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Best proximity point theorems for α-ψ-proximal contractions in intuitionistic fuzzy metric spaces

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Abstract
The aim of this paper is to introduce and study certain new concepts of α-ψ-proximal contractions in an intuitionistic fuzzy metric space. Then we establish certain best proximity point theorems for such proximal contractions in intuitionistic fuzzy metric spaces. As an application, we deduce best proximity and fixed point results in partially ordered intuitionistic fuzzy metric spaces. Several interesting consequences of our obtained results are presented in the form of new fixed point theorems which contain some recent fixed point theorems as special cases. Moreover, we discuss some illustrative examples to highlight the realized improvements.

MSC: 47H10; 54H25

Keywords: α-proximal admissible mapping; fuzzy α-ψ-proximal contractions; best proximity point; intuitionistic fuzzy ordered metric space

1 Introduction

Many problems arising in different areas of mathematics, such as optimization, variational analysis, and differential equations, can be modeled as fixed point equations of the form $Tx = x$. If T is not a self-mapping, the equation $Tx = x$ could have no solutions and, in this case, it is of a certain interest to determine an element x that is in some sense closest to Tx. Fan's best approximation theorem [1] asserts that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X and $T : K \to X$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Tx) : y \in K\}$, where d is a metric on X.

A best approximation theorem guarantees the existence of an approximate solution, a best proximity point theorem is contemplated for solving the problem to find an approximate solution which is optimal. Given the nonempty closed subsets A and B of X, when a non-self-mapping $T : A \to B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. Best proximity point theorems provide the existence of an element x^* such that $d(x^*, Tx^*) = d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$; this element is called a best proximity point of T. Moreover, if the mapping under consideration is a self-mapping, we note that this best proximity theorem reduces to a fixed point. For more details, we refer to [2–6] and references therein.

The concept of fuzzy set was introduced by Zadeh [7] in 1965 and it is well known that there are many viewpoints of the notion of metric space in fuzzy topology. In 1975, Kramosil and Michálek [8] introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space. Clearly, this work provides an important basis for the construction of fixed point theory in fuzzy metric spaces. Afterwards, Grabiec [9] defined the completeness of the fuzzy metric space (now known as a G-complete fuzzy metric space) and extended the Banach contraction theorem to G-complete fuzzy metric spaces. Subsequently, George and Veeramani [10] modified the definition of the Cauchy sequence introduced by Grabiec. Meanwhile, they slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek and then defined a Hausdorff and first countable topology. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani (now known as an complete fuzzy metric space) has emerged as another characterization of completeness, and some fixed point theorems have also been constructed on the basis of this metric space. From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces; see [11–22] and the references therein. On the other hand the concept of intuitionistic fuzzy set was introduced by Atanassov [23] as generalization of fuzzy set. In 2004, Park introduced the notion of intuitionistic fuzzy metric space [24]. He showed that for each intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, the topology generated by the intuitionistic fuzzy metric (M, N) coincides with the topology generated by the fuzzy metric M. For more details on intuitionistic fuzzy metric space and related results we refer the reader to [24–31].

2 Mathematical preliminaries

Definition 1 A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2 A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:

(a) $\diamond$ is commutative and associative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Examples of a t-conorm are $a \diamond b = \max\{a, b\}$ and $a \diamond b = \min\{1, a + b\}$.

Definition 3 A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) + N(x, y, t) \leq 1$;
(ii) $M(x, y, 0) = 0$;
(iii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$;

(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(vi) $M(x,y,\cdot):(0,\infty) \to [0,1]$ is left continuous;
(vii) $\lim_{t\to\infty} M(x,y,t) = 1$;
(viii) $N(x,y,0) = 1$;
(ix) $N(x,y,t) = 0$ if and only if $x = y$;
(x) $N(x,y,t) = N(y,x,t)$;
(xi) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$;
(xii) $N(x,y,\cdot):(0,\infty) \to [0,1]$ is right continuous;
(xiii) $\lim_{t\to\infty} N(x,y,t) = 0$.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1 Note that, if (M,N) is an intuitionistic fuzzy metric on X and $\{x_n\}$ be a sequence in X such that $\lim_{m,n\to\infty} M(x_n,x_m,t) = 1$, then $\lim_{m,n\to\infty} N(x_n,x_m,t) = 0$. Indeed, from (i) of Definition 3 we know that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x,y \in X$ and all $t > 0$.

Definition 4 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Then

- a sequence $\{x_n\}$ is said to be Cauchy sequence whenever $\lim_{m,n\to\infty} M(x_n,x_m,t) = 1$ and $\lim_{m,n\to\infty} N(x_n,x_m,t) = 0$ for all $t > 0$;
- a sequence $\{x_n\}$ is said to converge $x \in X$, if $\lim_{m,n\to\infty} M(x_n,x,t) = 1$ and $\lim_{m,n\to\infty} N(x_n,x,t) = 0$ for all $t > 0$;
- $(X,M,N,*,\diamond)$ is called complete whenever every Cauchy sequence is convergent in X.

