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Several matrix trace inequalities on Hermitian and skew-Hermitian matrices

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Abstract
In this paper, we present several matrix trace inequalities on Hermitian and skew-Hermitian matrices, which play an important role in designing and analyzing interior-point methods (IPMs) for semidefinite optimization (SDO).

Keywords: Hermitian matrix; skew-Hermitian matrix; Hermitian positive definite matrix; semidefinite optimization; interior-point methods

1 Introduction

SDO is the generalization of linear optimization (LO), which is convex optimization over the intersection of an affine set and the cone of positive semidefinite matrices. Several IPMs designed for LO have been successfully extended to SDO [1, 2]. Some matrix trace inequalities are developed and applied in the analysis of IPMs for SDO (see [3–12]). In [3], Yang proved the arithmetic mean-geometric mean inequality for positive definite matrices, which was an open question proposed by Bellman in [4]; Neudecke used a different method in [5] to show a slightly relaxed version of Yang's result in [3]; In [6], Coope considered alternative proofs of some simple matrix trace inequalities in [3–5] and further studied properties of products of Hermitian and positive (semi)definite matrices; In [7], Yang gave a new proof of the result obtained by Yang in [3] and extended it to a generalized positive semidefinite matrix. Based on the work in [3–5], Chang established a matrix trace inequality for products of Hermitian matrices in [8], which partly answers a conjecture proposed by Bellman in [4]. In addition, Yang gave a matrix trace inequality for products of positive semidefinite matrices in [9]; In [10], Yang *et al.* established a matrix trace inequality for positive semidefinite matrices, which improved the result given by Yang in [9]. Although there have been many results on matrix trace inequality, some important matrix trace inequality problems have not been fully solved. In this paper, we will provide several matrix trace inequalities on Hermitian and skew-Hermitian matrices, which play an important role in designing and analyzing IPMs for SDO.

This paper is organized as follows: In Section 2, a matrix trace inequality on 2×2 Hermitian and skew-Hermitian matrices is provided, and its simple proof is given by using an elementary method. However, it is difficult for us to use this method to deal with the high-dimensional case. Based on Lemmas 2 and 3 in Section 2, the high-dimensional case will be shown as Corollary 2 in Section 3. In Section 4, some conclusions are made.

The following notations are used throughout the paper. $\mathbf{N}$, $\mathbf{R}^{+}$, $\mathbf{C}$, and $\mathbf{C}^{n}$ denote the set of natural numbers, the set of nonnegative real numbers, the set of complex numbers and the set of vectors with n components, respectively. $\mathbf{C}^{n\times n}$ is the space of all $n \times n$ matrices over $\mathbf{C}$. Define

$$\operatorname{Span}\{v_1, \dots, v_n\} = \{k_1 v_1 + \cdots + k_n v_n | k_i \in \mathbf{C}, i = 1, 2, \dots, n\},$$

where $v_1, v_2, \dots, v_n \in \mathbf{C}^n$. The vector inner product of $\alpha \in \mathbf{C}^n$ and $\beta \in \mathbf{C}^n$ is defined by (α, β). $|\cdot|$ and $\|\cdot\|$ denote modules for complex numbers and the 2-norm for vectors, respectively. For $A, B \in \mathbf{C}^{n\times n}$, A^* represents the conjugate transpose of A, $A \geq B$ ($A > B$) means that $A - B$ is positive semidefinite (positive definite). For any Hermitian positive definite matrix Q, the expression $Q^{\frac{1}{2}}$ denotes the Hermitian square root of Q. Similarly, the power Q^r can be defined for any $Q > 0$ and $r \in \mathbf{R}$.

2 Preliminary results

In this section, we will present a matrix trace inequality on 2×2 Hermitian and skew-Hermitian matrices.

