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The i th p -geominimal surface area

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Abstract

In this paper, we introduce the concept of i th p -geominimal surface area, which extends the notion of p -geominimal surface area by Lutwak. Further, we prove some of its properties and related inequalities for this new notion.

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1 Introduction

During the past three decades, the investigations of the classical affine surface area have received great attention from the articles [1–16] or books [17, 18]. Based on the classical affine surface area, Lutwak [19] introduced the notion of p -affine surface area and obtained some isoperimetric inequalities for p -affine surface area.

Geominimal surface area was introduced by Petty [11] more than three decades ago. As Petty stated, this concept serves as a bridge connecting affine differential geometry, relative differential geometry, and Minkowskian geometry. Based on the classical geominimal surface area, Lutwak [19] introduced the notion of p -geominimal surface area and obtained some inequalities for it. Regarding the studies of p -affine surface area and p -geominimal surface area as well as its dual object, also see [13, 20–27].

Let $\mathcal{C}^n$ denote the set of compact convex subsets of the Euclidean n -space $\mathbb{R}^n$. The subset of $\mathcal{C}^n$ consisting of convex bodies (compact, convex sets with non-empty interiors) will be denoted by $\mathcal{K}^n$. For the set of convex bodies containing the origin in their interiors, write $\mathcal{K}_o^n$, and let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroid lies at the origin. As usual, S^{n-1} denotes the unit sphere with unit ball B_n , ω_n the volume of B_n .

Petty [11] defined the geominimal surface area $G(K)$ of a body $K \in \mathcal{K}_o^n$ by

$$\omega_n^{\frac{1}{n}} G(K) = \inf \left\{ n V_1(K, Q) V(Q^*)^{\frac{1}{n}} : Q \in \mathcal{K}_o^n \right\}. \quad (1.1)$$

For $p \geq 1$ and $K \in \mathcal{K}_o^n$, Lutwak [19] defined the p -geominimal surface area $G_p(K)$ of K by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf \left\{ n V_p(K, Q) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n \right\}. \quad (1.2)$$

Moreover, Lutwak proved the following inequalities for the p -geominimal surface area.

Theorem 1.1 *Let $K \in \mathcal{K}_c^n$ and $p \geq 1$, then*

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p}, \quad (1.3)$$

with equality if and only if K is an ellipsoid.

Let $\mathcal{F}_o^n$ denote the subset of $\mathcal{K}_o^n$ which has a positive continuous function, and let $\Omega_p(K)$ denote the p -affine surface area of K .

Theorem 1.2 *Let $K \in \mathcal{F}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n, \quad (1.4)$$

with equality if and only if K is of p -elliptic type.

Theorem 1.3 *If $K \in \mathcal{K}_o^n$ and $1 \leq p \leq q$, then*

$$\left(\frac{G_p(K)^n}{n^n V(K)^{n-p}} \right)^{\frac{1}{p}} \leq \left(\frac{G_q(K)^n}{n^n V(K)^{n-q}} \right)^{\frac{1}{q}}, \quad (1.5)$$

with equality if and only if K is p -selfminimal.

The purpose of this paper is to further extend Lutwak's p -geominimal surface area to the i th p -geominimal surface area. The technique we will use is that of the method designed by Lutwak [19]. Now, we define the notion of i th p -geominimal surface area as follows:

$$\omega_n^{\frac{p}{n-i}} G_{p,i}(K) = \inf \left\{ n W_{p,i}(K, Q) \tilde{W}_i(Q^*)^{\frac{p}{n-i}} : Q \in \mathcal{K}_o^n \right\}, \quad (1.6)$$

where $i \in \{0, 1, \dots, n-1\}$.

The main results are stated as follows. First, we establish the extended versions of Theorems 1.1, 1.2 and 1.3 given by Theorems 1.4, 1.5 and 1.6.

Theorem 1.4 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then*

$$G_{p,i}(K)^{n-i} \leq n^{n-i} (\omega_n^{-1} \omega_i^2 \omega_{n-i}^2)^p \binom{n}{i}^{-2p} W_i(K)^{n-p-i}, \quad (1.7)$$

with equality for $i = 0$ if and only if K is an ellipsoid, for $1 \leq i < n$ if and only if all $(n-i)$ -dimensional convex bodies which are contained in K are balls.

If $i = 0$, (1.7) is just inequality (1.3).

Let $\Omega_p^{(i)}(K)$ denote the $(i, 0)$ -type p -affine surface area of K (see Section 2.3).

Theorem 1.5 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{F}_{i,o}^n$, then*

$$\Omega_p^{(i)}(K)^{n+p-i} \leq (n\omega_n)^p G_{p,i}(K)^{n-i}, \quad (1.8)$$

with equality if and only if $K \in \mathcal{W}_{p,i}^n$. For the precise definition of $\mathcal{W}_{p,i}^n$, see Section 2.3.

Theorem 1.6 *If $K \in \mathcal{K}_o^n$ and $i \in \{0, 1, \dots, n-1\}$, then for $1 \leq p \leq q$,*

$$\left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} \leq \left(\frac{G_{q,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-q-i}} \right)^{1/q}, \quad (1.9)$$

with equality if and only if K is i th p -selfminimal.

The proofs of Theorems 1.4-1.6 will be given in Section 4 of this paper. Moreover, in Section 3 we also establish some properties of the i th p -geominimal surface area which may be required in the proofs of main results.

2 Background material for Brunn-Minkowski-Firey theory

2.1 Support function, radial function and polar of a convex body

For $\phi \in GL(n)$, let ϕ^t , ϕ^{-1} , and ϕ^{-t} denote the transpose, inverse, and inverse of the transpose of ϕ . For $K \in \mathcal{K}^n$, let $h(K, \cdot) : \mathbb{R}^n \rightarrow (-\infty, \infty)$ denote the support function of $K \in \mathcal{K}^n$, i.e., for $x \in \mathbb{R}^n$,

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

where $u \cdot x$ denotes the standard inner product of u and x . For $\phi \in GL(n)$, then obviously $h(\phi K, x) = h(K, \phi^t x)$. For the sake of convenience, we write h_K rather than $h(K, \cdot)$ for the support function of K . Apparently, for $K, L \in \mathcal{K}^n$, $K \subseteq L$ if and only if $h_K \leq h_L$. The set $\mathcal{K}^n$ will be viewed as equipped with the Hausdorff metric d defined by $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere $C(S^{n-1})$.

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L, u) = \rho_L(u) = \max\{\lambda > 0 : \lambda u \in L\}.$$

If $\rho(L, \cdot)$ is continuous and positive, L will be called a star body, and $\mathcal{S}_o^n$ will be used to denote the class of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Apparently, for $K, L \in \mathcal{S}_o^n$, $K \subseteq L$ if and only if $\rho_K \leq \rho_L$. Two star bodies K and L are said to be dilates (of one another) if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$. Let $\tilde{\delta}$ denote the radial Hausdorff metric as follows: if $K, L \in \mathcal{S}_o^n$, then $\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty$.

For $K \in \mathcal{K}_o^n$, the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

Obviously, we have $(K^*)^* = K$. For $K \in \mathcal{K}_o^n$, the support and radial functions of the polar body K^* of K are defined respectively by (see [18, 28])

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^*}(u) = \frac{1}{h_K(u)}$$

for all $u \in S^{n-1}$.

Define the Santaló product of $K \in \mathcal{K}_o^n$ by $V(K)V(K^*)$. The Blaschke-Santaló inequality (see [18, 28]) is one of the fundamental affine isoperimetric inequalities. It states that if $K \in \mathcal{K}_c^n$, then

$$V(K)V(K^*) \leq \omega_n^2, \quad (2.1)$$

with equality if and only if K is an ellipsoid.

2.2 The mixed p -quermassintegrals and dual mixed p -quermassintegrals

For $K \in \mathcal{K}^n$ and $i \in \{0, 1, \dots, n-1\}$, the quermassintegrals $W_i(K)$ of K are defined by (see [29])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS_i(K, u). \quad (2.2)$$

From (2.2), we easily see that $W_0(K) = V(K)$.

In the literature they have two representations, the quermassintegrals $W_i(K)$ and the intrinsic volumes $V_i(K)$, and we shall use both throughout. They are defined by

$$V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B_n, \dots, B_n}_i) = W_i(K) = \frac{\omega_i}{\binom{n}{i}} V_{n-i}(K), \quad i = 0, 1, \dots, n. \quad (2.3)$$

For real $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the Firey linear combination $\alpha \cdot K +_p \beta \cdot L$ is defined by (see [30])

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

Note that ‘ $\cdot_p$ ’ rather than ‘ $\cdot_p$ ’ is written for Firey scalar multiplication.

For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$, and real $p \geq 1$, the mixed p -quermassintegrals $W_{p,i}(K, L)$ of K and L , $i \in \{0, 1, \dots, n-1\}$, are defined by (see [29])

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}.$$

Obviously, for $p = 1$, $W_{1,i}(K, L)$ is just the classical mixed quermassintegrals $W_i(K, L)$. For $i = 0$, the mixed p -quermassintegrals $W_{p,0}(K, L)$ are just the p -mixed volume $V_p(K, L)$.