Definition 5 [28] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. We say the mapping $T:X\to X$ is t-uniformly continuous if for each $0<\epsilon<1$, there exists $0<\delta<1$, such that $M(x,y,t) \geq 1-\delta$ and $N(x,y,t) \leq \delta$ implies $M(Tx,Ty,t) \geq 1-\epsilon$ and $N(Tx,Ty,t) \leq \epsilon$ for all $x,y \in X$ and for all $t > 0$.

Lemma 1 [32] *Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space and T be a t-uniformly continuous mapping on X. If $x_n \to x$ as $n\to\infty$, then $Tx_n \to Tx$ as $n \to \infty$.*

Lemma 2 [32] *Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. If $x_n \to x$ and $y_n \to y$ as $n\to\infty$, then $M(x_n,y_n,t) \to M(x,y,t)$ and $N(x_n,y_n,t) \to N(x,y,t)$, $n\to\infty$, for all $t > 0$.*

Definition 6 [27, 33] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. The fuzzy metric (M,N) is called triangular whenever,

$$\frac{1}{M(x,y,t)} - 1 \leq \frac{1}{M(x,z,t)} - 1 + \frac{1}{M(z,y,t)} - 1$$

and

$$N(x,y,t) \leq N(x,z,t) + N(z,y,t)$$

for all $x,y,z \in X$ and all $t > 0$.

On the other hand, Samet *et al.* [34] defined the notion of α-admissible mappings as follows.

Definition 7 Let T be a self-mapping on X and $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if

$$x,y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1.$$

Salimi *et al.* [35] generalized the notion of α-admissible mappings in the following ways.

Definition 8 [35] Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x,y \in X, \quad \alpha(x,y) \ge \eta(x,y) \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge \eta(Tx,Ty).$$

Note that if we take $\eta(x,y) = 1$ then this definition reduces to Definition 7. Also, if we take, $\alpha(x,y) = 1$ then we say that T is an η-subadmissible mapping.

Definition 9 [5] A non-self-mapping $T : A \to B$ is called α-η-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \ge \eta(x_1,x_2), \\ d(u_1,Tx_1) = d(A,B), \\ d(u_2,Tx_2) = d(A,B) \end{cases} \quad \Longrightarrow \quad \alpha(u_1,u_2) \ge \eta(u_1,u_2)$$

for all $x_1,x_2,u_1,u_2 \in A$, where $\alpha, \eta : A \times A \to [0,\infty)$. Also, if we take $\eta(x,y) = 1$ for all $x,y \in A$ then we say T is an α-proximal admissible mapping.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

3 Main results

In [34] the authors consider the family Ψ of non-decreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Let A and B be nonempty subsets of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$. We denote by $A_0(t)$ and $B_0(t)$ the following sets:

$$\begin{aligned} A_0(t) &= \bigl\{x \in A : M(x,y,t) = M(A,B,t) \text{ for some } y \in B\bigr\}, \\ B_0(t) &= \bigl\{y \in B : M(x,y,t) = M(A,B,t) \text{ for some } x \in A\bigr\}, \end{aligned} \tag{3.1}$$

where $M(A,B,t) = \sup\{M(x,y,t) : x \in A, y \in B\}$.

Definition 10 Let A and B be two nonempty subsets of intuitionistic fuzzy metric spaces $(X,M,N,*,\diamond)$. Let, $T : A \to B$, $\alpha : A \times A \times (0,\infty) \to [0,\infty)$. We say that T is α-proximal admissible if for $x_1,x_2,u_1,u_2 \in A$ with

$$\begin{cases} \alpha(x_1,x_2,t) \ge t, \\ M(u_1,Tx_1,t) = M(A,B,t), \\ M(u_2,Tx_2,t) = M(A,B,t) \end{cases} \quad \text{we have} \quad \alpha(u_1,u_2,t) \ge t$$

for all $t > 0$.