Lemma 1 *Let $N > 0$ and M be 2×2 Hermitian and skew-Hermitian matrices, respectively. Then*

$$\operatorname{tr}\bigl(\bigl((N + M)^*(N + M)\bigr)^{-\frac{1}{2}}\bigr) \leq \operatorname{tr}\bigl(N^{-1}\bigr).$$

Proof From $N > 0$, there exists a unitary matrix U such that

$$N = U \Lambda U^*,$$

where

$$\Lambda = \operatorname{diag}\{n_1, n_2\} \quad \text{and} \quad n_1 \geq n_2 > 0.$$

Hence,

$$\operatorname{tr}\bigl(N^{-1}\bigr) = \operatorname{tr}\bigl(\Lambda^{-1}\bigr)$$

and

$$\operatorname{tr}\bigl(\bigl((N + M)^*(N + M)\bigr)^{-\frac{1}{2}}\bigr) = \operatorname{tr}\bigl(\bigl(\bigl(\Lambda + U^* M U\bigr)^*\bigl(\Lambda + U^* M U\bigr)\bigr)^{-\frac{1}{2}}\bigr),$$

where $U^* M U$ is a skew-symmetric matrix.

Without loss of generality, let

$$N = \operatorname{diag}\{n_1, n_2\} \quad \text{and} \quad M = \begin{bmatrix} 0 & m \\ -\bar{m} & 0 \end{bmatrix},$$

where $n_1 \geq n_2 > 0$. Then

$$N + M = \begin{bmatrix} n_1 & m \\ -\bar{m} & n_2 \end{bmatrix}.$$

Thus

$$(N+M)(N+M)^* = \begin{bmatrix} n_1^2 + |m|^2 & m(n_2 - n_1) \\ \bar{m}(n_2 - n_1) & n_2^2 + |m|^2 \end{bmatrix}.$$

From

$$\det\bigl(\lambda I_2 - (N+M)(N+M)^*\bigr) = 0,$$

it follows that

$$\lambda^2 - \bigl(n_1^2 + n_2^2 + 2|m|^2\bigr)\lambda + \bigl(n_1 n_2 + |m|^2\bigr)^2 = 0. \tag{1}$$

Suppose λ_1 and λ_2 are two roots of (1), where $\lambda_1 \geq \lambda_2$. According to Weda's theorem, we obtain

$$\begin{cases} \lambda_1 + \lambda_2 = n_1^2 + n_2^2 + 2|m|^2, \\ \lambda_1 \lambda_2 = n_1^2 n_2^2 + 2n_1^2|m|^2 + |m|^4. \end{cases} \tag{2}$$

From (2), it is easy to compute that

$$\begin{aligned} \left(\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}}\right)^2 &= \frac{\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\lambda_2}}{\lambda_1\lambda_2} \\ &= \frac{n_1^2 + n_2^2 + 2|m|^2 + 2(n_1 n_2 + |m|^2)}{(n_1 n_2 + |m|^2)^2} \\ &= \frac{(n_1 + n_2)^2 + 4|m|^2}{(n_1 n_2 + |m|^2)^2}. \end{aligned} \tag{3}$$

On the other hand, it is clear that

$$0 \leq 2n_1 n_2\bigl(n_1^2 + n_2^2\bigr)|m|^2 + (n_1 + n_2)^2|m|^4. \tag{4}$$

According to (4), it follows that

$$4|m|^2 n_1^2 n_2^2 \leq 2n_1 n_2 (n_1 + n_2)^2 |m|^2 + (n_1 + n_2)^2 |m|^4.$$

This implies

$$n_1^2 n_2^2 (n_1 + n_2)^2 + 4|m|^2 n_1^2 n_2^2 \leq \bigl(n_1 n_2 + |m|^2\bigr)^2 (n_1 + n_2)^2. \tag{5}$$

Thus, (5) can be written as

$$\frac{(n_1 + n_2)^2 + 4|m|^2}{(n_1 n_2 + |m|^2)^2} \leq \frac{(n_1 + n_2)^2}{n_1^2 n_2^2}.$$

This, together with (3), yields

$$\left(\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}}\right)^2 \leq \frac{(n_1 + n_2)^2}{n_1^2 n_2^2} = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^2,$$

i.e.,

$$\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}} \le \frac{1}{n_1} + \frac{1}{n_2}.$$

Hence,

$$\operatorname{tr}\bigl(\bigl((N+M)^*(N+M)\bigr)^{-1/2}\bigr) \le \operatorname{tr}\bigl(N^{-1}\bigr). \qquad \square$$

Remark 1 In this lemma, Weda's theorem is applied to prove the matrix trace inequality of 2×2 Hermitian and skew-Hermitian matrices. However, it is difficult for us to extend this result to the high-dimensional case, because it is too complex for us to compute the matrix traces of $n \times n$ Hermitian and skew-Hermitian matrices by using Weda's theorem, when $n \ge 3$.