For $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$ and each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1} such that the mixed p -quermassintegrals $W_{p,i}(K, L)$ have the following integral representation (see [29]):

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_{p,i}(K, v) \quad (2.4)$$

for all $L \in \mathcal{K}_o^n$. It turns out that the measure $S_{p,i}(K, \cdot)$, $i \in \{0, 1, \dots, n-1\}$, on S^{n-1} is absolutely continuous with respect to $S_i(K, \cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h^{1-p}(K, \cdot). \quad (2.5)$$

Together with (2.2) and (2.4), for $K \in \mathcal{K}_o^n$, $p \geq 1$, we have $W_{p,i}(K, K) = W_i(K)$.

An immediate consequence of the definition of Firey linear combination, and the integral representation (2.4), is that for $Q \in \mathcal{K}_o^n$, the mixed p -quermassintegrals $W_{p,i}(Q, \cdot) : \mathcal{K}_o^n \rightarrow (0, \infty)$ are Firey linear.

Proposition 2.1 *Suppose $K, L, Q \in \mathcal{K}_o^n$ and $\lambda, \mu \geq 0$. If $p \geq 1, i \in \{0, 1, \dots, n-1\}$, then*

$$W_{p,i}(Q, \lambda \cdot K + \mu \cdot L) = \lambda W_{p,i}(Q, K) + \mu W_{p,i}(Q, L). \quad (2.6)$$

It will be helpful to introduce the following notation. Define the inner radius $r(K)$ and the outer radius $R(K)$ of $K \in \mathcal{K}_o^n$ by

$$r(K) = \max\{\lambda > 0 : \lambda B_n \subset K\} \quad \text{and} \quad R(K) = \min\{\lambda > 0 : K \subset \lambda B_n\}.$$

Recall that B_n is the unit ball centered at the origin. Thus,

$$r(K) = \min_{u \in S^{n-1}} h_K(u) \quad \text{and} \quad R(K) = \max_{u \in S^{n-1}} h_K(u).$$

Obviously, the body K is contained in the closure of the annulus $R(K)B_n \setminus r(K)B_n$. Note that the notions of inner and outer radii as defined here are not translation invariant.

The next proposition shows that the functional $W_{p,i}(K, \cdot)^{1/p} : \mathcal{K}_o^n \rightarrow (0, \infty)$ is Lipschitzian. This observation will be needed in Sections 3 and 4.

Proposition 2.2 *If $K, L, Q \in \mathcal{K}_o^n$ and $p \geq 1, i \in \{0, 1, \dots, n-1\}$, then*

$$|W_{p,i}(Q, K)^{1/p} - W_{p,i}(Q, L)^{1/p}| \leq |h_K - h_L|_\infty W_i(Q)^{1/p} / r(Q). \quad (2.7)$$

Proof The Minkowski integral inequality, together with (2.4) and (2.5), gives

$$\begin{aligned} W_{p,i}(Q, K)^{1/p} &\leq W_{p,i}(Q, L)^{1/p} + \left[\frac{1}{n} \int_{S^{n-1}} |h_K(u) - h_L(u)|^p h(Q, u)^{1-p} dS_i(Q, u) \right]^{1/p} \\ &\leq W_{p,i}(Q, L)^{1/p} + |h_K - h_L|_\infty W_i(Q)^{1/p} \max_{u \in S^{n-1}} h_Q(u)^{-1}. \quad \square \end{aligned}$$

For $K \in \mathcal{S}_o^n$ and any real i , the i th dual quermassintegrals $\tilde{W}_i(K)$ of K are defined by (see [18, 28])

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) dS(u). \quad (2.8)$$

Obviously, $\tilde{W}_0(K) = V(K)$.

An immediate consequence of the definition of i th dual quermassintegrals is as follows.

Proposition 2.3 *If $p \geq 1$ and $i \in \mathbb{R}^n$, then the functional $\tilde{W}_i(\cdot) : \mathcal{S}_o^n \rightarrow (0, \infty)$ is continuous.*

For $K, L \in \mathcal{S}_o^n, p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the p -harmonic radial combination $\lambda \diamond K \hat{+}_{-p} \mu \diamond L \in \mathcal{S}_o^n$ is defined by (see [19])

$$\rho(\lambda \diamond K \hat{+}_{-p} \mu \diamond L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}.$$

If $K, L \in \mathcal{K}_o^n$ (rather than being in $\mathcal{S}_o^n$), then

$$\lambda \diamond K \hat{+}_{-p} \mu \diamond L = (\lambda \cdot K^* +_p \mu \cdot L^*)^*.$$

For $K, L \in \mathcal{S}_o^n$, $\varepsilon > 0$, $p \geq 1$, and real $i \neq n$, the dual mixed p -quermassintegrals $\tilde{W}_{-p,i}(K, L)$ of K and L are defined by (see [31])

$$\frac{n-i}{-p} \tilde{W}_{-p,i}(K, L) = \lim_{\varepsilon \rightarrow 0^+} \frac{\tilde{W}_i(K \hat{+}_{-p} \varepsilon \diamond L) - \tilde{W}_i(K)}{\varepsilon}. \quad (2.9)$$

If $i = 0$, we easily see that (2.9) is just the definition of dual p -mixed volume, i.e., $\tilde{W}_{-p,0}(K, L) = \tilde{V}_{-p}(K, L)$.

From (2.9), the integral representation of the dual mixed p -quermassintegrals is given by Wang and Leng [31]: If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and real $i \neq n$, $i \neq n + p$, then

$$\tilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u) \rho_L^{-p}(u) dS(u). \quad (2.10)$$

Together with (2.8) and (2.10), for $K \in \mathcal{S}_o^n$, $p \geq 1$, and $i \neq n$, $n + p$, it follows that $\tilde{W}_{-p,i}(K, K) = \tilde{W}_i(K)$.

Further, Wang and Leng [31] proved the following analog of the Minkowski inequality for the dual mixed p -quermassintegrals.

Lemma 2.4 *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then for $i < n$ or $i > n + p$,*

$$\tilde{W}_{-p,i}(K, L)^{n-i} \geq \tilde{W}_i(K)^{n+p-i} \tilde{W}_i(L)^{-p}, \quad (2.11)$$

with equality in every inequality if and only if K and L are dilates of each other. For $n < i < n + p$, inequality (2.11) is reverse.

Another consequence of Lemma 2.4 will be needed.

Lemma 2.5 ([32]) *Suppose $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu > 0$. If real $i < n$ or $n < i < n + p$, then*

$$\tilde{W}_i(\lambda \diamond K \hat{+}_{-p} \mu \diamond L)^{-p/(n-i)} \geq \lambda \tilde{W}_i(K)^{-p/(n-i)} + \mu \tilde{W}_i(L)^{-p/(n-i)}, \quad (2.12)$$

with equality in every inequality if and only if K and L are dilates of each other. For $i > n + p$, inequality (2.12) is reverse.

2.3 The i th p -curvature function and i th p -curvature image

A convex body $K \in \mathcal{K}^n$ is said to have a continuous i th curvature function $f_i(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ if its mixed surface area measure $S_i(K, \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative (see [29])

$$\frac{dS_i(K, \cdot)}{dS} = f_i(K, \cdot) \quad \text{for } i \in \{0, 1, \dots, n-1\}. \quad (2.13)$$

Let $\mathcal{F}_i^n$, $\mathcal{F}_{i,o}^n$, $\mathcal{F}_{i,c}^n$ denote the sets of all bodies in $\mathcal{K}^n$, $\mathcal{K}_o^n$, $\mathcal{K}_c^n$, respectively, that have an i th positive continuous curvature function. In particular, $\mathcal{F}_0^n := \mathcal{F}^n$, $\mathcal{F}_{0,o}^n := \mathcal{F}_o^n$, $\mathcal{F}_{0,c}^n := \mathcal{F}_c^n$.

If ∂K is a regular C^2 -hypersurface with (everywhere) positive principal curvatures, then $K \in \mathcal{F}_i^n$ for all i , and the curvature functions of K are proportional to the elementary symmetric functions of the principal radii of curvature (viewed as functions of the outer normals) of K . Thus, $f_0(K, u)$ is the reciprocal Gauss curvature of ∂K at the point of ∂K whose outer normal is u , while $f_{n-2}(K, u)$ is proportional to the arithmetic mean of the radii of curvature of ∂K at the point whose outer normal is u .

A convex body $K \in \mathcal{K}_o^n$ is said to have a p -curvature function $f_p(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ if its p -surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative (see [19])

$$\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot). \quad (2.14)$$

Lutwak [19] showed the notion of p -curvature image as follows: For each $K \in \mathcal{F}_o^n$ and $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the p -curvature image of K , by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K, \cdot). \quad (2.15)$$

Note that for $p = 1$, this definition is different from the classical curvature image (see [19]).

Recently, Liu *et al.* [33], Lu and Wang [34] as well as Ma and Liu [35, 36] independently introduced the concept of i th p -curvature function of $K \in \mathcal{K}_o^n$ as follows: Let $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, a convex body $K \in \mathcal{K}_o^n$ is said to have an i th p -curvature function $f_{p,i}(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ if its i th p -surface area measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S and has the Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS} = f_{p,i}(K, \cdot). \quad (2.16)$$

If the i th surface area measure $S_i(K, \cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S , we have

$$f_{p,i}(K, \cdot) = h(K, \cdot)^{1-p} f_i(K, \cdot). \quad (2.17)$$

According to the concept of i th p -curvature function of a convex body, Lu and Wang [34] and Ma [26] introduced independently the concept of i th p -curvature image of a convex body as follows: For each $K \in \mathcal{F}_{i,o}^n$, $i \in \{0, 1, \dots, n-1\}$ and real $p \geq 1$, define $\Lambda_{p,i} K \in \mathcal{S}_o^n$, the i th p -curvature image of K , by

$$\rho(\Lambda_{p,i} K, \cdot)^{n+p-i} = \frac{\tilde{W}_i(\Lambda_{p,i} K)}{\omega_n} f_{p,i}(K, \cdot). \quad (2.18)$$

The unusual normalization of definition (2.18) is chosen so that, for the unit ball B_n , it follows that $\Lambda_{p,i} B_n = B_n$. From definitions (2.15), (2.18) and formula (2.17), if $i = 0$, then $\Lambda_{p,0} K = \Lambda_p K$.