Let A and B be nonempty subsets of an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ and $T:A\to B$ be a non-self-mapping. We define $\mathcal{M}^T(x,y,u,v,t)$ and $\mathcal{N}^T(x,y,u,v,t)$ as follows:

$$\mathcal{M}^T(x,y,u,v,t)=\max\left\{\frac{1}{M(x,y,t)},\frac{1}{2}\left[\frac{1}{M(x,u,t)}+\frac{1}{M(y,v,t)}\right],\right.$$
$$\left.\frac{1}{2}\left[\frac{1}{M(x,v,t)}+\frac{1}{M(y,u,t)}-1\right]\right\}$$

and

$$\mathcal{N}^T(x,y,u,v,t)=\max\{M(x,u,t),M(y,v,t),M(x,v,t),M(y,u,t)\}.$$

Definition 11 Let A and B be nonempty subsets of an intuitionistic fuzzy metric spaces $(X,M,N,*,\diamond)$. Let $T:A\to B$ be a non-self-mapping and $\alpha:A\times A\times(0,\infty)\to[0,\infty)$ be a function. We say T is a α-ψ-proximal contractive mapping if for $x,y,u,v\in A$,

$$\left.\begin{array}{l}\alpha(x,y,t)\geq t,\\ M(u,Tx,t)=M(A,B,t),\\ M(v,Ty,t)=M(A,B,t)\end{array}\right\}$$
$$\Longrightarrow\quad\frac{1}{M(u,v,t)}-1\leq\psi\bigl(\mathcal{M}^T(x,y,u,v,t)-\mathcal{N}^T(x,y,u,v,t)\bigr)\tag{3.2}$$

holds for all $t>0$, where $\psi\in\Psi$.

Theorem 1 *Let A and B be nonempty subsets of a complete triangular intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ such that $A_0(t)$ is nonempty for all $t>0$. Let $T:A\to B$ be a t-uniformly continuous non-self-mapping satisfying the following assertions*:

(i) *T is an α-proximal admissible mapping and $T(A_0(t))\subseteq B_0(t)$ for all $t>0$;*
(ii) *T is a α-ψ-proximal contractive mapping;*
(iii) *for any sequence $\{y_n\}$ in $B_0(t)$ and $x\in A$ satisfying $M(x,y_n,t)\to M(A,B,t)$ as $n\to+\infty$, then $x\in A_0(t)$ for all $t>0$;*
(iv) *there exist elements x_0 and x_1 in $A_0(t)$ such that*

$$M(x_1,Tx_0,t)=M(A,B,t)\quad\textit{and}\quad\alpha(x_0,x_1,t)\geq t\quad\textit{for all } t>0.$$

Then there exists $x^*\in A$ such that $M(x^*,Tx^*,t)=M(A,B,t)$, for all $t>0$, that is, T has a best proximity point $x^*\in A$.

(v) *Moreover, if $M(x,Tx,t)=M(A,B,t)$, $M(y,Ty,t)=M(A,B,t)$ implies $\alpha(x,y,t)\geq t$ for all $t>0$, then T has a unique best proximity point.*

Proof By condition (iv) there exist elements x_0 and x_1 in $A_0(t)$ such that

$$M(x_1,Tx_0,t)=M(A,B,t)\quad\text{and}\quad\alpha(x_0,x_1,t)\geq t\quad\text{for all } t>0.$$

On the other hand $T(A_0(t))\subseteq B_0(t)$, so there exists $x_2\in A_0(t)$ such that

$$M(x_2,Tx_1,t)=M(A,B,t).$$

Now, since T is α-proximal admissible mapping, so we have $\alpha(x_1, x_2, t) \geq t$. That is,

$$M(x_2, Tx_1, t) = M(A, B, t), \qquad \alpha(x_1, x_2, t) \geq t.$$

Again, since $T(A_0(t)) \subseteq B_0(t)$, there exists $x_3 \in A_0(t)$ such that

$$M(x_3, Tx_2, t) = M(A, B, t).$$

Thus we have

$$M(x_2, Tx_1, t) = M(A, B, t), \qquad M(x_3, Tx_2, t) = M(A, B, t), \qquad \alpha(x_1, x_2, t) \geq t.$$

Again since T is α-proximal admissible mapping, so $\alpha(x_2, x_3, t) \geq t$. Hence,

$$M(x_3, Tx_2, t) = M(A, B, t), \qquad \alpha(x_2, x_3, t) \geq t.$$

Continuing this process, we get

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \qquad \alpha(x_n, x_{n+1}, t) \geq 1 \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$.

Now from (3.2) with $u = y = x_n$, $v = x_{n+1}$ and $x = x_{n-1}$, we get

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \leq \psi\bigl(\mathcal{M}^T(x_{n-1}, x_n, x_n, x_{n+1}, t) - \mathcal{N}^T(x_{n-1}, x_n, x_n, x_{n+1}, t)\bigr) \tag{3.4}$$

