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Solution sensitivity of generalized nonlinear parametric (A, η, m) -proximal operator system of equations in Hilbert spaces

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Dedicated to Professor Shih-Sen Chang's 80th birthday.

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Abstract

By using the new parametric resolvent operator technique associated with (A, η, m) -monotone operators, the purpose of this paper is to analyze and establish an existence theorem for a new class of generalized nonlinear parametric (A, η, m) -proximal operator system of equations with non-monotone multi-valued operators in Hilbert spaces. The results presented in this paper generalize the sensitivity analysis results of recent work on strongly monotone quasi-variational inclusions, nonlinear implicit quasi-variational inclusions, and nonlinear mixed quasi-variational inclusion systems in Hilbert spaces.

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1 Introduction

Recently, since the study of the sensitivity (analysis) of solutions for variational inclusion (operator equation) problems involving strongly monotone and relaxed cocoercive mappings under suitable second order and regularity assumptions is an increasing interest, there are many motivated researchers basing their work on the generalized resolvent operator (equation) techniques, which is used to develop powerful and efficient numerical techniques for solving (mixed) variational inequalities, related optimization, control theory, operations research, transportation network modeling, and mathematical programming problems. It is well known that the project technique and the resolvent operator technique can be used to establish an equivalence between (mixed) variational inequalities, variational inclusions, and resolvent equations. See, for example, [1–35] and the references therein.

In this paper, we consider the following system of (A, η, m) -proximal operator equations: For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, find $(z, t), (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $u \in S(x, \omega)$ and

$$\begin{cases} E(x, y, \omega) + \rho^{-1}R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) = 0, \\ F(u, y, \lambda) + \varrho^{-1}R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) = 0, \end{cases} \quad (1.1)$$

where Ω and Λ are two nonempty open subsets of real Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , in which the parameter ω and λ take values, respectively, $S : \mathcal{H}_1 \times \Omega \rightarrow 2^{\mathcal{H}_1}$ is a set-valued operator, $E : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \rightarrow \mathcal{H}_1$, $F : \mathcal{H}_1 \times \mathcal{H}_2 \times \Lambda \rightarrow \mathcal{H}_2$, $f : \mathcal{H}_1 \times \Omega \rightarrow \mathcal{H}_1$, $g : \mathcal{H}_2 \times \Lambda \rightarrow \mathcal{H}_2$, $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \rightarrow \mathcal{H}_1$, and $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda \rightarrow \mathcal{H}_2$ are nonlinear single-valued operators, $A_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$, $A_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$, $M : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \rightarrow 2^{\mathcal{H}_1}$ and $N : \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda \rightarrow 2^{\mathcal{H}_2}$ are any nonlinear operators such that for all $(z, \Omega) \in \mathcal{H}_1 \times \Omega$, $M(\cdot, z, \omega) : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ is an (A_1, η_1, m_1) -monotone operator with $f(\mathcal{H}_1, \omega) \cap \text{dom}(M(\cdot, z, \omega)) \neq \emptyset$ and for all $(t, \lambda) \in \mathcal{H}_2 \times \Lambda$, $N(\cdot, t, \lambda) : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ is an (A_2, η_2, m_2) -monotone operator with $g(\mathcal{H}_2, \lambda) \cap \text{dom}(N(\cdot, t, \lambda)) \neq \emptyset$, respectively, $R_{\rho, A_1}^{M(\cdot, x, \omega)} = I - A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)})$, I is the identity operator, $R_{\varrho, A_2}^{N(\cdot, y, \lambda)} = I - A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)})$, $A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)}(z)) = A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)})(z)$, $A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t)) = A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)})(t)$, $J_{\rho, A_1}^{M(\cdot, x, \omega)} = (A_1 + \rho M(\cdot, x, \omega))^{-1}$ and $J_{\varrho, A_2}^{N(\cdot, y, \lambda)} = (A_2 + \varrho N(\cdot, y, \lambda))^{-1}$ for all $x, z \in \mathcal{H}_1$, $y, t \in \mathcal{H}_2$, and $(\omega, \lambda) \in \Omega \times \Lambda$.

For appropriate and suitable choices of $S, E, F, M, N, f, g, A_i, \eta_i$, and \mathcal{H}_i for $i = 1, 2$, one sees that problem (1.1) is a generalized version of some problems, which includes a number (systems) of (parametric) quasi-variational inclusions, (parametric) generalized quasi-variational inclusions, (parametric) quasi-variational inequalities, (parametric) implicit quasi-variational inequalities studied by many authors as special cases; see, [1, 2, 5, 6, 8, 10–13, 15–21, 25, 28, 32–35] and the references therein.

Example 1.1 If $S : \mathcal{H}_1 \times \Omega \rightarrow \mathcal{H}_1$ is a single-valued operator, then for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, problem (1.1) reduces to the following problem of finding $(x, y), (z, t) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that:

$$\begin{cases} E(x, y, \omega) + \rho^{-1}R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) = 0, \\ F(S(x, \omega), y, \lambda) + \varrho^{-1}R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) = 0. \end{cases} \tag{1.2}$$

Example 1.2 If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $A_1 = A_2 = A$, $S = I$, $E = F$, $M = N$, $x = y$ and $\omega = \lambda$, then problem (1.2) reduces to finding $x, z \in \mathcal{H}$ such that

$$E(x, x, \omega) + \rho^{-1}R_{\rho, A}^{M(\cdot, x, \omega)}(z) = 0. \tag{1.3}$$

Problem (1.3) is equivalent to the following nonlinear equation:

$$x = J_{\rho, A}^{M(\cdot, x, \omega)}(z), \quad z = A(x) - \rho E(x, x, \omega),$$

which can be rewritten as the following generalized strongly monotone mixed quasi-variational inclusion:

$$0 \in E(x, x, \omega) + M(x, x, \omega),$$

and studied by Verma [34, 35] when M is A -monotone and (A, η) -monotone with respect to first variable.

Example 1.3 ([36]) Let \mathcal{H} be a real Hilbert space and $M : \text{dom}(M) \subset \mathcal{H} \rightarrow \mathcal{H}$ be an operator on \mathcal{H} such that M is monotone and $R(I + M) = \mathcal{H}$. Then based on the Yosida approximation $M_\rho = \frac{1}{\rho}(I - (I + \rho M)^{-1})$, for each given $u_0 \in \text{dom}(M)$, there exists exactly one

continuous function $u : [0, 1) \rightarrow \mathcal{H}$ such that the following first-order evolution equation:

$$\begin{cases} u'(t) + Mu(t) = 0, & 0 < t < \infty, \\ u(0) = u_0, \end{cases}$$

where the derivative $u'(t)$ exists in the sense of weak convergence, that is,

$$\frac{u(t+h) - u(t)}{h} \rightharpoonup u'(t) \quad \text{as } h \rightarrow 0$$

holds for all $t \in (0, \infty)$.