An immediate consequence of the definition of i th p -curvature image and the integral representations of $W_{p,i}$ and $\tilde{W}_{-p,i}$ is the following proposition.

Proposition 2.6 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$ and $K \in \mathcal{F}_{i,0}^n$, then*

$$W_{p,i}(K, Q^*) = \omega_n \tilde{W}_{-p,i}(\Lambda_{p,i}K, Q) / \tilde{W}_i(\Lambda_{p,i}K) \quad (2.19)$$

for all $Q \in \mathcal{S}_o^n$.

Recently, Ma [26] introduced the concept of $(i, 0)$ -type p -affine surface area as follows: Let $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, the $(i, 0)$ -type p -affine surface area $\Omega_p^{(i)}(K)$ of $K \in \mathcal{F}_{i,0}^n$ is defined by

$$\Omega_p^{(i)}(K) = \int_{S^{n-1}} f_{p,i}(K, u)^{\frac{n-i}{n+p-i}} dS(u).$$

An immediate consequence of the definition of i th p -curvature image and the integral representations of $\Omega_p^{(i)}$ and $\tilde{W}_i$ is the following proposition.

Proposition 2.7 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{F}_{i,0}^n$, then*

$$\Omega_p^{(i)}(K) = n\omega_n^{\frac{n-i}{n+p-i}} \tilde{W}_i(\Lambda_{p,i}K)^{\frac{p}{n+p-i}}. \quad (2.20)$$

A body $K \in \mathcal{F}_o^n$ is of i th p -elliptic type if the function $f_{p,i}(K, \cdot)^{1/(n+p-i)}$ is the support function of a convex body in $\mathcal{K}_o^n$, i.e., K is of i th p -elliptic type if there exists a body $Q \in \mathcal{K}_o^n$ such that

$$f_{p,i}(K, \cdot) = h(Q, \cdot)^{-(n+p-i)}.$$

Define

$$\mathcal{W}_{p,i}^n = \{K \in \mathcal{F}_{i,0}^n : \text{there exists } Q \in \mathcal{K}_o^n \text{ with } f_{p,i}(K, \cdot) = h(Q, \cdot)^{-(n+p-i)}\}.$$

An immediate consequence of the definition of $\mathcal{W}_{p,i}^n$ and the definition of $\Lambda_{p,i}$ is the following.

Proposition 2.8 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{F}_{i,0}^n$, then*

$$K \in \mathcal{W}_{p,i}^n \quad \text{if and only if} \quad \Lambda_{p,i}K \in \mathcal{K}_o^n.$$

3 The i th p -geominimal surface area

Let $O(n)$ denote an orthogonal transformation group in $\mathbb{R}^n$. We will give the following lemmas and propositions.

Lemma 3.1 ([29]) *Suppose $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, then for any $\phi \in O(n)$,*

$$W_{p,i}(\phi K, \phi L) = W_{p,i}(K, L).$$

Lemma 3.2 ([37]) *Suppose $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and real $i \in \mathbb{R}$ as well as $i \neq n$, $i \neq n+p$, then, for any $\phi \in O(n)$,*

$$\tilde{W}_{-p,i}(\phi K, \phi L) = \tilde{W}_{-p,i}(K, L).$$

Specifically,

$$\tilde{W}_i(\phi K) = \tilde{W}_i(K) \quad \text{for any } \phi \in O(n).$$

An immediate consequence of the definition of $G_{p,i}$ and Lemma 3.1 and Lemma 3.2 is the following.

Proposition 3.3 *Suppose $K \in \mathcal{K}_o^n$. If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$ and $\phi \in O(n)$, then*

$$G_{p,i}(\phi K) = G_{p,i}(K).$$

Lemma 3.4 *If $p \geq 1$, and K_j is a sequence of bodies in $\mathcal{K}_o^n$ such that $K_j \rightarrow K_0 \in \mathcal{K}_o^n$, then for $i \in \{0, 1, \dots, n-1\}$, $S_{p,i}(K_j, \cdot) \rightarrow S_{p,i}(K_0, \cdot)$ weakly.*

Proof Suppose $f \in C(S^{n-1})$. Since $K_j \rightarrow K_0$, by the definition of support function, $h_{K_j} \rightarrow h_{K_0}$ uniformly on S^{n-1} . Since the continuous function h_{K_0} is positive, h_{K_j} are uniformly bounded away from 0. It follows that $h_{K_j}^{1-p} \rightarrow h_{K_0}^{1-p}$ uniformly on S^{n-1} , and thus

$$fh_{K_j}^{1-p} \rightarrow fh_{K_0}^{1-p} \quad \text{uniformly on } S^{n-1}.$$

But $K_j \rightarrow K_0$ also implies that

$$S_i(K_j, \cdot) \rightarrow S_i(K_0, \cdot) \quad \text{weakly on } S^{n-1}$$

follows from the weak continuity of surface area measures (see, for example, Schneider [12, 19]). Hence,

$$\int_{S^{n-1}} f(u) h(K_j, u)^{1-p} dS_i(K_j, u) \rightarrow \int_{S^{n-1}} f(u) h(K_0, u)^{1-p} dS_i(K_0, u),$$

or equivalently,

$$\int_{S^{n-1}} f(u) dS_{p,i}(K_j, u) \rightarrow \int_{S^{n-1}} f(u) dS_{p,i}(K_0, u). \quad \square$$

Lemma 3.5 *Suppose $K_j \rightarrow K_0 \in \mathcal{K}_o^n$ and $L_j \rightarrow L_0 \in \mathcal{K}_o^n$. If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, then $W_{p,i}(K_j, L_j) \rightarrow W_{p,i}(K_0, L_0)$.*

Proof Since $h_{L_j} \rightarrow h_{L_0}$ uniformly on S^{n-1} , and h_L is continuous, the h_{L_i} are uniformly bounded on S^{n-1} . Hence,

$$h_{L_j}^p \rightarrow h_{L_0}^p \quad \text{uniformly on } S^{n-1}.$$

By Lemma 3.4, $K_j \rightarrow K_0$ implies that

$$S_{p,i}(K_j, \cdot) \rightarrow S_{p,i}(K_0, \cdot) \quad \text{weakly on } S^{n-1}.$$

Hence,

$$\int_{S^{n-1}} h_{L_j}^p dS_{p,i}(K_j, u) \rightarrow \int_{S^{n-1}} h_{L_0}^p dS_{p,i}(K_0, u). \quad \square$$

By the definition of dual mixed p -quermassintegrals and the continuity of the radial function, we have the following.

Lemma 3.6 *Suppose $K_j \rightarrow K_0 \in \mathcal{S}_0^n$ and $L_j \rightarrow L_0 \in \mathcal{S}_0^n$. If $p \geq 1$, $i \in \mathbb{R}$, and $i \neq n$, $i \neq n + p$, then $\tilde{W}_{-p,i}(K_j, L_j) \rightarrow \tilde{W}_{-p,i}(K_0, L_0)$.*

An immediate consequence of the definition of $\Omega_p^{(i)}$ and Lemma 3.5 and Lemma 3.6 is the following.

Proposition 3.7 *For $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, the function $G_{p,i} : \mathcal{K}_o^n \rightarrow (0, \infty)$ is upper semicontinuous.*

Suppose $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$. For $p \geq 1$, define $W_{p,i}(K, L^*)$ by

$$W_{p,i}(K, L^*) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^{-p} dS_{p,i}(K, u).$$

Since $h_{Q^*} = 1/\rho_Q$ for $Q \in \mathcal{K}_o^n$, it follows from the integral representation (2.4) that, if L happens to belong to $\mathcal{K}_o^n$ (rather than just to $\mathcal{S}_o^n$), the new definition of $W_{p,i}(K, L^*)$ agrees with the old definition.

An immediate consequence of Lemma 3.5 is as follows.

Lemma 3.8 *If $p \geq 1$ and $L \in \mathcal{S}_o^n$, then $W_{p,i}(\cdot, L^*) : \mathcal{K}_o^n \rightarrow (0, \infty)$ is continuous.*

The following simple fact will be needed.

Proposition 3.9 *Suppose $K_j \in \mathcal{K}_o^n$, $K_j \rightarrow L \in \mathcal{C}_o^n$ and $0 \leq i \leq n$. If the sequence $\tilde{W}_i(K_j^*)$ is bounded, then $L \in \mathcal{K}_o^n$.*

Proof Note that for $A \in \mathcal{S}_o^n$ and $0 < i < n$, $i \in \mathbb{R}$, it follows that $\tilde{W}_i(A) \leq V(A)^{(n-i)/n} \omega_n^{i/n}$ with equality if and only if A is an n -ball centered at the origin (see [38]). We chose two non-negative real numbers c_0, c_1 such that $c_1 \omega_n^{i/n} V(K_j^*)^{(n-i)/n} \leq \tilde{W}_i(K_j^*) \leq c_0$ for all j . Since L is compact, there exists a real r_0 such that $L \subset r_0 B_n$, and since $K_j \rightarrow L$, the number r_0 may be chosen so that $K_j \subset r_0 B_n$ for all i , as well. (Recall that B_n is the unit ball centered at the origin.)