for all $t > 0$ and all $n \in \mathbb{N}$ where

$$\begin{aligned}
&\mathcal{M}^T(x_{n-1}, x_n, x_n, x_{n+1}, t) \\
&\quad = \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right],\right. \\
&\qquad \left.\frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_{n+1}, t)} + \frac{1}{M(x_n, x_n, t)} - 1\right]\right\} \\
&\quad = \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right], \frac{1}{2M(x_{n-1}, x_{n+1}, t)}\right\} \\
&\quad \leq \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right],\right. \\
&\qquad \left.\frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} - 1 + \frac{1}{M(x_n, x_{n+1}, t)} - 1\right] + \frac{1}{2}\right\} \\
&\quad \leq \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right],\right. \\
&\qquad \left.\frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right] - \frac{1}{2}\right\} \\
&\quad = \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{2}\left[\frac{1}{M(x_{n-1}, x_n, t)} + \frac{1}{M(x_n, x_{n+1}, t)}\right]\right\} \\
&\quad \leq \max\left\{\frac{1}{M(x_{n-1}, x_n, t)}, \frac{1}{M(x_n, x_{n+1}, t)}\right\}.
\end{aligned}$$

This implies

$$\mathcal{M}^T(x_{n-1},x_n,x_n,x_{n+1},t) \le \max\left\{\frac{1}{M(x_{n-1},x_n,t)},\frac{1}{M(x_n,x_{n+1},t)}\right\}. \tag{3.5}$$

Also we have

$$\begin{aligned}&\mathcal{N}^T(x_{n-1},x_n,x_n,x_{n+1},t)\\ &\quad= \max\bigl\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t),M(x_{n-1},x_{n+1},t),M(x_n,x_n,t)\bigr\}\\ &\quad= \max\bigl\{M(x_{n-1},x_n,t),M(x_n,x_{n+1},t),M(x_{n-1},x_{n+1},t),1\bigr\} = 1.\end{aligned} \tag{3.6}$$

Thus, from (3.4), (3.5), and (3.6) we have

$$\begin{aligned}\frac{1}{M(x_n,x_{n+1},t)} - 1 &\le \psi\bigl(\mathcal{M}^T(x_{n-1},x_n,x_n,x_{n+1},t) - \mathcal{N}^T(x_{n-1},x_n,x_n,x_{n+1},t)\bigr)\\ &\le \psi\left(\max\left\{\frac{1}{M(x_{n-1},x_n,t)},\frac{1}{M(x_n,x_{n+1},t)}\right\} - 1\right).\end{aligned}$$

Now if $\max\{\frac{1}{M(x_{n-1},x_n,t)},\frac{1}{M(x_n,x_{n+1},t)}\} = \frac{1}{M(x_n,x_{n+1},t)}$, then we get

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \psi\left(\frac{1}{M(x_n,x_{n+1},t)} - 1\right) < \frac{1}{M(x_n,x_{n+1},t)} - 1,$$

which is a contradiction. Hence,

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \psi\left(\frac{1}{M(x_{n-1},x_n,t)} - 1\right)$$

for all $n \in \mathbb{N}$ and $t > 0$. So we deduce

$$\frac{1}{M(x_n,x_{n+1},t)} - 1 \le \psi^n\left(\frac{1}{M(x_0,x_1,t)} - 1\right)$$

for all $n \in \mathbb{N}$ and $t > 0$. Fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\sum_{n\ge N}\psi^n\left(\frac{1}{M(x_0,x_1,t)} - 1\right) < \epsilon.$$

Let $m,n \in \mathbb{N}$ with $m > n \ge N$. Then by triangular inequality we get

$$\frac{1}{M(x_n,x_m,t)} - 1 \le \sum_{k=n}^{m-1}\left[\frac{1}{M(x_k,x_{k+1},t)} - 1\right] \le \sum_{n\ge N}\psi^n\left(\frac{1}{M(x_0,x_1,t)} - 1\right) < \epsilon.$$

Consequently, $\lim_{m,n\to\infty}[\frac{1}{M(x_n,x_m,t)} - 1] = 0$, *i.e.*, $\lim_{m,n\to\infty} M(x_n,x_m,t) = 1$. Hence $\{x_n\}$ is a Cauchy sequence. Now, since $(X,M,N,*,\diamond)$ is a complete intuitionistic fuzzy metric space, so there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is t-uniformly continuous, so by Lemmas 1 and 2, we have

$$M\bigl(x^*, Tx^*, t\bigr) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(A,B,t).$$

That is, x^* is a best proximity of T. We show that x^* is unique best proximity point of T. Assume, to the contrary, that there exists $t_0 > 0$ such that $0 < M(x^*, w, t_0) < 1$ and $w \neq x^*$ is another best proximity point of T, that is, $M(x^*, Tx^*, t) = M(A, B, t)$ and $M(w, Tw, t) = M(A, B, t)$ for all $t > 0$. Now if condition (v) holds, then, from (3.2), we have

$$\frac{1}{M(x^*, w, t_0)} - 1 \le \psi\big(\mathcal{M}^T\big(x^*, w, x^*, w, t_0\big) - \mathcal{N}^T\big(x^*, w, x^*, w, t_0\big)\big),$$

