists x_0 in X such that $(x_0, Tx_0) \in E(G)$,*
(ii) *T is an implicit relation type G-contraction,*
(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

5.3 Implicit relation type ordered contraction

Theorem 11 ([3], Theorem 3.2) *Let $(X,d,\preceq)$ be a partially ordered complete metric space. Assume that $T : X \to X$ is a self-mapping that satisfies the following conditions*:

(i) *there exists x_0 in X such that $x_0 \preceq Tx_0$,*
(ii) *for all $x, y \in X$ with $x \preceq y$ we have*

$$F\bigl(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\bigr) \leq 0,$$

where $F \in \mathcal{F}$,

(iii) *either T is continuous or if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 12 *Let $(X, d, \preceq)$ be complete metric space. Assume $a + b + 2c + 2d < 1$. Also, suppose that $T : X \to X$ is a self-mapping that satisfies the following conditions:*

(i) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,*
(iii) *for $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq ad(x, y) + b\frac{[1 + d(x, Tx)]d(y, Ty)}{1 + d(x, y)} + c\big[d(x, Tx) + d(y, Ty)\big] + d\big[d(x, Ty) + d(y, Tx)\big].$$

Then T has a fixed point.

Corollary 13 *Let $(X, d, \preceq)$ be complete metric space. Assume that $T : X \to X$ is a self-mapping that satisfies the following conditions:*

(i) *there exist element $x_0 \in X$ such that $x_0 \preceq Tx_0$,*
(ii) *if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,*
(iii) *for $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq \psi\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right) + L\min\big\{d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\big\},$$

where $\psi \in \Psi$. Then T has a fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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