For each j , let

$$r_j = r(K_j) = \min_{u \in S^{n-1}} h(K_j, u) = h(K_j, u_j),$$

where $u_j \in S^{n-1}$ is any point where this minimum is attained. Since $\rho(K_j^*, u_j) = 1/h(K_j, u_j) = r_j^{-1}$, it follows that K_j^* contains the point $r_j^{-1} u_j$. Since $K_j \subset r_0 B_n$, it follows that $\rho(K_j^*, u_j) = 1/h(K_j, u_j) \geq 1/r_0$.

Thus, K_j^* contains the right cone whose apex is $r_j^{-1} u_j$ and whose base is an $(n-1)$ -dimensional ball of radius r_0^{-1} that lies in the subspace orthogonal to u_j . Thus, for $0 \leq i \leq n$,

$$\frac{\omega_{n-1}}{n} r_0^{1-n} r_j^{-1} \leq V(K_j^*) \leq c_1^{-\frac{n}{n-i}} \omega_n^{-\frac{i}{n-i}} \tilde{W}_i(K_j^*)^{\frac{n}{n-i}} \leq (c_0 c_1^{-1})^{\frac{n}{n-i}} \omega_n^{-\frac{i}{n-i}},$$

and hence $r_j \geq \frac{1}{n} \omega_{n-1} \omega_n^{\frac{i}{n-1}} r_0^{1-n} (c_0 c_1^{-1})^{-\frac{n}{n-1}}$. Hence the ball, centered at the origin, of radius $\frac{1}{n} \omega_{n-1} \omega_n^{\frac{i}{n-1}} r_0^{1-n} (c_0 c_1^{-1})^{-\frac{n}{n-1}}$ is contained in each K_j , and thus this ball is contained in L as well. $\square$

For $K \in \mathcal{K}_o^n$, let $P_{p,i}K$ denote the compact convex set whose support function, for $x \in \mathbb{R}^n$, is given by

$$h(P_{p,i}K, x)^p = \frac{1}{n} \int_{S^{n-1}} 2^{-p} (|x \cdot u| + x \cdot u)^p dS_{p,i}(K, u).$$

The fact that the function $h(P_{p,i}K, \cdot)$ is convex and hence is the support function of a compact convex set is a direct consequence of the Minkowski integral inequality. Obviously, $h(P_{p,i}K, \cdot) \geq 0$ on S^{n-1} . That $h(P_{p,i}K, \cdot) > 0$ on S^{n-1} follows from the fact that the surface area measure of a convex body cannot be concentrated on a closed hemisphere of S^{n-1} . If it were the case that $h(P_{p,i}K, \cdot) = 0$, then $S_{p,i}(K, \cdot)$, and thus the i th surface area measure $S_i(K, \cdot)$ would be concentrated on a closed hemisphere bounded by the great sphere of S^{n-1} that is orthogonal to u_0 . Since $h(P_{p,i}, \cdot)$ is positive, $P_{p,i}K \in \mathcal{K}_o^n$.

For $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$, let $w_i(K|u^\perp)$ denote the $(n-1)$ -dimensional quermassintegrals of $K|u^\perp$, the image of the orthogonal projection of K onto the $(n-1)$ -dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u . The i th projection body $\Pi_i K \in \mathcal{K}_o^n$ of $K \in \mathcal{K}_o^n$, $i = 0, 1, \dots, n-1$, is the body whose support function is given by

$$h(\Pi_i K, u) = w_i(K|u^\perp) = \frac{1}{2} \int_{S^{n-1}} |u \cdot u'| dS_i(K, u')$$

for $u \in S^{n-1}$. (See the survey of Lutwak [39].) Since for $K \in \mathcal{K}_o^n$,

$$\int_{S^{n-1}} u dS_i(K, u) = 0,$$

it follows from (2.5) that

$$nP_{1,i}K = \Pi_i K.$$

As noted previously, $h(P_{p,i}K, \cdot) > 0$ on S^{n-1} . However a slightly stronger statement will be needed in this section.

Lemma 3.10 For $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then

$$h(P_{p,i}K, u) \geq W_i(K)^{(1-p)/p} w_i(K|u^\perp)/n$$

for all $u \in S^{n-1}$.

Proof Since from (2.5),

$$h(P_{p,i}K, x)^p = \frac{1}{n} \int_{S^{n-1}} \left(\frac{|x \cdot u| + x \cdot u}{2h(K, u)} \right)^p h(K, u) dS_i(K, u)$$

and

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS_i(K, u),$$

it follows from Jensen's inequality that for all $u \in S^{n-1}$,

$$W_i(K)^{-1/p} h(P_{p,i}K, u) \geq W_i(K)^{-1} h(P_{1,i}K, u).$$

To complete the proof, recall that $nh(P_{1,i}K, u) = h(\Pi_i K, u) = w_i(K|u^\perp)$. $\square$

Proposition 3.11 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, there exists a unique body $\bar{K} \in \mathcal{K}_o^n$ such that*

$$G_{p,i}(K) = nW_{p,i}(K, \bar{K}) \quad \text{and} \quad \tilde{W}_i(\bar{K}^*) = \omega_n.$$

Proof Choose c_1 such that $h(P_{p,i}K, \cdot)^p \geq c_1 > 0$ on S^{n-1} . From the definition of $G_{p,i}(K)$, there exists a sequence $M_j \in \mathcal{K}_o^n$ such that $\tilde{W}_i(M_j^*) = \omega_n$ with $W_{p,i}(K, B_n) \geq W_{p,i}(K, M_j)$ for all j , and

$$nW_{p,i}(K, M_j) \rightarrow G_{p,i}(K).$$

To see that $M_j \in \mathcal{K}_o^n$ are uniformly bounded, let

$$R_j = R(M_j) = \rho(M_j, u_j) = \max\{\rho(M_j, u) : u \in S^{n-1}\},$$

where u_j is any of the points in S^{n-1} at which this maximum is attained.

Since the support function of M_j dominates that of the convex set $e_j = \{\lambda u_j : 0 \leq \lambda \leq R_j\} \subset M_j$, and since the measure $S_{p,i}(K, \cdot)$ is positive, it follows that $W_{p,i}(K, M_j) \geq R_j^p h(P_{p,i}K, u_j)^p$. Hence,

$$W_{p,i}(K, B_n) \geq W_{p,i}(K, M_j) \geq R_j^p h(P_{p,i}K, u_j)^p \geq R_j^p c_1.$$

Since M_j are uniformly bounded, the Blaschke selection theorem guarantees the existence of a subsequence of M_j , which will also be denoted by M_j , and a compact convex $L \in \mathcal{C}^n$, such that $M_j \rightarrow L$. Since $\tilde{W}_i(M_j^*) = \omega_n$, Proposition 3.9 gives $L \in \mathcal{K}_o^n$. Now, $M_j \rightarrow L$ implies that $M_j^* \rightarrow L^*$, and since $\tilde{W}_i(M_j^*) = \omega_n$, it follows that $\tilde{W}_i(L^*) = \omega_n$. Lemma 3.5 can now be used to conclude that L will serve as the desired body $\bar{K}$.

The uniqueness of the minimizing body is easily demonstrated as follows. Suppose $L_1, L_2 \in \mathcal{K}_o^n$, such that $\tilde{W}_i(L_1^*) = \omega_n = \tilde{W}_i(L_2^*)$, and

$$\begin{aligned} W_{p,i}(K, L_1) \tilde{W}_i(L_1^*)^{p/(n-i)} &= \inf\{W_{p,i}(K, Q) \tilde{W}_i(Q^*)^{p/(n-i)} : Q \in \mathcal{K}_o^n\} \\ &= W_{p,i}(K, L_2) \tilde{W}_i(L_2^*)^{p/(n-i)}. \end{aligned}$$

Define $L \in \mathcal{K}_o^n$ by

$$L = \frac{1}{2} \cdot L_1 +_p \frac{1}{2} \cdot L_2.$$

Proposition 2.1 shows that

$$W_{p,i}(K, L) = W_{p,i}(K, L_1) = W_{p,i}(K, L_2).$$

Since, obviously,

$$L^* = \frac{1}{2} \diamond L_1^* \hat{+}_p \frac{1}{2} \diamond L_2^*$$

and $\tilde{W}_i(L_1^*) = \omega_n = \tilde{W}_i(L_2^*)$, it follows from Lemma 2.5 that

$$\tilde{W}_i(L^*) \leq \omega_n$$

with equality if and only if $L_1 = L_2$. Thus,

$$W_{p,i}(K, L) \tilde{W}_i(L^*)^{p/(n-i)} < W_{p,i}(K, L_1) \tilde{W}_i(L_1^*)^{p/(n-i)}$$

is the contradiction that would arise if it were the case that $L_1 \neq L_2$. $\square$

The unique body whose existence is guaranteed by Proposition 3.11 will be denoted by $T_{p,i}K$ and will be called the i th p -Petty body of K . The polar body of $T_{p,i}K$ will be denoted by $T_{p,i}^*K$ rather than $(T_{p,i}K)^*$. When $p = 1$, the subscript will often be suppressed. Thus, for $K \in \mathcal{K}_o^n$, $p \geq 1$, and $i \in \{0, 1, \dots, n-1\}$, the body $T_{p,i}K$ is defined by

$$G_{p,i}(K) = nW_{p,i}(K, T_{p,i}K) \quad \text{and} \quad \tilde{W}_i(T_{p,i}^*K) = \omega_n.$$

The next proposition shows that the mapping $T_{p,i} : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$ is an orthogonal transformation invariant mapping.