where

$$\begin{aligned}\mathcal{M}^T\big(x^*, w, x^*, w, t_0\big) &= \max\left\{\frac{1}{M(x^*, w, t_0)}, \frac{1}{2}\left[\frac{1}{M(x^*, x^*, t_0)} + \frac{1}{M(w, w, t_0)}\right],\right.\\ &\qquad \left.\frac{1}{2}\left[\frac{1}{M(x^*, w, t_0)} + \frac{1}{M(w, x^*, t_0)} - 1\right]\right\}\\ &= \max\left\{\frac{1}{M(x^*, w, t_0)}, 1, \frac{1}{M(x^*, w, t_0)} - \frac{1}{2}\right\}\\ &= \frac{1}{M(x^*, w, t_0)}\end{aligned}$$

and

$$\mathcal{N}^T\big(x^*, w, x^*, w, t_0\big) = \max\big\{M\big(x^*, x^*, t_0\big), M(w, w, t_0), M\big(x^*, w, t_0\big), M\big(w, x^*, t_0\big)\big\} = 1.$$

Therefore,

$$\frac{1}{M(x^*, w, t_0)} - 1 \le \psi\left(\frac{1}{M(x^*, w, t_0)} - 1\right) < \frac{1}{M(x^*, w, t_0)} - 1,$$

which is a contradiction. Hence, $M(x^*, w, t_0) = 1$ for all $t > 0$. *i.e.*, $x^* = w$. Thus T has unique best proximity point. □

Theorem 2 *Let A and B be nonempty subsets of a complete triangular intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \to B$ be a non-self-mapping satisfying the following assertions*:

(i) *T is an α-proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;*
(ii) *T is a α-ψ-proximal contractive mapping such that ψ is continuous*;
(iii) *for any sequence $\{y_n\}$ in $B_0(t)$ and $x \in A$ satisfying $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0(t)$ for all $t > 0$;*
(iv) *there exist elements x_0 and x_1 in $A_0(t)$ such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all } t > 0 \quad \text{and} \quad \alpha(x_0, x_1, t) \ge t;$$

(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \ge t$ for all $t > 0$ and n with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x, t) \ge t$ for all $t > 0$ and all n.*

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$, for all $t > 0$, that is, T has a best proximity point $x^* \in A$.

(vi) *Moreover, if $M(x, Tx, t) = M(A, B, t)$, $M(y, Ty, t) = M(A, B, t)$ implies $\alpha(x, y, t) \ge t$ for all $t > 0$, then T has a unique best proximity point.*

Proof Following the same lines in the proof of Theorem 1, we can construct a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \qquad \alpha(x_n, x_{n+1}, t) \geq t \quad \text{for all } n \in \mathbb{N} \tag{3.7}$$

and $x_n \to x^*$ as $n \to \infty$, that is, $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$, for all $t > 0$. Moreover,

$$\begin{aligned} M(A, B, t) &= M(x_{n+1}, Tx_n, t) \\ &\geq M\big(x_{n+1}, x^*, t\big) \star M\big(x^*, Tx_n, t\big) \\ &\geq M\big(x_{n+1}, x^*, t\big) \star M\big(x^*, x_{n+1}, t\big) \star M(x_{n+1}, Tx_n, t) \\ &= M\big(x_{n+1}, x^*, t\big) \star M\big(x^*, x_{n+1}, t\big) \star M(A, B, t). \end{aligned}$$

This implies

$$\begin{aligned} M(A, B, t) &\geq M\big(x_{n+1}, x^*, t\big) \star M\big(x^*, Tx_n, t\big) \\ &\geq M\big(x_{n+1}, x^*, t\big) \star M\big(x^*, x_{n+1}, t\big) \star M(A, B, t). \end{aligned}$$

Passing to the limit as $n \to +\infty$ in the above inequality, we get

$$M(A, B, t) \geq 1 \star \lim_{n\to+\infty} M\big(x^*, Tx_n, t\big) \geq 1 \star 1 \star M(A, B, t),$$

that is,

$$\lim_{n\to+\infty} M\big(x^*, Tx_n, t\big) = M(A, B, t)$$

and so, by condition (iii), $x^* \in A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, then there exists $z \in A_0(t)$ such that $M(z, Tx^*, t) = M(A, B, t)$. Also from (iv) we have $\alpha(x_n, x^*, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$.