In order to prove the high-dimensional case of Lemma 1, we need to give the following two lemmas.

Lemma 2 [13] *Let $A \in \mathbf{C}^{n\times n}$. Then $x^*(A + A^*)x \le 2\sqrt{x^*A^*Ax}$, $\forall x \in \mathbf{C}^n$, $\|x\| = 1$.*

Lemma 3 [13] *Let $H = H^* \in \mathbf{C}^{n\times n}$ with eigenvalues $\lambda_1(H) \ge \lambda_2(H) \ge \cdots \ge \lambda_n(H)$ and their corresponding orthonormal eigenvectors $v_1, v_2, \ldots, v_n$, respectively. Define*

$$V = \operatorname{span}\{v_p, v_{p+1}, \ldots, v_q\}, \quad 1 \le p \le q \le n.$$

Then

$$\lambda_p(H) \ge v^*Hv \ge \lambda_q(H)$$

for any $v \in V$ with $\|v\| = 1$.

3 Main results

In this section, we develop several matrix trace inequalities on Hermitian and skew-Hermitian matrices. Furthermore, Lemma 1 is extended to the high-dimensional case as Corollary 2.

Theorem 1 *Let*

$$A \in \mathbf{C}^{n\times n}, \quad H = \frac{A + A^*}{2}.$$

If $\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$ and $\lambda_1(H), \lambda_2(H), \ldots, \lambda_n(H)$ are singular values of A and eigenvalues of H, respectively. They are arranged in such a way that

$$\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A), \qquad \lambda_1(H) \ge \lambda_2(H) \ge \cdots \ge \lambda_n(H).$$

Then

$$\sigma_k(A) \ge \lambda_k(H),$$

where $k = 1, 2, \ldots, n$.

Proof From the definition of singular values of A, it follows that

$$\sigma_1^2(A), \qquad \sigma_2^2(A), \qquad \ldots, \qquad \sigma_n^2(A)$$

are eigenvalues of A^*A, and their corresponding orthonormal eigenvectors of A^*A are denoted as $u_1, u_2, \ldots, u_n$, respectively. Suppose $v_1, v_2, \ldots, v_n$ are orthonormal eigenvectors of H, and

$$\lambda_1(H), \qquad \lambda_2(H), \qquad \ldots, \qquad \lambda_n(H)$$

are eigenvalues of H corresponding to these orthonormal eigenvectors, respectively. Define

$$W_{1k} = \operatorname{Span}\{v_k, v_{k+1}, \ldots, v_n\}$$

and

$$W_{kn} = \operatorname{Span}\{u_1, u_2, \ldots, u_k\},$$

where $k = 1, 2, \ldots, n$. It is clear that

$$\dim(W_{1k}) = n - k + 1 \quad \text{and} \quad \dim(W_{kn}) = k.$$

Then

$$\dim(W_{1k}) + \dim(W_{kn}) = n + 1.$$

According to the dimensional formula, it follows that

$$\dim(W_{1k} + W_{kn}) + \dim(W_{1k} \cap W_{kn}) = \dim(W_{1k}) + \dim(W_{kn}).$$

This implies that

$$\dim(W_{1k} \cap W_{kn}) \neq \{0\}.$$

From Lemma 3, it follows that

$$\sigma_k^2(A) \geq w^*\left(A^*A\right)w \quad \text{and} \quad w^*Hw \geq \lambda_k(A)$$

for any $w \in W_{1k} \cap W_{kn}$ with $\|w\| = 1$. According to Lemma 2, we obtain

$$\sigma_k(A) \geq \sqrt{w^*\left(A^*A\right)w} \geq w^*Hw \geq \lambda_k(H),$$

where $k = 1, 2, \ldots, n$. This completes the proof. □

Remark 2 In this section, we prove Theorem 1 by using a simple and elementary method, which is slightly different from the proof method in [14]. This theorem reveals the relationship between singular values and eigenvalue of matrices. Based on it, several matrix trace inequalities on Hermitian and skew-Hermitian matrices will be obtained immediately.