Proposition 3.12 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then for $\phi \in O(n)$,*

$$T_{p,i}\phi K = \phi T_{p,i}K.$$

Proof From the definition of $T_{p,i}$ and Proposition 3.3,

$$nW_{p,i}(K, T_{p,i}K) = G_{p,i}(K) = G_{p,i}(\phi K) = nW_{p,i}(\phi K, T_{p,i}\phi K).$$

By Lemma 3.1,

$$W_{p,i}(K, T_{p,i}K) = W_{p,i}(\phi K, T_{p,i}\phi K) = W_{p,i}(K, \phi^{-1}T_{p,i}\phi K).$$

The uniqueness part of Proposition 3.11 shows that $T_{p,i}K = \phi^{-1}T_{p,i}\phi K$, which is the desired result. $\square$

In order to prove Lemma 3.14, the following fact will be needed.

Lemma 3.13 ([40]) *If $K \in \mathcal{K}^n$ and $i \in \{1, 2, \dots, n-1\}$, then*

$$W_i(K) \geq \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}},$$

with equality if and only if K is an n -ball.

The following crude bound on the size of $T_{p,i}K$ will be helpful.

Lemma 3.14 *Suppose $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$. If $r, R > 0$ are such that*

$$rB_n \subset K \subset RB_n,$$

then

$$h(T_{p,i}K, u) \leq \frac{n\omega_i\omega_{n-i}}{\binom{n}{i}\omega_{n-1}} (R/r)^{n-i}$$

for all $u \in S^{n-1}$.

Proof From the integral representation (2.4) and formulas (2.2) and (2.5), the trivial estimate follows:

$$W_{p,i}(K, B_n) \leq \frac{W_i(K)}{\min_{u \in S^{n-1}} h_K^p(u)} = \frac{\omega_i V_{n-i}(K)}{\binom{n}{i} \min_{u \in S^{n-1}} h_K^p(u)} \leq \frac{\omega_i \omega_{n-i} R^{n-i}}{\binom{n}{i} r^p}.$$

From the minimality property of $T_{p,i}K$, it follows that

$$W_{p,i}(K, T_{p,i}K) \leq W_{p,i}(K, B_n).$$

Let u_0 be any point in S^{n-1} such that

$$\rho(T_{p,i}K, u_0) = \max\{\rho(T_{p,i}K, u) : u \in S^{n-1}\} = R(T_{p,i}K).$$

Since the support function of $T_{p,i}K$ dominates that of the convex set $e_0 = \{\lambda u_0 : 0 \leq \lambda \leq R(T_{p,i}K)\} \subset T_{p,i}K$, it follows that

$$R(T_{p,i}K)^p h(P_{p,i}K, u_0)^p \leq W_{p,i}(K, T_{p,i}K).$$

But from Lemma 3.10, Lemma 3.13 and formula (2.3), it follows that

$$\begin{aligned} h(P_{p,i}K, u_0)^p &\geq W_i(K)^{1-p} w_i(K|u_0^\perp)^p / n^p \\ &= n^{-p} \binom{n}{i}^{p-1} \omega_i^{1-p} V_{n-i}(K)^{1-p} w_i(K|u_0^\perp)^p \\ &\geq n^{-p} \binom{n}{i}^{p-1} \omega_i^{1-p} V_{n-i}(K)^{1-p} \omega_{n-1}^{\frac{ip}{n-1}} v(K|u_0^\perp)^{\frac{(n-1-i)p}{n-1}} \\ &\geq n^{-p} \binom{n}{i}^{p-1} \omega_i^{1-p} \omega_{n-1}^{\frac{ip}{n-1}} (R^{n-i} \omega_{n-i})^{1-p} (r^{n-1} \omega_{n-1})^{\frac{(n-1-i)p}{n-1}}. \end{aligned}$$

The desired result is obtained by combining all these inequalities. $\square$

If the outer radii of a sequence of bodies are uniformly bounded from above and the inner radii of the sequence are bounded away from 0, then the same is true for the radii of the i th p -Petty bodies of the sequence. This is contained in the following lemma.

Lemma 3.15 *Suppose $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$. If $K_j \in \mathcal{K}_o^n$ is a family of bodies for which there exist $r, R > 0$ such that*

$$rB_n \subset K_j \subset RB_n \quad \text{for all } j,$$

then there exist $r', R' > 0$ such that

$$r'B_n \subset T_{p,i}K_j \subset R'B_n \quad \text{for all } j.$$

Proof The existence of $R' > 0$, and thus the fact that $T_{p,i}K_j$ are uniformly bounded, is contained in Lemma 3.14. Let $r_j = r(T_{p,i}K_j)$ denote the inner radius of $T_{p,i}K_j$. Thus,

$$r_j = \min_{u \in S^{n-1}} h(T_{p,i}K_j, u) = h(T_{p,i}K_j, u_j),$$

where $u_j \in S^{n-1}$ is any point where this minimum is attained. Suppose that the infimum of r_j is 0. Thus, there exists a subsequence of $T_{p,i}K_j$, which will not be relabeled, such that

$$h(T_{p,i}K_j, u_j) \rightarrow 0.$$

The Blaschke selection theorem, in conjunction with Proposition 3.9, demonstrates the existence of $M \in \mathcal{K}_o^n$ such that for a subsequence of $T_{p,i}K_j$, which will also not be relabeled,

$$T_{p,i}K_j \rightarrow M.$$

But $h(T_{p,i}K_j) \rightarrow 0$ and $|h_{T_{p,i}K_j} - h_M|_\infty \rightarrow 0$ imply that $h_M(u_j) \rightarrow 0$, which is impossible since the continuous function h_M is positive. $\square$

The case $i = 0$ of the following proposition is due to Lutwak [19]. The proof of this proposition is based on the one given by Petty and Lutwak.

Proposition 3.16 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, then the functional $G_{p,i} : \mathcal{K}_o^n \rightarrow (0, \infty)$ is continuous.*

Proof That $G_{p,i}$ is upper semicontinuous follows immediately from Lemma 3.8: The i th p -geominimal surface area

$$\omega_n^{p/(n-i)} G_{p,i} : \mathcal{K}_o^n \rightarrow (0, \infty)$$

is defined as the infimum of continuous functions

$$n W_{p,i}(\cdot, Q^*) \tilde{W}_i(Q)^{p/(n-i)} : \mathcal{K}_o^n \rightarrow (0, \infty)$$

as Q ranges over $\mathcal{K}_o^n$.

To see that $G_{p,i}$ is lower semicontinuous at $K_0 \in \mathcal{K}_o^n$, let $K_j \in \mathcal{K}_o^n$ be a sequence of bodies such that $K_j \rightarrow K_0$ with $G_{p,i}(K_j) \rightarrow l \in \mathbb{R}$. It will be shown that $l \geq G_{p,i}(K_0)$, and thus

$$\liminf G_{p,i}(K_j) \geq G_{p,i}(K_0).$$

By Lemma 3.15 the $T_{p,i}K_j$ are uniformly bounded. The Blaschke selection theorem, in conjunction with Proposition 3.9, yields the existence of a body $M \in \mathcal{K}_o^n$ and a subsequence of $T_{p,i}K_j$, which will not be relabeled, such that $T_{p,i}K_j \rightarrow M$ and $\tilde{W}_i(M^*) = \omega_n$. Lemma 3.5 and the facts that $K_j \rightarrow K_0$ and $T_{p,i}K_j \rightarrow M$ may be used to conclude that $G_{p,i}(K_j) = nW_{p,i}(K_j, T_{p,i}K_j) \rightarrow nW_{p,i}(K_0, M)$. Now $nW_{p,i}(K_0, M) = l$, since $G_{p,i}(K_j) = l$. But the definition of $G_{p,i}(K_0)$ shows that

$$\omega_n^{p/(n-i)} l = nW_{p,i}(K_0, M) \tilde{W}_i(M^*)^{p/(n-i)} \geq \omega_n^{p/(n-i)} G_{p,i}(K_0)$$

and completes the argument. $\square$

The case $i = 0$ of the following result is due to Lutwak [19].

Proposition 3.17 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, then the map $T_{p,i} : \mathcal{K}_o^n \rightarrow \mathcal{K}_o^n$ is continuous.*

Proof Suppose $K_j \in \mathcal{K}_o^n$ such that $K_j \rightarrow K_0 \in \mathcal{K}_o^n$. Let $T_{p,i}K_j$ denote a subsequence of $T_{p,i}K_j$. Since $K_0 \in \mathcal{K}_o^n$, Lemma 3.15 shows that $T_{p,i}K_j$ are uniformly bounded. The Blaschke selection theorem, in conjunction with Proposition 3.9, yields the existence of a body $M \in \mathcal{K}_o^n$ and a subsequence of $T_{p,i}K_j$, which will not be relabeled, such that $T_{p,i}K_j \rightarrow M$ and $\tilde{W}_i(M^*) = \omega_n$. Lemma 3.5 and the facts that $K_j \rightarrow K_0$ and $T_{p,i}K_j \rightarrow M$ may be used to conclude that $G_{p,i}(K_j) = nW_{p,i}(K_j, T_{p,i}K_j) \rightarrow nW_{p,i}(K_0, M)$. But by Proposition 3.16, $G_{p,i}(K_j) \rightarrow G_{p,i}(K_0)$. Hence, $G_{p,i}(K_0) = nW_{p,i}(K_0, M)$, and the uniqueness part of Proposition 3.11 shows that $T_{p,i}K_0 = M$.