On the other hand, Lan [27] introduced a new concept of (A, η) -monotone operators, which generalizes the (H, η) -monotonicity and A -monotonicity in Hilbert spaces and other existing monotone operators as special cases, and studied some properties of (A, η) -monotone operators and applied resolvent operators associated with (A, η) -monotone operators to approximate the solutions of a new class of nonlinear (A, η) -monotone operator inclusion problems with relaxed cocoercive operators in Hilbert spaces. Lan *et al.* [29] and Verma [34] introduced and studied a new class of parametric generalized relaxed cocoercive implicit quasi-variational inclusions with A -monotone operators, respectively. By using the parametric implicit resolvent operator technique for A -monotone, we analyzed solution sensitivity for this kind of generalized relaxed cocoercive inclusions in Hilbert spaces. In [31, 35], based on the (A, η) -resolvent operator technique, Verma and Lan introduced and investigated a sensitivity analysis for a class of generalized strongly monotone variational inclusions in Hilbert spaces, respectively. Furthermore, using the concept and technique of resolvent operators, Agarwal *et al.* [2] and Jeong [19] introduced and studied a new system of parametric generalized nonlinear mixed quasi-variational inclusions in a Hilbert space and in L_p ($p \geq 2$) spaces, respectively.

In this paper, we shall generalize the resolvent equations by introducing (A, η, m) -proximal operator equations in Hilbert spaces and establish a relationship between a class of parametric (A, η, m) -monotone variational inclusion systems and a class of generalized nonlinear parametric (A, η, m) -proximal operator system of equations. Further, we study sensitivity analysis of the solution set for the system (1.1) of (A, η, m) -proximal operator equations with non-monotone set-valued operators in Hilbert spaces.

Our results improve and generalize the results on the sensitivity analysis for generalized nonlinear mixed quasi-variational inclusions [2, 9, 22, 29, 33–35] and others. For more details, we recommend [4, 7, 10, 13, 14, 16, 17, 23, 24, 26, 32].

2 Preliminaries

In the sequel, let Λ be a nonempty open subset of a real Hilbert space \mathcal{H} in which the parameter λ take values.

Definition 2.1 An operator $T : \mathcal{H} \times \mathcal{H} \times \Lambda \rightarrow \mathcal{H}$ is said to be

- (i) m -relaxed monotone in the first argument if there exists a positive constant m such that

$$\langle T(x, u, \lambda) - T(y, u, \lambda), x - y \rangle \geq -m \|x - y\|^2$$

for all $(x, y, u, \lambda) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Lambda$;

(ii) s -cocoercive in the first argument if there exists a constant $s > 0$ such that

$$\langle T(x, u, \lambda) - T(y, u, \lambda), x - y \rangle \geq s \|T(x, u, \lambda) - T(y, u, \lambda)\|^2$$

for all $(x, y, u, \lambda) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Lambda$;

(iii) γ -relaxed cocoercive with respect to A in the first argument if there exists a positive constant γ such that

$$\langle T(x, u, \lambda) - T(y, u, \lambda), A(x) - A(y) \rangle \geq -\gamma \|T(x, u, \lambda) - T(y, u, \lambda)\|^2$$

for all $(x, y, u, \lambda) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Lambda$;

(iv) (ϵ, α) -relaxed cocoercive with respect to A in the first argument if there exist positive constants ϵ and α such that

$$\langle T(x, u, \lambda) - T(y, u, \lambda), A(x) - A(y) \rangle \geq -\alpha \|T(x, u, \lambda) - T(y, u, \lambda)\|^2 + \epsilon \|x - y\|^2$$

for all $(x, y, u, \lambda) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Lambda$.

In a similar way, we can define (relaxed) cocoercivity of the operator $T(\cdot, \cdot, \cdot)$ in the second argument.

Definition 2.2 An operator $T : \mathcal{H} \times \mathcal{H} \times \Lambda \rightarrow \mathcal{H}$ is said to be μ -Lipschitz continuous in the first argument if there exists a constant $\mu > 0$ such that

$$\|T(x, u, \lambda) - T(y, u, \lambda)\| \leq \mu \|x - y\|, \quad \forall (x, y, u, \lambda) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Lambda.$$

In a similar way, we can define Lipschitz continuity of the operator $T(\cdot, \cdot, \cdot)$ in the second and third argument.

Definition 2.3 Let $F : \mathcal{H} \times \Lambda \rightarrow 2^{\mathcal{H}}$ be a multi-valued operator. Then F is said to be τ - $\hat{\mathbf{H}}$ -Lipschitz continuous in the first argument if there exists a constant $\tau > 0$ such that

$$\hat{\mathbf{H}}(F(x, \lambda), F(y, \lambda)) \leq \tau \|x - y\|, \quad \forall x, y \in \mathcal{H}, \lambda \in \Lambda,$$

where $\hat{\mathbf{H}} : 2^{\mathcal{H}} \times 2^{\mathcal{H}} \rightarrow (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric, i.e.,

$$\hat{\mathbf{H}}(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{x \in B} \inf_{y \in A} \|x - y\| \right\}, \quad \forall A, B \in 2^{\mathcal{H}}.$$

In a similar way, we can define $\hat{\mathbf{H}}$ -Lipschitz continuity of the operator $F(\cdot, \cdot)$ in the second argument.

Lemma 2.1 ([37]) Let (\mathcal{X}, d) be a complete metric space and $T_1, T_2 : \mathcal{X} \rightarrow CB(\mathcal{X})$ be two set-valued contractive operators with same contractive constant $t \in (0, 1)$, i.e.,

$$\hat{\mathbf{H}}(T_i(x), T_i(y)) \leq td(x, y), \quad \forall x, y \in \mathcal{X}, i = 1, 2.$$

Then

$$\hat{H}(F(T_1), F(T_2)) \leq \frac{1}{1-t} \sup_{x \in \mathcal{X}} \hat{H}(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed point sets of T_1 and T_2 , respectively.

Definition 2.4 Let $A : \mathcal{H} \rightarrow \mathcal{H}$, $\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be two single-valued operators. Then a multi-valued operator $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is called (A, η, m) -monotone (so-called (A, η) -monotonicity [27, 35], (A, η) -maximal relaxed monotonicity [3]) if

- (i) M is m -relaxed η -monotone,
- (ii) $(A + \rho M)(\mathcal{H}) = \mathcal{H}$ for every $\rho > 0$.

Remark 2.1 For appropriate and suitable choices of m, A, η , and \mathcal{H} , it is easy to see that Definition 2.4 includes a number of definitions of monotone operators and monotone mappings (see [14, 27, 29, 30]).