Suppose there exists $t_0 > 0$ such that $M(x^*, z, t_0) < 1$. Then from (3.2) with $x = x_n$, $y = x^*$, $u = x_{n+1}$, and $v = z$ we get

$$\frac{1}{M(x_{n+1}, z, t_0)} - 1 \leq \psi\big(\mathcal{M}^T\big(x_n, x^*, x_{n+1}, z, t_0\big) - \mathcal{N}^T\big(x_n, x^*, x_{n+1}, z, t_0\big)\big). \tag{3.8}$$

On the other hand we know that

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{M}^T\big(x_n, x^*, x_{n+1}, z, t_0\big) \\ &\quad = \lim_{n\to\infty}\bigg(\max\bigg\{\frac{1}{M(x_n, x^*, t_0)}, \frac{1}{2}\bigg[\frac{1}{M(x_n, x_{n+1}, t_0)} + \frac{1}{M(x^*, z, t_0)}\bigg], \\ &\quad\quad \frac{1}{2}\bigg[\frac{1}{M(x_n, z, t_0)} + \frac{1}{M(x^*, x_{n+1}, t_0)} - 1\bigg]\bigg\}\bigg) \\ &\quad = \max\bigg\{\frac{1}{M(x^*, x^*, t_0)}, \frac{1}{2}\bigg[\frac{1}{M(x^*, x^*, t_0)} + \frac{1}{M(x^*, z, t_0)}\bigg], \\ &\quad\quad \frac{1}{2}\bigg[\frac{1}{M(x^*, z, t_0)} + \frac{1}{M(x^*, x^*, t_0)} - 1\bigg]\bigg\} \end{aligned}$$

$$= \max\left\{1, \frac{1}{2}\left[1 + \frac{1}{M(x^*, z, t_0)}\right], \frac{1}{2}\left[\frac{1}{M(x^*, z, t_0)}\right]\right\}$$
$$= \frac{1}{2}\left[1 + \frac{1}{M(x^*, z, t_0)}\right]$$

and

$$\lim_{n\to\infty} \mathcal{N}^T(x_n, x^*, x_{n+1}, z, t_0)$$
$$= \lim_{n\to\infty} \max\{M(x_n, x_{n+1}, t_0), M(x^*, z, t), M(x_n, z, t_0), M(x^*, x_{n+1}, t_0)\}$$
$$= \max\{1, M(x^*, z, t_0), M(x^*, z, t_0), 1\} = 1.$$

Now by taking the limit as $n \to \infty$ in (3.8) we get

$$\frac{1}{M(x^*, z, t_0)} - 1 \le \psi\left(\frac{1}{2}\left[1 + \frac{1}{M(x^*, z, t_0)}\right] - 1\right) < \frac{1}{2}\left[1 + \frac{1}{M(x^*, z, t_0)}\right] - 1,$$

which implies $\frac{1}{2M(x^*,z,t_0)} < \frac{1}{2}$, *i.e.*, $M(x^*, z, t_0) > 1$, which is a contradiction. Hence, $M(x^*, z, t) = 1$ for all $t > 0$. So, $x^* = z$. Therefore, T has a best proximity point. □

Example 1 Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$. Consider $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Moreover, consider $A = (-\infty, -1]$, $B = [1, +\infty)$ and define $T : A \to B$ by

$$Tx = \begin{cases} -x^3 + 2, & \text{if } x \in (-\infty, -14), \\ 2x^4 + 5, & \text{if } x \in [-14, -12), \\ 4x^4 + 5, & \text{if } x \in [-12, -10), \\ -x^5 + 6, & \text{if } x \in [-10, -8), \\ 10, & \text{if } x \in [-8, -6), \\ |x^3| + 1, & \text{if } x \in [-6, -4), \\ -x + |(x + 3)(x + 4)|, & \text{if } x \in [-4, -2), \\ 1, & \text{if } x \in [-2, -1]. \end{cases}$$

Also, define $\alpha : X \times X \times (0, \infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 2t, & \text{if } x, y \in [-2, -1], \\ \frac{1}{2}t, & \text{otherwise,} \end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{1}{2}t \quad \text{for all } t \ge 0.$$

Clearly, $M(A, B, t) = \sup\{M(x, y, t)\ x \in A, y \in B\} = \frac{t}{t+2}$. Hence,

$$A_0(t) = \left\{x \in A : M(x, y, t) = M(A, B, t) = \frac{t}{t + 2} \text{ for some } y \in B\right\} = \{-1\},$$
$$B_0(t) = \left\{y \in B : M(x, y, t) = M(A, B, t) = \frac{t}{t + 2} \text{ for some } x \in A\right\} = \{1\}.$$

It is immediate to show that $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$, $M(-1, T(-1), t) = M(A, B, t)$ and $\alpha(-1, -1, t) \geq t$. Suppose

$$\begin{cases} \alpha(x, y, t) \geq t, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t), \end{cases}$$

then

$$\begin{cases} x, y \in [-2, -1], \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Hence, $u = v = -1$, that is, $\alpha(u, v, t) \geq t$. Therefore T is an α-proximal admissible mapping. Further,

$$\frac{1}{M(u, v, t)} - 1 = 0 \leq \psi\left(\mathcal{M}^T(x, y, u, v, t) - \mathcal{N}^T(x, y, u, v, t)\right),$$

that is, T is an α-ψ-proximal contractive mapping. Moreover, if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$ such that $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq [-2, -1]$ and hence $x \in [-2, -1]$. Consequently, $\alpha(x_n, x, t) \geq t$ for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$. Therefore all the conditions of Theorem 2 hold and T has a unique best proximity point. Here $z = -1$ is the best proximity point of T.