Hence, every subsequence of the sequence $T_{p,i}K_j$ has a subsequence converging to $T_{p,i}K_0$. $\square$

4 The i th p -geominimal surface area ratio

In [19], Lutwak defined the p -geominimal area ratio of K by

$$\left(\frac{G_p(K)^n}{n^n V(K)^{n-p}} \right)^{1/p}.$$

For $K \in \mathcal{K}_o^n$, we define the i th p -geominimal area ratio of K as

$$\left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p},$$

and define the i th Santaló product of $K \in \mathcal{K}_o^n$ by $W_i(K) \tilde{W}_i(K^*)$.

The i th p -geominimal area ratio does not exceed the i th Santaló product divided by ω_n . To see this, just take $Q = K$ in the definition of i th p -geominimal surface area

$$\omega_n^{\frac{p}{n-i}} G_{p,i}(K) = \inf \left\{ nW_{p,i}(K, Q) \tilde{W}_i(Q^*)^{\frac{p}{n-i}} : Q \in \mathcal{K}_o^n \right\}$$

and get the following.

Proposition 4.1 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then*

$$\omega_n \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} \leq W_i(K) \tilde{W}_i(K^*).$$

An immediate consequence of Proposition 4.1 is as follows.

Theorem 4.2 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then*

$$G_{p,i}(K)^{n-i} \leq n^{n-i} \omega_n^{-p} W_i(K)^{n-p-i} \tilde{W}_i(K^*)^p. \quad (4.1)$$

Lemma 4.3 ([38]) *If $K \in \mathcal{K}_o^n$ and $i \in \{1, \dots, n-1\}$, then*

$$\tilde{W}_i(K) \leq W_i(K), \quad (4.2)$$

with equality if and only if K is an n -ball (centered at the origin).

Proof of Theorem 1.4 Inequality (4.1), together with (4.2), (2.3) and (2.1), yields

$$\begin{aligned} G_{p,i}(K)^{n-i} &\leq n^{n-i} \omega_n^{-p} W_i(K)^{n-p-i} [W_i(K) \tilde{W}_i(K^*)]^p \\ &\leq n^{n-i} \omega_n^{-p} W_i(K)^{n-p-i} [W_i(K) W_i(K^*)]^p \\ &= n^{n-i} \omega_n^{-p} W_i(K)^{n-p-i} \omega_i^{2p} [V_{n-i}(K) V_{n-i}(K^*)]^p / \binom{n}{i}^{2p} \\ &\leq n^{n-i} (\omega_n^{-1} \omega_i^2 \omega_{n-i}^2)^p \binom{n}{i}^{-2p} W_i(K)^{n-p-i}. \end{aligned}$$

According to the conditions of equality in inequalities (4.2) and (2.1), we know that for $i = 0$ equality of inequality (1.7) holds if and only if K is an ellipsoid for $1 \leq i < n$ if and only if all $(n-i)$ -dimensional convex bodies contained in K are balls. $\square$

Theorem 4.4 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{F}_{i,o}^n$, then*

$$G_{p,i}(K)^{n-i} \geq n^{n-i} \omega_n^{n-p-i} \tilde{W}_i(\Lambda_{p,i} K)^p,$$

with equality if and only if $K \in \mathcal{W}_{p,i}^n$.

Proof Since $G_{p,i}(K) = n W_{p,i}(K, T_{p,i} K)$ and $\tilde{W}_i(T_{p,i}^* K) = \omega_n$, Proposition 2.6 gives

$$G_{p,i}(K) = n \omega_n \tilde{W}_{-p,i}(\Lambda_{p,i} K, T_{p,i}^* K) / \tilde{W}_i(\Lambda_{p,i} K).$$

Apply inequality (2.11) and get

$$G_{p,i}(K)^{n-i} \geq n^{n-i} \omega_n^{n-p-i} \tilde{W}_i(\Lambda_{p,i} K)^p,$$

with equality if and only if $\Lambda_{p,i} K$ and $T_{p,i}^* K$ are dilates of each other. Since $T_{p,i} K \in \mathcal{K}_o^n$, equality implies that $\Lambda_{p,i} K \in \mathcal{K}_o^n$, and by Proposition 2.8 this means that $K \in \mathcal{W}_{p,i}^n$.

To see that if $K \in \mathcal{W}_{p,i}^n$, there is equality in the inequality of the theorem, combine Proposition 2.6 with the definition of $G_{p,i}(K)$ to get

$$\omega_n^{p/(n-i)} G_{p,i}(K) = \inf \{ n \omega_n \tilde{W}_{-p,i}(\Lambda_{p,i}K, Q^*) \tilde{W}_i(Q^*)^{p/(n-i)} / \tilde{W}_i(\Lambda_{p,i}K) : Q \in \mathcal{K}_o^n \}.$$

Since $K \in \mathcal{K}_o^n$, by Proposition 2.8, $\Lambda_{p,i}K \in \mathcal{K}_o^n$. Thus, $Q = \Lambda_{p,i}^*K$ gives $n \omega_n \tilde{W}_i(\Lambda_{p,i}K)^{p/(n-i)} \geq \omega_n^{p/(n-i)} G_{p,i}(K)$ and demonstrates the desired equality in the inequality. $\square$

An immediate consequence of Theorem 4.4 and Proposition 2.7 is Theorem 1.5.

Suppose $1 \leq p < q$ and $K \in \mathcal{K}_o^n$ with $L \in \mathcal{S}_o^n$. Since

$$\rho_L^{-p} h_K^{1-p} = [\rho_L^{-q} h_K^{1-q}]^{p/q} h_K^{(q-p)/q},$$

the Hölder inequality yields the following.

Proposition 4.5 *Suppose $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $i \in \{0, 1, \dots, n-1\}$. If $1 \leq p < q$, then*

$$\left(\frac{W_{p,i}(K, L^*)}{W_i(K)} \right)^{1/p} \leq \left(\frac{W_{q,i}(K, L^*)}{W_i(K)} \right)^{1/q},$$

with equality if and only if there exists $c > 0$ such that $\rho_L = c/h_K$ almost everywhere with respect to $S_i(K, \cdot)$.

Suppose $1 \leq p < q$, and $K \in \mathcal{K}_o^n$ with $L \in \mathcal{S}_o^n$. From the integral representation of $W_{p,i}(K, L^*)$, the easy estimate follows:

$$|W_{p,i}(K, L^*) - W_{q,i}(K, L^*)| \leq W_{p,i}(K, L^*) \max_{u \in S^{n-1}} |[\rho_L(u) h_K(u)]^{p-q} - 1|.$$

This gives the following proposition.

Proposition 4.6 *Suppose $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $i \in \{0, 1, \dots, n-1\}$. Then the function defined on $[1, \infty)$ by*

$$p \mapsto W_{p,i}(K, L^*)$$

is continuous.

The definition of $G_{p,i}(K)$ of $K \in \mathcal{K}_o^n$, by

$$\left(\frac{G_{p,i}(K)}{n W_i(K)} \right)^{1/p} = \inf \{ [W_{p,i}(K, Q^*) / W_i(K)]^{1/p} : Q \in \mathcal{K}_o^n, \tilde{W}_i(Q) = \omega_n \},$$

together with Proposition 4.5, shows that the i th p -geominimal area ratios are monotone non-decreasing in p .

Proposition 4.7 *If $K \in \mathcal{K}_o^n$ and $i \in \{0, 1, \dots, n-1\}$, then for $1 \leq p \leq q$,*

$$\left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} \leq \left(\frac{G_{q,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-q-i}} \right)^{1/q}.$$

The equality conditions for the inequality of Proposition 4.7 are given in Theorem 1.6. Proposition 4.7 provides a key step in showing the following.

Proposition 4.8 *If $K \in \mathcal{K}_o^n$ and $i \in \{0, 1, \dots, n-1\}$, then the function defined on $[1, \infty)$ by*

$$p \mapsto G_{p,i}(K)$$

is continuous.

Proof Proposition 4.7 shows that the function $\psi : [1, \infty) \rightarrow (0, \infty)$ defined by

$$\psi(p) = \left(\frac{G_{p,i}(K)}{nW_i(K)} \right)^{1/p}$$

is monotone. The continuity of $p \mapsto G_{p,i}(K)$ will be demonstrated by establishing the continuity of ψ .