Proposition 2.1 ([27]) Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a r -strongly η -monotone operator, $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be an (A, η) -monotone operator. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Definition 2.5 Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a strictly η -monotone operator and $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be an (A, η, m) -monotone operator. The resolvent operator $J_{\eta, M}^{\rho, A} : \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$J_{\eta, M}^{\rho, A}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in \mathcal{H}.$$

Proposition 2.2 ([27]) Let \mathcal{H} be a q -uniformly smooth Banach space and $\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be τ -Lipschitz continuous, $A : \mathcal{H} \rightarrow \mathcal{H}$ be a r -strongly η -monotone operator and $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be an (A, η, m) -monotone operator. Then the resolvent operator $J_{\eta, M}^{\rho, A} : \mathcal{H} \rightarrow \mathcal{H}$ is $\frac{\tau}{r-\rho m}$ -Lipschitz continuous, i.e.,

$$\|J_{\eta, M}^{\rho, A}(x) - J_{\eta, M}^{\rho, A}(y)\| \leq \frac{\tau}{r - \rho m} \|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

In connection with the (A, η, m) -proximal operator equations system (1.1), we consider the following generalized parametric (A, η, m) -monotone variational inclusion system:

For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $u \in S(x, \omega)$ and

$$\begin{cases} 0 \in E(x, y, \omega) + M(x, x, \omega), \\ 0 \in F(u, y, \lambda) + N(y, y, \lambda). \end{cases} \quad (2.1)$$

Remark 2.2 For appropriate and suitable choices of $E, F, M, N, S, A_i, \eta_i$, and \mathcal{H}_i for $i = 1, 2$, it is easy to see that problem (2.1) includes a number (systems) of (parametric) quasi-variational inclusions, (parametric) generalized quasi-variational inclusions, (parametric) quasi-variational inequalities, (parametric) implicit quasi-variational inequalities studied by many authors as special cases; see, for example, [1–35] and the references therein.

Now, for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, the solution set $Q(\omega, \lambda)$ of problem (2.1) is denoted by

$$Q(\omega, \lambda) = \left\{ (z, t, x, y) \in \mathcal{H}_1 \times \mathcal{H}_2 : \exists u \in S(x, \omega) \right. \\
 \text{such that } E(x, y, \omega) + \rho^{-1}R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) = 0 \\
 \left. \text{and } F(u, y, \lambda) + \varrho^{-1}R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) = 0 \right\}.$$

In this paper, our aim is to study the behavior of the solution set $Q(\omega, \lambda)$ and the conditions on these operators $S, E, F, M, N, \eta_1, \eta_2, A_1, A_2$ under which the function $Q(\omega, \lambda)$ is continuous or Lipschitz continuous with respect to the parameter $(\omega, \lambda) \in \Omega \times \Lambda$.

3 Sensitivity analysis results

In the sequel, we first transfer problem (2.1) into a problem of finding the parametric fixed point of the associated (A, η, m) -resolvent operator.

Lemma 3.1 *For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, an element $(x, y) \in Q(\omega, \lambda)$ is a solution of problem (2.1) if and only if there are $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ and $u \in S(x, \omega)$ such that*

$$\begin{cases} x = J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)), \\ y = J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)), \end{cases} \quad (3.1)$$

where $J_{\rho, A_1}^{M(\cdot, x, \omega)} = (A_1 + \rho M(\cdot, x, \omega))^{-1}$ and $J_{\varrho, A_2}^{N(\cdot, y, \lambda)} = (A_2 + \varrho N(\cdot, y, \lambda))^{-1}$ are the corresponding resolvent operator in first argument of an (A_1, η_1) -monotone operator $M(\cdot, \cdot, \cdot)$, (A_2, η_2) -monotone operator $N(\cdot, \cdot, \cdot)$, respectively, A_i is an r_i -strongly monotone operator for $i = 1, 2$ and $\rho, \varrho > 0$.

Proof For each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, by the definition of the resolvent operators $J_{\rho, A_1}^{M(\cdot, x, \omega)} = (A_1 + \rho M(\cdot, x, \omega))^{-1}$ of $M(\cdot, x, \omega)$ and $J_{\varrho, A_2}^{N(\cdot, y, \lambda)} = (A_2 + \varrho N(\cdot, y, \lambda))^{-1}$ of $N(\cdot, y, \lambda)$, respectively, we know that there exist $x \in \mathcal{H}_1, y \in \mathcal{H}_2$, and $u \in S(x, \omega)$ such that (3.1) holds if and only if

$$A_1(x) - \rho E(x, y, \omega) \in A_1(x) + \rho M(x, x, \omega), \\
 A_2(y) - \varrho F(u, y, \lambda) \in A_2(y) + \varrho N(y, y, \lambda),$$

i.e.,

$$0 \in E(x, y, \omega) + M(x, x, \omega), \\
 0 \in F(u, y, \lambda) + N(y, y, \lambda).$$

It follows from the definition of $Q(\omega, \lambda)$ that $(x, y) \in Q(\omega, \lambda)$ is a solution of problem (2.1) if and only if there exist $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, and $u \in S(x, \omega)$ such that equation (3.1) holds. \square

Now, we show that problem (1.1) is equivalent to problem (2.1).

Lemma 3.2 *The problem (1.1) has a solution (z, t, x, y, u) with $u \in S(x, \omega)$ if and only if problem (2.1) has a solution (x, y, u) with $u \in S(x, \omega)$, where*

$$x = J_{\rho, A_1}^{M(\cdot, x, \omega)}(z), \quad y = J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) \tag{3.2}$$

and

$$\begin{aligned} z &= A_1(x) - \rho E(x, y, \omega), \\ t &= A_2(y) - \varrho F(u, y, \lambda). \end{aligned}$$

Proof Let (x, y, u) with $u \in S(x, \omega)$ be a solution of problem (2.1). Then, by Lemma 3.1, it is a solution of the following system of equations:

$$\begin{cases} x = J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)), \\ y = J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)). \end{cases}$$

By using the fact $R_{\rho, A_1}^{M(\cdot, x, \omega)} = I - A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)})$, $R_{\varrho, A_2}^{N(\cdot, y, \lambda)} = I - A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)})$ and (3.1), we have

$$\begin{aligned} &R_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)) \\ &= A_1(x) - \rho E(x, y, \omega) - A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega))) \\ &= A_1(x) - \rho E(x, y, \omega) - A_1(x) \\ &= -\rho E(x, y, \omega) \end{aligned}$$

and

$$\begin{aligned} &R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)) \\ &= A_2(y) - \varrho F(u, y, \lambda) - A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda))) \\ &= A_2(y) - \varrho F(u, y, \lambda) - A_2(y) \\ &= -\varrho F(u, y, \lambda), \end{aligned}$$

which imply that

$$\begin{aligned} E(x, y, \omega) + \rho^{-1} R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) &= 0, \\ F(u, y, \lambda) + \varrho^{-1} R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) &= 0 \end{aligned}$$

with $z = A_1(x) - \rho E(x, y, \omega)$ and $t = A_2(y) - \varrho F(u, y, \lambda)$, i.e. (z, t, x, y, u) with $u \in S(x, \omega)$ is a solution of problem (1.1).