Theorem 3 *Let A and B be nonempty subsets of a complete triangular intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \to B$ be a t-uniformly continuous non-self-mapping. Assume that following assertions hold true*:

(i) *T is an α-proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$;*

(ii) *for $x, y, u, v \in A$,*

$$\left.\begin{aligned} &\alpha(x, y, t) \geq t, \\ &M(u, Tx, t) = M(A, B, t), \\ &M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \implies \frac{1}{M(u, v, t)} - 1 \leq \left(\frac{\frac{1}{M(x,v,t)} - 1 + \frac{1}{M(y,u,t)} - 1}{\frac{1}{M(x,v,t)} - 1 + \frac{1}{M(y,u,t)} - 1 + \frac{1}{t}}\right)\left(\frac{1}{M(x, y, t)} - 1\right) \quad (3.9)$$

holds for all $t > 0$;

(iii) *for any sequence $\{y_n\}$ in $B_0(t)$ and $x \in A$ satisfying $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0(t)$ for all $t > 0$;*

(iv) *there exist elements x_0 and x_1 in $A_0(t)$ such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{and} \quad \alpha(x_0, x_1, t) \geq t \quad \text{for all } t > 0.$$

Then there exists $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$, for all $t > 0$, that is, T has a best proximity point $x^* \in A$.

(v) *Moreover, if $M(x, Tx, t) = M(A, B, t)$, $M(y, Ty, t) = M(A, B, t)$ implies $\alpha(x, y, t) \geq t$ for all $t > 0$, then T has a unique best proximity point.*

Proof Following the same lines in the proof of Theorem 1, we can construct a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \qquad \alpha(x_n, x_{n+1}, t) \geq t \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

From (ii) with $u = y = x_n$, $v = x_{n+1}$ and $x = x_{n-1}$, we get

$$\begin{aligned}
&\frac{1}{M(x_n, x_{n+1}, t)} - 1 \\
&\quad \leq \left(\frac{\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 + \frac{1}{M(x_n, x_n, t)} - 1}{\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 + \frac{1}{M(x_n, x_n, t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \\
&\quad = \left(\frac{\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1}{\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \\
&\quad \leq \left(\frac{\frac{1}{M(x_{n-1}, x_n, t)} - 1 + \frac{1}{M(x_n, x_{n+1}, t)} - 1}{\frac{1}{M(x_{n-1}, x_n, t)} - 1 + \frac{1}{M(x_n, x_{n+1}, t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right).
\end{aligned} \tag{3.11}$$

As in the proof of Theorem 2.2 of [27], we deduce that $\{x_n\}$ is a Cauchy sequence. The completeness of $(X, M, N, *, \diamond)$ ensures that the sequence $\{x_n\}$ converges to some $x^* \in X$, that is $\lim_{n\to+\infty} M(x_n, x^*, t) = 1$. Since T is t-uniformly continuous, so by Lemmas 1 and 2, we have

$$M\big(x^*, Tx^*, t\big) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(A, B, t).$$

That is, x^* is a best proximity of T. Now we show that x^* is unique best proximity point of T. Suppose, to the contrary, that there exists $t_0 > 0$ such that $0 < M(x^*, w, t_0) < 1$ and $w \neq x^*$ is another best proximity point of T, that is, $M(x^*, Tx^*, t) = M(A, B, t)$ and $M(w, Tw, t) = M(A, B, t)$ for all $t > 0$. Now if condition (v) holds, then, from (ii), we have

$$\begin{aligned}
\frac{1}{M(x^*, w, t)} - 1 &\leq \left(\frac{\frac{1}{M(x^*, w, t)} - 1 + \frac{1}{M(w, x^*, t)} - 1}{\frac{1}{M(x^*, w, t)} - 1 + \frac{1}{M(w, x^*, t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x^*, w, t)} - 1 \right) \\
&< \frac{1}{M(x^*, w, t)} - 1,
\end{aligned}$$

which is a contradiction. Hence, $w = x^*$. That is, T has a unique best proximity point. □

Theorem 4 *Let A and B be nonempty subsets of a complete triangular intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ such that $A_0(t)$ is nonempty for all $t > 0$. Let $T : A \to B$ be a non-self-mapping. Assume that the following assertions hold true*:

- (i) *T is an α-proximal admissible mapping and $T(A_0(t)) \subseteq B_0(t)$ for all $t > 0$*;
- (ii) (3.9) *holds for all $t > 0$*;
- (iii) *for any sequence $\{y_n\}$ in $B_0(t)$ and $x \in A$ satisfying $M(x, y_n, t) \to M(A, B, t)$ as $n \to +\infty$, then $x \in A_0(t)$ for all $t > 0$*;
- (iv) *there exist elements x_0 and x_1 in $A_0(t)$ such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all } t > 0 \quad \text{and} \quad \alpha(x_0, x_1, t) \geq t;$$

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq t$ *for all* n *and all* $t > 0$ *such that* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x, t) \geq t$ *for all* n *and all* $t > 0$.