Suppose $p_j \rightarrow p_0$. By Proposition 3.11, there exist $T_{p_j,i}K \in \mathcal{K}_0^n$ such that $\tilde{W}_i(T_{p_j,i}^*K) = \omega_n$ and

$$G_{p_j,i}(K) = nW_{p_j,i}(K, T_{p_j,i}K) \leq nW_{p_j,i}(K, B_n).$$

First assume that $p_j \geq p_0$ for all j . From the definition of i th p -geominimal surface area and Proposition 4.5, it follows that

$$\begin{aligned} \psi(p_0) &= \left(\frac{W_{p_0,i}(K, T_{p_0,i}K)}{W_i(K)} \right)^{1/p_0} \\ &\leq \left(\frac{W_{p_0,i}(K, T_{p_j,i}K)}{W_i(K)} \right)^{1/p_0} \\ &\leq \left(\frac{W_{p_j,i}(K, T_{p_j,i}K)}{W_i(K)} \right)^{1/p_j} \\ &= \psi(p_j) \\ &\leq \left(\frac{W_{p_j,i}(K, T_{p_0,i}K)}{W_i(K)} \right)^{1/p_j}. \end{aligned}$$

The continuity of the function $p \mapsto [W_{p,i}(K, T_{p_0,i}K)/W_i(K)]^{1/p}$ shows that

$$\left(\frac{W_{p_j,i}(K, T_{p_0,i}K)}{W_i(K)} \right)^{1/p_j} \rightarrow \left(\frac{W_{p_0,i}(K, T_{p_0,i}K)}{W_i(K)} \right)^{1/p_0},$$

and hence

$$\psi(p_j) \rightarrow \psi(p_0).$$

Now assume that $p_j \leq p_0$ for all j . That $\psi(p_j) \rightarrow \psi(p_0)$ will be proven by showing that every subsequence of $\psi(p_j)$ has a subsequence converging to $\psi(p_0)$. Let $\psi(p_j)$ denote a subsequence of $\psi(p_j)$.

Lemma 3.14 shows that $T_{p,i}K$ are uniformly bounded. Thus, the Blaschke selection theorem and Proposition 3.9 can now be used to deduce the existence of a subsequence of $T_{p,i}K$, which will also be denoted by $T_{p,i}K$, and a body $\bar{K} \in \mathcal{K}_o^n$, with $\tilde{W}_i(\bar{K}^*) = \omega_n$, such that $T_{p,i}K \rightarrow \bar{K}$. Obviously,

$$\begin{aligned} & \left| \left(\frac{W_{p,i}(K, T_{p,i}K)}{W_i(K)} \right)^{1/p_j} - \left(\frac{W_{p_0,i}(K, \bar{K})}{W_i(K)} \right)^{1/p_0} \right| \\ & \leq \left| \left(\frac{W_{p,i}(K, T_{p,i}K)}{W_i(K)} \right)^{1/p_j} - \left(\frac{W_{p,i}(K, \bar{K})}{W_i(K)} \right)^{1/p_j} \right| \\ & \quad + \left| \left(\frac{W_{p,i}(K, \bar{K})}{W_i(K)} \right)^{1/p_j} - \left(\frac{W_{p_0,i}(K, \bar{K})}{W_i(K)} \right)^{1/p_0} \right|. \end{aligned}$$

By Proposition 4.6 the second term of this sum tends to 0. By Proposition 2.2 the first term in this sum is bounded by

$$|h(T_{p,i}, \cdot) - h(\bar{K}, \cdot)|_{\infty} / r(K),$$

and since $T_{p,i}K \rightarrow \bar{K}$, this also tends to 0. Hence,

$$\psi(p_j) \rightarrow \left(\frac{W_{p_0,i}(K, \bar{K})}{W_i(K)} \right)^{1/p_0} \geq \left(\frac{W_{p_0,i}(K, T_{p_0,i}K)}{W_i(K)} \right)^{1/p_0} = \psi(p_0),$$

where the inequality is justified by the definition of i th p -geominimal surface area. But by Proposition 4.8, ψ is monotone non-decreasing, and hence $\psi(p_j) \rightarrow \psi(p_0)$. $\square$

For $K \in \mathcal{K}_o^n$, let $\sigma(K) \subset S^{n-1}$ denote the compact set that is the support of the i th surface area measure $S_i(K, \cdot)$ of K ; i.e., $\omega = S^{n-1} \setminus \sigma(K)$ is the largest open subset of S^{n-1} for which $S_i(K, \omega) = 0$. Let $\nu(K) \subset S^{n-1}$ denote the set of extreme normal directions of ∂K .

Lemma 4.9 Suppose $K \in \mathcal{K}_o^n$ and $c > 0$. If $h_{T_{p,i}K} = ch_K$ almost everywhere with respect to $S_i(K, \cdot)$, then $h_{T_{p,i}K} = ch_K$ everywhere.

Proof Since $h_{T_{p,i}K}$ and ch_K are continuous, and $h_{T_{p,i}K} = ch_K$ almost everywhere with respect to $S_i(K, \cdot)$, it follows that $h_{T_{p,i}K} = ch_K$ on $\sigma(K)$. But $\nu(K) \subset \sigma(K)$, and hence

$$h_{T_{p,i}K} = ch_K \quad \text{on } \nu(K).$$

But

$$\begin{aligned} T_{p,i}K & \subset \bigcap_{u \in \nu(K)} \{x \in \mathbb{R}^n : u \cdot x \leq h_{T_{p,i}K}(u)\} \\ & \subset \bigcap_{u \in \nu(K)} \{x \in \mathbb{R}^n : u \cdot x \leq h_{cK}(u)\} \\ & \subset cK \end{aligned}$$

shows that $K^* \subset cT_{p,i}^*K$. Since $\tilde{W}_i(T_{p,i}^*K) = \omega_n$, it follows that $\tilde{W}_i(K^*) \leq c^{n-i} \omega_n$ with equality if and only if $cK = T_{p,i}K$.

We now show that indeed there is equality in this inequality, and hence $cK = T_{p,i}K$. First note that since $h_{T_{p,i}K} = ch_K$ almost everywhere with respect to $S_i(K, \cdot)$, it follows that $h_{T_{p,i}K} = ch_K$ almost everywhere with respect to $S_{p,i}(K, \cdot)$. Hence, from the integral representation (2.4) it follows that

$$G_{p,i}(K) = nW_{p,i}(K, T_{p,i}K) = nW_{p,i}(K, cK) = c^p nW_i(K).$$

From this and the definition of i th p -geominimal surface area, it follows that

$$c^p nW_i(K) = G_{p,i}(K) \leq nW_{p,i}(K, K) (\tilde{W}_i(K^*)/\omega_n)^{p/(n-i)} = nW_i(K) (\tilde{W}_i(K^*)/\omega_n)^{p/(n-i)}.$$

Hence $\tilde{W}_i(K^*) \geq c^{n-i} \omega_n$. □

Proposition 4.1 states that the i th p -geominimal ratio is always dominated by the Santaló product (divided by ω_n); i.e., for $K \in \mathcal{K}_o^n$, $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$,

$$\left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} \leq W_i(K) \tilde{W}_i(K^*)/\omega_n.$$

The main result of this section is that in the limit (as $p \rightarrow \infty$) these two quantities are equal.

Theorem 4.10 *If $p \geq 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then*

$$\lim_{p \rightarrow \infty} \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} = \sup_{p \geq 1} \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} = W_i(K) \tilde{W}_i(K^*)/\omega_n.$$

Proof Since the first equality is an immediate consequence of Proposition 4.7, only the second equality needs to be demonstrated.

Proposition 3.11 guarantees the existence of $T_{p,i}K \in \mathcal{K}_o^n$ such that $\tilde{W}_i(T_{p,i}^*K) = \omega_n$ and $G_{p,i}(K) = nW_{p,i}(K, T_{p,i}K) \leq nW_{p,i}(K, B_n)$ for all p .

By Lemma 3.14 it follows that there exists $c > 0$ such that $T_{p,i}K \subset cB_n$ for all p . The Blaschke selection theorem and Proposition 3.9 may be used to deduce the existence of a subsequence of $T_{p,i}K$, which will also be denoted by $T_{p,i}K$, and a body $K_0 \in \mathcal{K}_o^n$ with $\tilde{W}_i(K_0^*) = \omega_n$ such that $T_{p,i}K \rightarrow K_0$ as $p \rightarrow \infty$.

Define $r(K, K_0) = \max\{\lambda > 0 : \lambda K_0 \subset K\}$. Now,

$$\begin{aligned} & \left| \left(\frac{W_{p,i}(K, T_{p,i}K)}{W_i(K)} \right)^{1/p} - \frac{1}{r(K, K_0)} \right| \\ & \leq \left| \left(\frac{W_{p,i}(K, T_{p,i}K)}{W_i(K)} \right)^{1/p} - \left(\frac{W_{p,i}(K, K_0)}{W_i(K)} \right)^{1/p} \right| \\ & \quad + \left| \left(\frac{W_{p,i}(K, K_0)}{W_i(K)} \right)^{1/p} - \frac{1}{r(K, K_0)} \right|. \end{aligned}$$

Proposition 2.2 shows that the first term in this sum is dominated by

$$|h(T_{p,i}K, \cdot) - h(K_0, \cdot)|_\infty / r(K),$$

and hence tends to 0 as $p \rightarrow \infty$. To see that the second term of this sum tends to 0, note that from (2.4) and (2.5) it follows that

$$\begin{aligned} & \lim_{p \rightarrow \infty} \left(\frac{W_{p,i}(K, K_0)}{W_i(K)} \right)^{1/p} \\ &= \lim_{p \rightarrow \infty} \left(\frac{1}{W_i(K)} \int_{S^{n-1}} \left(\frac{h_{K_0}(u)}{h_K(u)} \right)^p h_K(u) dS_i(K, u) \right)^{1/p} \\ &= \max_{u \in \sigma(K)} \frac{h_{K_0}(u)}{h_K(u)}, \end{aligned}$$

where the maximum is taken over the support of the measure $S_i(K, \cdot)$. But

$$\max_{u \in \sigma(K)} \frac{h_{K_0}(u)}{h_K(u)} = \frac{1}{r(K, K_0)}$$

is easily verified: First abbreviate $\max_{u \in \sigma(K)} [h_{K_0}(u)/h_K(u)] = c_0$. From the definition of $r(K, K_0)$, it follows that $h_{K_0}/h_K \leq 1/r(K, K_0)$ on S^{n-1} and hence $c_0 \leq 1/r(K, K_0)$. From the definition of c_0 , it follows that $h_{K_0} \leq c_0 h_K$ on $\sigma(K)$. Since $\nu(K)$, the set of extreme normal directions of ∂K is a subset of $\sigma(K)$ (see, e.g., Schneider [41]), $h_{K_0} \leq c_0 h_K$ on $\nu(K)$. Thus,

$$\begin{aligned} K_0 &\subset \bigcap_{u \in \nu(K)} \{x \in \mathbb{R}^n : u \cdot x \leq h_{K_0}(u)\} \\ &\subset \bigcap_{u \in \nu(K)} \{x \in \mathbb{R}^n : u \cdot x \leq c_0 h_K(u)\} \\ &\subset c_0 K. \end{aligned}$$

But $K_0 \subset c_0 K$ implies that $r(K, K_0) \geq 1/c_0$, and thus $c_0 = 1/r(K, K_0)$.