Conversely, letting (z, t, x, y, u) with $u \in S(x, \omega)$ is a solution of problem (1.1), then

$$\begin{cases} \rho E(x, y, \omega) = -R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) = A_1(J_{\rho, A_1}^{M(\cdot, x, \omega)}(z)) - z, \\ \varrho F(u, y, \lambda) = -R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) = A_2(J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t)) - t. \end{cases} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \rho E(x, y, \omega) &= A_1 \left(J_{\rho, A_1}^{M(\cdot, x, \omega)} (A_1(x) - \rho E(x, y, \omega)) \right) - A_1(x) + \rho E(x, y, \omega), \\ \varrho F(u, y, \lambda) &= A_2 \left(J_{\varrho, A_2}^{N(\cdot, y, \lambda)} (A_2(y) - \varrho F(u, y, \lambda)) \right) - A_2(y) + \varrho F(u, y, \lambda), \end{aligned}$$

which imply that

$$\begin{aligned} A_1(x) &= A_1 \left(J_{\rho, A_1}^{M(\cdot, x, \omega)} (A_1(x) - \rho E(x, y, \omega)) \right), \\ A_2(y) &= A_2 \left(J_{\varrho, A_2}^{N(\cdot, y, \lambda)} (A_2(y) - \varrho F(u, y, \lambda)) \right) \end{aligned}$$

and so

$$\begin{aligned} x &= J_{\rho, A_1}^{M(\cdot, x, \omega)} (A_1(x) - \rho E(x, y, \omega)), \\ y &= J_{\varrho, A_2}^{N(\cdot, y, \lambda)} (A_2(y) - \varrho F(u, y, \lambda)), \end{aligned}$$

i.e., (x, y, u) with $u \in S(x, \omega)$ is a solution of problem (2.1). □

Alternative proof Let

$$z = A_1(x) - \rho E(x, y, \omega), \quad t = A_2(y) - \varrho F(u, y, \lambda).$$

Then, by (3.2), we know

$$x = J_{\rho, A_1}^{M(\cdot, x, \omega)}(z), \quad y = J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t)$$

and

$$z = A_1 \left(J_{\rho, A_1}^{M(\cdot, x, \omega)}(z) \right) - \rho E(x, y, \omega), \quad t = A_2 \left(J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) \right) - \varrho F(u, y, \lambda).$$

Since $A_1 \left(J_{\rho, A_1}^{M(\cdot, x, \omega)}(z) \right) = A_1 \left(J_{\rho, A_1}^{M(\cdot, x, \omega)} \right)(z)$ and $A_2 \left(J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) \right) = A_2 \left(J_{\varrho, A_2}^{N(\cdot, y, \lambda)} \right)(t)$, we have

$$E(x, y, \omega) + \rho^{-1} R_{\rho, A_1}^{M(\cdot, x, \omega)}(z) = 0, \quad F(u, y, \lambda) + \varrho^{-1} R_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t) = 0,$$

the required problem (1.1). □

We now invoke Lemmas 3.1 and 3.2 to suggest the following sensitivity analysis results for the system of (A, η, m) -proximal operator equations (1.1).

Theorem 3.1 *Let $A_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ be r_i -strongly monotone and s_i -Lipschitz continuous for all $i = 1, 2$, $S : \mathcal{H}_1 \times \Omega \rightarrow CB(\mathcal{H}_1)$ be κ - \hat{H} -Lipschitz continuous in the first variable, $M : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \rightarrow 2^{\mathcal{H}_1}$ be (A_1, η_1) -monotone with constant m_1 in the first variable, and $N : \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda \rightarrow 2^{\mathcal{H}_2}$ be (A_2, η_2) -monotone with constant m_2 in the first variable. Let $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be τ_1 -Lipschitz continuous, $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be τ_2 -Lipschitz continuous, $E : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \rightarrow \mathcal{H}_1$ be (γ_1, α_1) -relaxed cocoercive with respect to A_1 and μ_1 -Lipschitz continuous in the first variable, $F : \mathcal{H}_1 \times \mathcal{H}_2 \times \Lambda \rightarrow \mathcal{H}_2$ be (γ_2, α_2) -relaxed cocoercive with*

respect to A_2 and μ_2 -Lipschitz continuous in the second variable, and let E be β_2 -Lipschitz continuous in the second variable, and F be β_1 -Lipschitz continuous in the first variable. If

$$\begin{aligned} & \|J_{\rho, A_1}^{M(\cdot, x, \omega)}(z) - J_{\rho, A_1}^{M(\cdot, y, \omega)}(z)\| \\ & \leq v_1 \|x - y\|, \quad \forall (x, y, z, \omega) \in \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega, \end{aligned} \tag{3.4}$$

$$\begin{aligned} & \|J_{\varrho, A_2}^{N(\cdot, x, \lambda)}(z) - J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(z)\| \\ & \leq v_2 \|x - y\|, \quad \forall (x, y, z, \lambda) \in \mathcal{H}_2 \times \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda, \end{aligned} \tag{3.5}$$

with $v_i < 1$ for $i = 1, 2$ and there exist constants $\rho \in (0, \frac{r_1}{m_1})$, $\varrho \in (0, \frac{r_2}{m_2})$ such that

$$\begin{cases} \tau_1^2 (s_1^2 - 2\rho\gamma_1 + \rho^2\mu_1^2 + 2\rho\alpha_1\mu_1^2) < (r_1 - \rho m_1)^2 (1 - v_1 - \frac{\varrho\beta_1\kappa\tau_2}{r_2 - \varrho m_2})^2, \\ \tau_2^2 (s_2^2 - 2\varrho\gamma_2 + \varrho^2\mu_2^2 + 2\varrho\alpha_2\mu_2^2) < (r_2 - \varrho m_2)^2 (1 - v_2 - \frac{\rho\beta_2\tau_1}{r_1 - \rho m_1})^2, \end{cases} \tag{3.6}$$

then, for each $(\omega, \lambda) \in \Omega \times \Lambda$, the following results hold:

- (1) the solution set $Q(\omega, \lambda)$ of problem (1.1) is nonempty;
- (2) $Q(\omega, \lambda)$ is a closed subset in $\mathcal{H}_1 \times \mathcal{H}_2$.

Proof In the sequel, from (3.1), we first define operators $\Phi_\rho : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \rightarrow \mathcal{H}_1$ and $\Psi_\varrho : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \rightarrow \mathcal{H}_2$ as follows:

$$\begin{aligned} \Phi_\rho(x, y, \omega, \lambda) &= J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)), \\ \Psi_\varrho(x, y, \omega, \lambda) &= J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(x, y, \lambda)) \end{aligned} \tag{3.7}$$

for all $(x, y, \omega, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda$.