Theorem 2 *Let* $A \in \mathbf{C}^{n\times n}$, $H = \frac{A+A^*}{2}$. *The following matrix trace inequalities are satisfied.*

(I) *If* $H \geq 0$, *then*

$$\operatorname{tr}\bigl((A^*A)^r\bigr) \geq \operatorname{tr}\bigl(H^{2r}\bigr), \quad \forall r \in \mathbf{R}^+.$$

(II) *If* $H > 0$, *then*

$$\operatorname{tr}\bigl((A^*A)^{-r}\bigr) \leq \operatorname{tr}\bigl(H^{-2r}\bigr), \quad \forall r \in \mathbf{R}^+.$$

(III)

$$\operatorname{tr}\bigl((A^*A)^{\frac{2k+1}{2}}\bigr) \geq \operatorname{tr}\bigl(H^{2k+1}\bigr), \quad \forall k \in \mathbf{N}.$$

Proof (I) From Theorem 1, it follows that

$$\begin{aligned}
\operatorname{tr}\bigl((A^*A)^r\bigr) &= \sum_{i=1}^{n} \lambda_i\bigl((A^*A)^r\bigr) \\
&= \sum_{i=1}^{n} \bigl(\lambda_i(A^*A)\bigr)^r \\
&= \sum_{i=1}^{n} \bigl(\sigma_i(A)\bigr)^{2r} \\
&\geq \sum_{i=1}^{n} \bigl(\lambda_i(H)\bigr)^{2r} \\
&= \sum_{i=1}^{n} \lambda_i\bigl(H^{2r}\bigr) \\
&= \operatorname{tr}\bigl(H^{2r}\bigr), \quad \forall r \in \mathbf{R}^+.
\end{aligned}$$

Similar to the proof of (I), we can easily verify that (II) and (III) hold. This completes the proof. □

Let

$$A = M + N, \quad H = \frac{A + A^*}{2},$$

where $N \in \mathbf{C}^{n\times n}$ and $M \in \mathbf{C}^{n\times n}$ be Hermitian and skew-Hermitian matrices, respectively. By using Theorem 2, the following conclusions will be obtained immediately.

Corollary 1 *Let* $N \in \mathbf{C}^{n\times n}$ *and* $M \in \mathbf{C}^{n\times n}$ *be Hermitian and skew-Hermitian matrices, respectively. The following matrix trace inequalities are satisfied.*

(I) *If* $H \geq 0$, *then*

$$\operatorname{tr}\left(\left((N+M)^*(N+M)\right)^r\right) \geq \operatorname{tr}\left(N^{2r}\right).$$

(II) *If* $H > 0$, *then*

$$\operatorname{tr}\left(\left((N+M)^*(N+M)\right)^{-r}\right) \leq \operatorname{tr}\left(N^{-2r}\right).$$

(III)

$$\operatorname{tr}\left(\left((N+M)^*(N+M)\right)^{\frac{2k+1}{2}}\right) \geq \operatorname{tr}\left(N^{2k+1}\right).$$

Based on Corollary 1, the following results can be obtained when $r = \frac{1}{2}$. This implies that the proof of the general case of Lemma 1 is completed.

Corollary 2 *Let* $N \in \mathbf{C}^{n\times n}$ *and* $M \in \mathbf{C}^{n\times n}$ *be a Hermitian and skew-Hermitian matrices, respectively. The following matrix trace inequalities are satisfied.*

(I) *If* $N \geq 0$, *then*

$$\operatorname{tr}\left(\left((N+M)^*(N+M)\right)^{\frac{1}{2}}\right) \geq \operatorname{tr}(N).$$

(II) *If* $N > 0$, *then*

$$\operatorname{tr}\left(\left((N+M)^*(N+M)\right)^{-\frac{1}{2}}\right) \leq \operatorname{tr}\left(N^{-1}\right).$$

4 Conclusions

This paper proves several matrix trace inequalities on Hermitian and skew-Hermitian matrices. These matrix trace inequalities can be applied to design and analyze interior-point methods (IPMs) for semidefinite optimization (SDO). In addition, matrix trace inequalities have many potential applications in control theory, for example, stabilization of time-delay systems (see [15–18]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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