From the definition of $r(K, K_0)$ it follows that $r(K, K_0)K^* \subset K_0^*$ and hence

$$r(K, K_0) \tilde{W}_i(K^*)^{1/(n-i)} \leq \tilde{W}_i(K_0^*)^{1/(n-i)} = \omega_n^{1/(n-i)}.$$

Thus,

$$\lim_{p \rightarrow \infty} \left(\frac{G_{p,i}(K)}{n W_i(K)} \right)^{1/p} = \lim_{p \rightarrow \infty} \left(\frac{W_{p,i}(K, T_{p,i}K)}{W_i(K)} \right)^{1/p} = \frac{1}{r(K, K_0)} \geq \left(\frac{\tilde{W}_i(K^*)}{\omega_n} \right)^{1/(n-i)}.$$

But from Proposition 4.1 it follows that

$$\lim_{p \rightarrow \infty} \left(\frac{G_{p,i}(K)}{n W_i(K)} \right)^{1/p} \leq \left(\frac{\tilde{W}_i(K^*)}{\omega_n} \right)^{1/(n-i)},$$

and this completes the proof. $\square$

A body in $\mathcal{K}_o^n$ will be called i th p -selfminimal if $T_{p,i}K$ and K are dilates of each other. From this definition and Proposition 3.12 it follows that the class of i th p -selfminimal bodies is an orthogonal transformation invariant class of bodies.

The next proposition characterizes the i th p -selfminimal bodies as those bodies whose i th p -geominimal ratio is their i th Santaló product.

Proposition 4.11 *If $p \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_o^n$, then*

$$\omega_n \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} \leq W_i(K) \tilde{W}_i(K^*),$$

with equality if and only if K is i th p -selfminimal.

Proof The inequality is just Proposition 4.1. To obtain the equality conditions, first assume that K is i th p -selfminimal. Now, $T_{p,i}K = cK$ implies that

$$G_{p,i}(K) = nW_{p,i}(K, T_{p,i}K) = nW_{p,i}(K, cK) = nc^p W_i(K).$$

But $T_{p,i}K = cK$ implies that $cT_{p,i}^*K = K^*$, and since $\tilde{W}_i(T_{p,i}^*K) = \omega_n$, it follows that $c^{n-i} = \tilde{W}_i(K^*)/\omega_n$. This shows that there is equality in the inequality.

Suppose now that there is equality in the inequality of the proposition; *i.e.*,

$$G_{p,i}(K)^{n-i} = n^{n-i} [\tilde{W}_i(K^*)/\omega_n]^p W_i(K)^{n-i}.$$

Since $G_{p,i}(K) = nW_{p,i}(K, T_{p,i}K)$ and $W_i(K) = W_{p,i}(K, K)$, rewrite the last displayed identity as follows:

$$W_{p,i}(K, T_{p,i}K) = W_{p,i}(K, [\tilde{W}_i(K^*)/\omega_n]^{1/(n-i)} K).$$

Since the i th dual quermassintegrals of the polar body of $[\tilde{W}_i(K^*)/\omega_n]^{1/(n-i)} K$ are ω_n , it follows from the uniqueness of $T_{p,i}K$ that

$$T_{p,i}K = [\tilde{W}_i(K^*)/\omega_n]^{1/(n-i)} K,$$

and thus that K is i th p -selfminimal. □

An immediate consequence of Proposition 4.11 and Proposition 4.7 is the fact that an i th p -selfminimal body is i th q -selfminimal for all $q \geq p$.

An immediate consequence of Propositions 4.11 and 4.7 is that if $K \in \mathcal{K}_o^n$ is i th p -selfminimal, then

$$\left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-p-i}} \right)^{1/p} = \left(\frac{G_{q,i}(K)^{n-i}}{n^{n-i} W_i(K)^{n-q-i}} \right)^{1/q}$$

for all $q \geq p$.

As the next proposition shows, this property characterizes the i th p -selfminimal bodies among bodies in $\mathcal{K}_o^n$; *i.e.*, if for some $q > p$ the i th q -geominimal area ratio of a body in $\mathcal{K}_o^n$ is equal to its i th p -geominimal area ratio, then the body must be an i th p -selfminimal body (and thus in $\mathcal{K}_s^n$ denote the set of convex bodies having their Santaló point at the origin).

Proof of Theorem 1.6 In light of Proposition 4.7, only the equality conditions need to be established. Assume that there is equality in this inequality. From Propositions 3.11 and

4.5, and the definition of i th p -geominimal surface area, it follows that

$$\left(\frac{G_{q,i}(K)}{nW_i(K)} \right)^{1/q} = \left(\frac{W_{q,i}(K, T_{q,i}K)}{W_i(K)} \right)^{1/q} \geq \left(\frac{W_{p,i}(K, T_{q,i}K)}{W_i(K)} \right)^{1/p} \geq \left(\frac{G_{p,i}(K)}{nW_i(K)} \right)^{1/p}.$$

The hypothesis forces:

$$\left(\frac{G_{q,i}(K)}{nW_i(K)} \right)^{1/q} = \left(\frac{W_{q,i}(K, T_{q,i}K)}{W_i(K)} \right)^{1/q} = \left(\frac{W_{p,i}(K, T_{q,i}K)}{W_i(K)} \right)^{1/p} = \left(\frac{G_{p,i}(K)}{nW_i(K)} \right)^{1/p}.$$

The middle equality and the equality conditions of Proposition 4.5 yield $c > 0$ such that $h(T_{q,i}K, \cdot) = ch_K$ almost everywhere with respect to $S_i(K, \cdot)$. The right equality and the uniqueness of $T_{p,i}K$ show $T_{q,i}K = T_{p,i}K$. The fact that $T_{p,i}K = cK$ now follows from Lemma 4.9. $\square$

Theorem 1.6 contains a new characterization of selfminimal bodies as follows.

Proposition 4.12 *If $K \in \mathcal{K}_o^n$ and $i \in \{0, 1, \dots, n-1\}$, then for $p > 1$,*

$$\frac{G_i(K)^{n-i}}{n^{n-i}W_i(K)^{n-1-i}} \leq \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i}W_i(K)^{n-p-i}} \right)^{1/p},$$

with equality if and only if K is i th selfminimal.

When Proposition 4.7 is combined with Propositions 4.11 and 4.12, the result is that for $K \in \mathcal{K}_o^n$ and $1 \leq p \leq q$,

$$\frac{G_i(K)^{n-i}}{n^{n-i}W_i(K)^{n-1-i}} \leq \left(\frac{G_{p,i}(K)^{n-i}}{n^{n-i}W_i(K)^{n-p-i}} \right)^{1/p} \leq \left(\frac{G_{q,i}(K)^{n-i}}{n^{n-i}W_i(K)^{n-q-i}} \right)^{1/q} \leq W_i(K)\tilde{W}_i(K^*)/\omega_n.$$

Finally, we propose the following two open questions.

Conjecture 4.13 *Suppose $K \in \mathcal{K}_c^n$, $i \in \{0, 1, \dots, n-1\}$, and $p \geq 1$. Does it follow that*

$$G_{p,i}(K)^{n-i} \leq n^{n-i}\omega_n^p W_i(K)^{n-p-i}? \quad (4.3)$$

With equality in inequality for $i = 0$ if and only if K is an ellipsoid; for $0 < i \leq n-1$ if and only if K is a ball.

Obviously, the case $i = 0$ of Conjecture 4.13 is just the inequalities for the p -geominimal surface area by Lutwak (see [19]).

Conjecture 4.14 *Suppose $K \in \mathcal{K}_c^n$ and $i \in \{0, 1, \dots, n-1\}$. Does it follow that*

$$W_i(K)\tilde{W}_i(K^*) \leq \omega_n^2? \quad (4.4)$$

With equality in inequality for $i = 0$ if and only if K is an ellipsoid; for $0 < i \leq n-1$ if and only if K is a ball.

Obviously, the case $i = 0$ of Conjecture 4.14 is just the Blaschke-Santaló inequality (see [18, 28]).

Competing interests

The author declares that they have no competing interests.

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