Now, define a norm $\|\cdot\|_1$ on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$\|(x, y)\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2.$$

It is easy to see that $(\mathcal{H}_1 \times \mathcal{H}_2, \|\cdot\|_1)$ is a Banach space (see [14]). By (3.7), for any given $\rho > 0$ and $\varrho > 0$, define an operator $G : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \rightarrow 2^{\mathcal{H}_1} \times 2^{\mathcal{H}_2}$ by

$$\begin{aligned} G_{\rho, \varrho}(x, y, \omega, \lambda) &= \{(\Phi_\rho(x, y, \omega, \lambda), \Psi_\varrho(x, y, \omega, \lambda)) : u \in S(x, \omega), \\ & \quad \forall (x, y, \omega, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda\}. \end{aligned}$$

For any $(x, y, \omega, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda$, since $S(x, \omega) \in CB(\mathcal{H}_1)$, $A_1, A_2, \eta_1, \eta_2, E, F, J_{\rho, A_1}^{M(\cdot, x, \omega)}$, $J_{\varrho, A_2}^{N(\cdot, y, \lambda)}$ are continuous, we have $G_{\rho, \varrho}(x, y, \omega, \lambda) \in CB(\mathcal{H}_1 \times \mathcal{H}_2)$. Now, for each fixed $(\omega, \lambda) \in \Omega \times \Lambda$, we prove that $G_{\rho, \varrho}(x, y, \omega, \lambda)$ is a multi-valued contractive operator.

In fact, for any $(x, y, \omega, \lambda), (\hat{x}, \hat{y}, \omega, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda$ and $(a_1, a_2) \in G_{\rho, \varrho}(x, y, \omega, \lambda)$, there exists $u \in S(x, \omega)$ such that

$$\begin{aligned} a_1 &= J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)), \\ a_2 &= J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(x, y, \lambda)). \end{aligned}$$

Note that $S(\hat{x}, \omega) \in CB(\mathcal{H}_1)$, it follows from Nadler's result [38] that there exists $\hat{u} \in S(\hat{x}, \omega)$ such that

$$\|u - \hat{u}\| \leq \hat{\mathbf{H}}(S(x, \omega), S(\hat{x}, \omega)). \tag{3.8}$$

Setting

$$b_1 = J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(\hat{x}) - \rho E(\hat{x}, \hat{y}, \omega)),$$

$$b_2 = J_{\rho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(\hat{y}) - \rho F(\hat{u}, \hat{y}, \lambda)),$$

then we have $(b_1, b_2) \in G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)$. It follows from (3.4) and Proposition 2.2 that

$$\begin{aligned} \|a_1 - b_1\| &= \|J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)) - J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(\hat{x}) - \rho E(\hat{x}, \hat{y}, \omega))\| \\ &\leq \|J_{\rho, A_1}^{M(\cdot, x, \omega)}(A_1(x) - \rho E(x, y, \omega)) - J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(x) - \rho E(x, y, \omega))\| \\ &\quad + \|J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(x) - \rho E(x, y, \omega)) - J_{\rho, A_1}^{M(\cdot, \hat{x}, \omega)}(A_1(\hat{x}) - \rho E(\hat{x}, \hat{y}, \omega))\| \\ &\leq v_1 \|x - \hat{x}\| + \frac{\tau_1}{r_1 - \rho m_1} \|A_1(x) - \rho E(x, y, \omega) - (A_1(\hat{x}) - \rho E(\hat{x}, \hat{y}, \omega))\| \\ &\leq v_1 \|x - \hat{x}\| + \frac{\rho \tau_1}{r_1 - \rho m_1} \|E(\hat{x}, y, \omega) - E(\hat{x}, \hat{y}, \omega)\| \\ &\quad + \frac{\tau_1}{r_1 - \rho m_1} \|A_1(x) - A_1(\hat{x}) - \rho [E(x, y, \omega) - E(\hat{x}, y, \omega)]\|. \end{aligned} \tag{3.9}$$

By the assumptions of E, A_1 , we have

$$\|E(\hat{x}, y, \omega) - E(\hat{x}, \hat{y}, \omega)\| \leq \beta_2 \|y - \hat{y}\| \tag{3.10}$$

and

$$\begin{aligned} &\|A_1(x) - A_1(\hat{x}) - \rho [E(x, y, \omega) - E(\hat{x}, y, \omega)]\|^2 \\ &\leq \|A_1(x) - A_1(\hat{x})\|^2 + \rho^2 \|E(x, y, \omega) - E(\hat{x}, y, \omega)\|^2 \\ &\quad - 2\rho \langle E(x, y, \omega) - E(\hat{x}, y, \omega), A_1(x) - A_1(\hat{x}) \rangle \\ &\leq \|A_1(x) - A_1(\hat{x})\|^2 + \rho^2 \|E(x, y, \omega) - E(\hat{x}, y, \omega)\|^2 \\ &\quad - 2\rho (-\alpha_1 \|E(x, y, \omega) - E(\hat{x}, y, \omega)\|^2 + \gamma_1 \|x - \hat{x}\|^2) \\ &\leq (s_1^2 - 2\rho\gamma_1 + \rho^2\mu_1^2 + 2\rho\alpha_1\mu_1^2) \|x - \hat{x}\|^2. \end{aligned} \tag{3.11}$$

Combining (3.9)-(3.11), we have

$$\|a_1 - b_1\| \leq \theta_1 \|x - \hat{x}\| + \vartheta_1 \|y - \hat{y}\|, \tag{3.12}$$

where

$$\theta_1 = v_1 + \frac{\tau_1}{r_1 - \rho m_1} \sqrt{s_1^2 - 2\rho\gamma_1 + \rho^2\mu_1^2 + 2\rho\alpha_1\mu_1^2}, \quad \vartheta_1 = \frac{\rho\beta_2\tau_1}{r_1 - \rho m_1}.$$

Similarly, by the assumptions of S , A_2 , F , and (3.8), we obtain

$$\begin{aligned}
 & \|a_2 - b_2\| \\
 &= \left\| J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)) - J_{\varrho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(\hat{y}) - \varrho F(\hat{u}, \hat{y}, \lambda)) \right\| \\
 &\leq \left\| J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)) - J_{\varrho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)) \right\| \\
 &\quad + \left\| J_{\varrho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(y) - \varrho F(u, y, \lambda)) - J_{\varrho, A_2}^{N(\cdot, \hat{y}, \lambda)}(A_2(\hat{y}) - \varrho F(\hat{u}, \hat{y}, \lambda)) \right\| \\
 &\leq \nu_2 \|y - \hat{y}\| + \frac{\tau_2}{r_2 - \varrho m_2} \|A_2(y) - \varrho F(u, y, \lambda) - (A_2(\hat{y}) - \varrho F(\hat{u}, \hat{y}, \lambda))\| \\
 &\leq \nu_2 \|y - \hat{y}\| + \frac{\varrho \tau_2}{r_2 - \varrho m_2} \|F(u, y, \lambda) - F(\hat{u}, y, \lambda)\| \\
 &\quad + \frac{\tau_2}{r_2 - \varrho m_2} \|A_2(y) - A_2(\hat{y}) - \varrho (F(\hat{u}, y, \lambda) - F(\hat{u}, \hat{y}, \lambda))\| \\
 &\leq \theta_2 \|x - \hat{x}\| + \vartheta_2 \|y - \hat{y}\|, \tag{3.13}
 \end{aligned}$$

where

$$\theta_2 = \frac{\varrho \beta_1 \kappa \tau_2}{r_2 - \varrho m_2}, \quad \vartheta_2 = \nu_2 + \frac{\tau_2}{r_2 - \varrho m_2} \sqrt{s_2^2 - 2\varrho \gamma_2 + \varrho^2 \mu_2^2 + 2\varrho \alpha_2 \mu_2^2}.$$

It follows from (3.12) and (3.13) that

$$\begin{aligned}
 \|a_1 - b_1\| + \|a_2 - b_2\| &\leq (\theta_1 + \theta_2) \|x - \hat{x}\| + (\vartheta_1 + \vartheta_2) \|y - \hat{y}\| \\
 &\leq \sigma (\|x - \hat{x}\| + \|y - \hat{y}\|), \tag{3.14}
 \end{aligned}$$

where

$$\sigma = \max\{\theta_1 + \theta_2, \vartheta_1 + \vartheta_2\}.$$

It follows from condition (3.6) that $\sigma < 1$. Hence, from (3.14), we get

$$\begin{aligned}
 d((a_1, a_2), G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)) &= \inf_{(b_1, b_2) \in G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)} (\|a_1 - b_1\| + \|a_2 - b_2\|) \\
 &\leq -\sigma \|(x, y) - (\hat{x}, \hat{y})\|.
 \end{aligned}$$

Since $(a_1, a_2) \in G_{\rho, \varrho}(x, y, \omega, \lambda)$ is arbitrary, we obtain

$$\sup_{(a_1, a_2) \in G_{\rho, \varrho}(x, y, \omega, \lambda)} d((a_1, a_2), G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)) \leq -\sigma \|(x, y) - (\hat{x}, \hat{y})\|.$$

By the same argument, we can prove

$$\sup_{(b_1, b_2) \in G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)} d((b_1, b_2), G_{\rho, \varrho}(x, y, \omega, \lambda)) \leq -\sigma \|(x, y) - (\hat{x}, \hat{y})\|.$$

It follows from the definition of the Hausdorff metric \hat{H} on $CB(\mathcal{H}_1 \times \mathcal{H}_2)$ that

$$\hat{H}(G_{\rho, \varrho}(x, y, \omega, \lambda), G_{\rho, \varrho}(\hat{x}, \hat{y}, \omega, \lambda)) \leq \sigma \|(x, y) - (\hat{x}, \hat{y})\|$$

for all $(x, \hat{x}, \omega) \in \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega, (y, \hat{y}, \lambda) \in \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda$, i.e., $G_{\rho, \varrho}(x, y, \omega, \lambda)$ is a multi-valued contractive operator, which is uniform with respect to $(\omega, \lambda) \in \Omega \times \Lambda$. By a fixed point theorem of Nadler [38], for each $(\omega, \lambda) \in \Omega \times \Lambda$, $G_{\rho, \varrho}(x, y, \omega, \lambda)$ has a fixed point $(x(\lambda), y(\lambda)) \in \mathcal{H}_1 \times \mathcal{H}_2$, i.e., $(x, y) \in G_{\rho, \varrho}(x, y, \omega, \lambda)$. By the definition of G , we know that there exists $u \in S(x, \omega)$ such that (3.1) holds. Thus, it follows from Lemma 3.1 that (x, y, u) with $u \in S(x, \omega)$ is a solution of problem (2.1). Hence, it follows from Lemma 3.2 that (z, t, x, y, u) with $u \in S(x, \omega)$ is a solution of problem (1.1). Therefore, $Q(\omega, \lambda) \neq \emptyset$ for all $(\omega, \lambda) \in \Omega \times \Lambda$.

Next, we prove the conclusion (2). For each $(\omega, \lambda) \in \Omega \times \Lambda$, let $\{(z_n, t_n, x_n, y_n)\} \subset Q(\omega, \lambda)$ and $z_n \rightarrow z_0, t_n \rightarrow t_0, x_n \rightarrow x_0, y_n \rightarrow y_0$ as $n \rightarrow \infty$. Then we know that there exists $u_n \in S(x_n, \omega)$ and

$$\begin{aligned} (x_n, y_n) &\in G_{\rho, \varrho}(x_n, y_n, \omega, \lambda), \\ z_n &= A_1(x_n) - \rho E(x_n, y_n, \omega), \quad t_n = A_2(y_n) - \varrho F(u_n, y_n, \lambda), \quad \forall n = 1, 2, \dots, \end{aligned}$$

and

$$z_0 = A_1(x_0) - \rho E(x_0, y_0, \omega), \quad t_0 = A_2(y_0) - \varrho F(u_0, y_0, \lambda).$$

By the proof of conclusion (1), we have

$$\hat{\mathbf{H}}(G_{\rho, \varrho}(x_n, y_n, \omega, \lambda), G_{\rho, \varrho}(x_0, y_0, \omega, \lambda)) \leq \sigma \|(x_n, y_n) - (x_0, y_0)\|, \quad \forall (\omega, \lambda) \in \Omega \times \Lambda.$$

It follows that

$$\begin{aligned} d((x_0, y_0), G_{\rho, \varrho}(x_0, y_0, \omega, \lambda)) &\leq \|(x_0, y_0) - (x_n, y_n)\| \\ &\quad + d((x_n, y_n), G_{\rho, \varrho}(x_n, y_n, \omega, \lambda)) \\ &\quad + \hat{\mathbf{H}}(G_{\rho, \varrho}(x_n, y_n, \omega, \lambda), G_{\rho, \varrho}(x_0, y_0, \omega, \lambda)) \\ &\leq (1 + \sigma) \|(x_n, y_n) - (x_0, y_0)\|. \end{aligned}$$

Hence, we have $(x_0, y_0) \in G_{\rho, \varrho}(x_0, y_0, \omega, \lambda)$ and $(x_0, y_0) \in Q(\omega, \lambda)$. Therefore, $Q(\omega, \lambda)$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$. □

Theorem 3.2 *Under the hypotheses of Theorem 3.1, further assume that*

- (i) *for any $x \in \mathcal{H}_1, \omega \rightarrow S(x, \omega)$ is l_S - $\hat{\mathbf{H}}$ -Lipschitz continuous (or continuous);*
- (ii) *for any $x, z \in \mathcal{H}_1, y, t \in \mathcal{H}_2, \omega \rightarrow E(x, y, \omega), \omega \rightarrow J_{\rho, A_1}^{M(\cdot, x, \omega)}(z), \lambda \rightarrow F(x, y, \lambda)$ and $\lambda \rightarrow J_{\varrho, A_2}^{N(\cdot, y, \lambda)}(t)$ both are Lipschitz continuous (or continuous) with Lipschitz constants l_E, l_{J_1}, l_F , and l_{J_2} , respectively.*

Then the solution set $Q(\omega, \lambda)$ of problem (1.1) is Lipschitz continuous (or continuous) from $\Omega \times \Lambda$ to $\mathcal{H}_1 \times \mathcal{H}_2$.

Proof From the hypotheses and Theorem 3.1, for any $(\omega, \lambda), (\bar{\omega}, \bar{\lambda}) \in \Omega \times \Lambda$, we know that $Q(\omega, \lambda)$ and $Q(\bar{\omega}, \bar{\lambda})$ are nonempty closed subsets of $\mathcal{H}_1 \times \mathcal{H}_2$. By the proof of Theorem 3.1, $G_{\rho, \varrho}(x, y, \omega, \lambda)$ and $G_{\rho, \varrho}(x, y, \bar{\omega}, \bar{\lambda})$ are both multi-valued contractive operators

with the same contraction constant $\sigma \in (0, 1)$ and have fixed points $(x(\omega, \lambda), y(\omega, \lambda))$ and $(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}))$, respectively. It follows from Lemmas 2.1 and 3.2 that

$$\hat{H}(Q(\omega, \lambda), Q(\bar{\omega}, \bar{\lambda})) \leq \frac{1}{1 - \sigma} \sup_{(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2} \hat{H}(G_{\rho, \varrho}(x(\omega, \lambda), y(\omega, \lambda), \omega, \lambda), G_{\rho, \varrho}(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega}, \bar{\lambda})). \quad (3.15)$$

Setting $(a_1, a_2) \in G_{\rho, \varrho}(x(\omega, \lambda), y(\omega, \lambda), \omega, \lambda)$, there exists $u(\omega, \lambda) \in S(x(\omega, \lambda), \omega)$ such that

$$a_1 = J_{\rho, A_1}^{M(\cdot, x(\omega, \lambda), \omega)}(A_1(x(\omega, \lambda)) - \rho E(x(\omega, \lambda), y(\omega, \lambda), \omega)),$$

$$a_2 = J_{\varrho, A_2}^{N(\cdot, y(\omega, \lambda), \lambda)}(A_2(y(\omega, \lambda)) - \varrho F(u(\omega, \lambda), y(\omega, \lambda), \lambda)).$$

Since $S(x(\omega, \lambda), \omega), S(x(\bar{\omega}, \bar{\lambda}), \bar{\omega}) \in CB(\mathcal{H}_1)$, it follows from Nadler's result [38] that there exists $u(\bar{\omega}, \bar{\lambda}) \in S(x(\bar{\omega}, \bar{\lambda}), \bar{\omega})$ such that

$$\|u(\omega, \lambda) - u(\bar{\omega}, \bar{\lambda})\| \leq \hat{H}(S(x(\omega, \lambda), \omega), S(x(\bar{\omega}, \bar{\lambda}), \bar{\omega})). \quad (3.16)$$

Let

$$b_1 = J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \bar{\omega})}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega})),$$

$$b_2 = J_{\varrho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})}(A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})).$$

Then we have $(b_1, b_2) \in G_{\rho, \varrho}(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega}, \bar{\lambda})$. It follows from the assumptions on $J_{\rho, A_1}^{M(\cdot, \cdot, \cdot)}, E, A_1$, and S that

$$\begin{aligned} \|a_1 - b_1\| &= \|J_{\rho, A_1}^{M(\cdot, x(\omega, \lambda), \omega)}(A_1(x(\omega, \lambda)) - \rho E(x(\omega, \lambda), y(\omega, \lambda), \omega)) \\ &\quad - J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \bar{\omega})}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega}))\| \\ &\leq \|J_{\rho, A_1}^{M(\cdot, x(\omega, \lambda), \omega)}(A_1(x(\omega, \lambda)) - \rho E(x(\omega, \lambda), y(\omega, \lambda), \omega)) \\ &\quad - J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \omega)}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \omega))\| \\ &\quad + \|J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \omega)}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \omega)) \\ &\quad - J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \bar{\omega})}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \omega))\| \\ &\quad + \|J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \bar{\omega})}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \omega)) \\ &\quad - J_{\rho, A_1}^{M(\cdot, x(\bar{\omega}, \bar{\lambda}), \bar{\omega})}(A_1(x(\bar{\omega}, \bar{\lambda})) - \rho E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega}))\| \\ &\leq \theta_1 \|x(\omega, \lambda) - x(\bar{\omega}, \bar{\lambda})\| + \vartheta_1 \|y(\omega, \lambda) - y(\bar{\omega}, \bar{\lambda})\| + l_1 \|\omega - \bar{\omega}\| \\ &\quad + \frac{\rho \tau_1}{r_1 - \rho m_1} \|E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \omega) - E(x(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\omega})\| \\ &\leq \theta_1 \|x(\omega, \lambda) - x(\bar{\omega}, \bar{\lambda})\| + \vartheta_1 \|y(\omega, \lambda) - y(\bar{\omega}, \bar{\lambda})\| + k_1 \|\omega - \bar{\omega}\|, \end{aligned} \quad (3.17)$$

where θ_1 and ϑ_1 are the constants of (3.12) and

$$k_1 = l_1 + \frac{\rho \tau_1 l_E}{r_1 - \rho m_1}.$$

Similarly, by the assumptions on $g, J_{\rho, A_2}^{N(\cdot, \cdot, \cdot)}, F, A_2$ and S ,

$$\begin{aligned}
 \|a_2 - b_2\| &= \left\| J_{\rho, A_2}^{N(\cdot, y(\omega, \lambda), \lambda)} (A_2(y(\omega, \lambda)) - \varrho F(u(\omega, \lambda), y(\omega, \lambda), \lambda)) \right. \\
 &\quad \left. - J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})) \right\| \\
 &\leq \left\| J_{\rho, A_2}^{N(\cdot, y(\omega, \lambda), \lambda)} (A_2(y(\omega, \lambda)) - \varrho F(u(\omega, \lambda), y(\omega, \lambda), \lambda)) \right. \\
 &\quad \left. - J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \lambda)) \right\| \\
 &\quad + \left\| J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \lambda)) \right. \\
 &\quad \left. - J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})) \right\| \\
 &\quad + \left\| J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \lambda)) \right. \\
 &\quad \left. - J_{\rho, A_2}^{N(\cdot, y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})} (A_2(y(\bar{\omega}, \bar{\lambda})) - \varrho F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})) \right\| \\
 &\leq \theta_2 \|x(\omega, \lambda) - x(\bar{\omega}, \bar{\lambda})\| + \vartheta_2 \|y(\omega, \lambda) - y(\bar{\omega}, \bar{\lambda})\| + l_{j_2} \|\lambda - \bar{\lambda}\| \\
 &\quad + \frac{\varrho \tau_2}{r_2 - \rho m_2} \|F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \lambda) - F(u(\bar{\omega}, \bar{\lambda}), y(\bar{\omega}, \bar{\lambda}), \bar{\lambda})\| \\
 &\leq \theta_2 \|x(\omega, \lambda) - x(\bar{\omega}, \bar{\lambda})\| + \vartheta_2 \|y(\omega, \lambda) - y(\bar{\omega}, \bar{\lambda})\| + k_2 \|\lambda - \bar{\lambda}\|, \tag{3.18}
 \end{aligned}$$

where θ_2 and ϑ_2 are the constants of (3.13) and

$$k_2 = l_{j_2} + \frac{\varrho \tau_2 l_F}{r_2 - \rho m_2}.$$

It follows from (3.17), (3.18) and (3.1) that

$$\begin{aligned}
 \|a_1 - b_1\| + \|a_2 - b_2\| &\leq (\theta_1 + \theta_2) \|x(\omega, \lambda) - x(\bar{\omega}, \bar{\lambda})\| \\
 &\quad + (\vartheta_1 + \vartheta_2) \|y(\omega, \lambda) - y(\bar{\omega}, \bar{\lambda})\| \\
 &\quad + k_1 \|\omega - \bar{\omega}\| + k_2 \|\lambda - \bar{\lambda}\| \\
 &\leq \sigma (\|a_1 - b_1\| + \|a_2 - b_2\|) + k_1 \|\omega - \bar{\omega}\| + k_2 \|\lambda - \bar{\lambda}\|,
 \end{aligned}$$

where σ is the constant of (3.13), which implies that

$$\|a_1 - b_1\| + \|a_2 - b_2\| \leq \Theta (\|\omega - \bar{\omega}\| + \|\lambda - \bar{\lambda}\|), \tag{3.19}$$

where

$$\Theta = \frac{1}{1 - \sigma} \max\{k_1, k_2\}.$$

Hence, from (3.19), we obtain

$$\sup_{(a_1, a_2) \in G_{\rho, \varrho}(x, y, \omega, \lambda)} d((a_1, a_2), G_{\rho, \varrho}(x, y, \bar{\omega}, \bar{\lambda})) \leq \Theta \|(\omega, \lambda) - (\bar{\omega}, \bar{\lambda})\|.$$

By using a similar argument as above, we get

$$\sup_{(b_1, b_2) \in G_{\rho, \varrho}(x, y, \bar{\omega}, \bar{\lambda})} d(G_{\rho, \varrho}(x, y, \omega, \lambda), (b_1, b_2)) \leq \Theta \|(\omega, \lambda) - (\bar{\omega}, \bar{\lambda})\|.$$

It follows that

$$\hat{H}(G_{\rho, \varrho}(x, y, \omega, \lambda), G_{\rho, \varrho}(x, y, \bar{\omega}, \bar{\lambda})) \leq \Theta \|(\omega, \lambda) - (\bar{\omega}, \bar{\lambda})\|$$

for all $(x, y, \omega, \bar{\omega}, \lambda, \bar{\lambda}) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Omega \times \Lambda \times \Lambda$. Thus, (3.15) implies

$$\hat{H}(Q(\omega, \lambda), Q(\bar{\omega}, \bar{\lambda})) \leq \frac{\Theta}{1 - \sigma} \|(\omega, \lambda) - (\bar{\omega}, \bar{\lambda})\|.$$

This proves that $Q(\omega, \lambda)$ is Lipschitz continuous in $(\omega, \lambda) \in \Omega \times \Lambda$. If each operator under conditions (i) and (ii) is assumed to be continuous in $(\omega, \lambda) \in \Omega \times \Lambda$, then by a similar argument as above, we can show that $S(\lambda)$ is continuous in $(\omega, \lambda) \in \Omega \times \Lambda$. \square

Remark 3.1 In Theorems 3.1 and 3.2, if E, F are strongly monotone in the first and second variable, i.e., when $\alpha_i = 0$ ($i = 1, 2$) in Theorems 3.1 and 3.2, respectively, then we can obtain the corresponding results. Our results improve and generalize the well-known results in [2, 29, 33–35].

4 Application

In this section, we give an application.

Lemma 4.1 ([39]) *Let $\phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function. Then $J_\alpha^\partial \phi = (I + \alpha \partial \phi)^{-1}$ is nonexpansive for any constant $\alpha > 0$.*

Theorem 4.1 *Let \mathcal{H}_i be a real Hilbert space and $\phi_i : \mathcal{H}_i \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function for $i = 1, 2$. Suppose that $E : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \rightarrow \mathcal{H}_1$ is γ_1 -strongly monotone and μ_1 -Lipschitz continuous in the first variable, and is β_2 -Lipschitz continuous in the second variable, $F : \mathcal{H}_1 \times \mathcal{H}_1 \times \Lambda \rightarrow \mathcal{H}_2$ is γ_2 -strongly monotone and μ_2 -Lipschitz continuous in the second variable, and is β_1 -Lipschitz continuous in the first variable. If there exist positive constants ρ and ϱ such that*

$$\begin{cases} \varrho \beta_1 + \sqrt{s_1^2 - 2\rho\gamma_1 + \rho^2\mu_1^2} < 1, \\ \rho \beta_2 + \sqrt{s_2^2 - 2\varrho\gamma_2 + \varrho^2\mu_2^2} < 1, \end{cases}$$

then, for each $(\omega, \lambda) \in \Omega \times \Lambda$:

- (1) $(x^*, y^*) \in \mathbb{R}^2$ is the unique solution of the following nonlinear problem:

$$\begin{cases} \langle E(x^*, y^*, \omega), x - x^* \rangle \geq \rho \phi_1(x^*, \omega) - \rho \phi_1(x, \omega), \\ \langle F(x^*, y^*, \lambda), y - y^* \rangle \geq \varrho \phi_2(y^*, \lambda) - \varrho \phi_2(y, \lambda). \end{cases} \tag{4.1}$$

- (2) *Moreover, the solution (x^*, y^*) of problem (4.1) is continuous (or Lipschitz continuous) from $\Omega \times \Lambda$ to \mathbb{R}^2 , if in addition, for any $x, z \in \mathcal{H}_1, y, t \in \mathcal{H}_2, \omega \rightarrow E(x, y, \omega), \omega \rightarrow J_\rho^\partial \phi_1(\cdot, \omega)(z), \lambda \rightarrow F(x, y, \lambda)$ and $\lambda \rightarrow J_\varrho^\partial \phi_2(\cdot, \lambda)(t)$ both are Lipschitz continuous (or continuous) with Lipschitz constants $l_E, l_{F_1}, l_F,$ and l_{F_2} , respectively.*

Proof Letting

$$\begin{aligned}\Phi_\rho(x, y, \omega, \lambda) &= J_\rho^{\partial\phi_1(\cdot, \omega)}(x - \rho E(x, y, \omega)), \\ \Psi_\rho(x, y, \omega, \lambda) &= J_\rho^{\partial\phi_2(\cdot, \lambda)}(y - \rho F(x, y, \lambda))\end{aligned}\tag{4.2}$$

for all $(x, y, \omega, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda$ and defining $\| \cdot \|_1$ on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$\|(x, y)\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2,$$

then it is easy to see that $(\mathcal{H}_1 \times \mathcal{H}_2, \| \cdot \|_1)$ is a Banach space (see [14]). Further, one can show that $G_{\rho, \rho}(x, y, \omega, \lambda) = (\Phi_\rho(x, y, \omega, \lambda), \Psi_\rho(x, y, \omega, \lambda))$ is a contractive operator and the rest of proof can be carried out by Theorems 3.1 and 3.2, and so it is omitted. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by JKK. JKK and HYL prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

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