Then there exists $x^* \in A$ *such that* $M(x^*, Tx^*, t) = M(A, B, t)$, *for all* $t > 0$, *that is,* T *has a best proximity point* $x^* \in A$.

(vi) *Moreover, if* $M(x, Tx, t) = M(A, B, t)$, $M(y, Ty, t) = M(A, B, t)$ *imply* $\alpha(x, y, t) \geq t$ *for all* $t > 0$, *then* T *has a unique best proximity point.*

Proof Following the same lines in the proof of Theorem 3, we can construct a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \qquad \alpha(x_n, x_{n+1}, t) \geq t \quad \text{for all } n \in \mathbb{N}, \tag{3.12}$$

$x_n \to x^*$ as $n \to \infty$, and there exists $z \in A_0(t)$ such that $M(z, Tx^*, t) = M(A, B, t)$. Also, $\alpha(x_n, x, t) \geq t$ for all n and all $t > 0$. Then from (ii) with $x = x_n$, $y = x^*$, $u = x_{n+1}$ and $v = z$ we get

$$\begin{aligned}\frac{1}{M(x_{n+1}, z, t)} - 1 &\leq \left(\frac{\frac{1}{M(x_n,z,t)} - 1 + \frac{1}{M(x^*,x_{n+1},t)} - 1}{\frac{1}{M(x_n,z,t)} - 1 + \frac{1}{M(x^*,x_{n+1},t)} - 1 + \frac{1}{t}}\right)\left(\frac{1}{M(x_n, x^*, t)} - 1\right) \\ &< \frac{1}{M(x_n, x^*, t)} - 1.\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality we get $\frac{1}{M(x^*,z,t)} - 1 = 0$, *i.e.*, $x^* = z$. Therefore x^* is a best proximity point of T. Uniqueness follows similarly as in Theorem 3. □

4 Best proximity point results in partially ordered intuitionistic fuzzy metric space

Fixed point theorems for monotone operators in partially ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [36–40] and references therein). The aim of this section is to deduce certain new best proximity results in the context of partially ordered intuitionistic fuzzy metric spaces.

Definition 12 Let A, B be two nonempty closed subsets of a partially ordered intuitionistic fuzzy metric space $(X, M, N, *, \diamond, \preceq)$. Then $T : A \to B$ is said to be a proximally order-preserving, if for all $x, y, u, v \in A$,

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t) \end{cases} \implies \quad u \preceq v$$

holds for all $t > 0$.

Theorem 5 *Let* A *and* B *be nonempty subsets of a partially ordered complete triangular intuitionistic fuzzy metric space* $(X, M, N, *, \diamond, \preceq)$ *such that* $A_0(t)$ *is nonempty for all* $t > 0$. *Let* $T : A \to B$ *be a t-uniformly continuous non-self-mapping satisfying the following assertions*:

(i) T *is proximally order-preserving and* $T(A_0(t)) \subseteq B_0(t)$ *for all* $t > 0$;

(ii) *for* $x, y, u, v \in A$,

$$\left.\begin{array}{l} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \implies \frac{1}{M(u, v, t)} - 1 \leq \psi\bigl(\mathcal{M}^T(x, y, u, v, t) - \mathcal{N}^T(x, y, u, v, t)\bigr) \tag{4.1}$$

holds for all $t > 0$, *where* $\psi \in \Psi$;

(iii) *for any sequence* $\{y_n\}$ *in* $B_0(t)$ *and* $x \in A$ *satisfying* $M(x, y_n, t) \to M(A, B, t)$ *as* $n \to +\infty$, *then* $x \in A_0(t)$ *for all* $t > 0$;

(iv) *there exist elements* x_0 *and* x_1 *in* $A_0(t)$ *such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all } t > 0 \text{ and } x_0 \preceq x_1.$$

Then there exists $x^* \in A$ *such that* $M(x^*, Tx^*, t) = M(A, B, t)$, *for all* $t > 0$, *that is*, T *has a best proximity point* $x^* \in A$.

Proof Define $\alpha : A \times A \times (0, \infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 2t, & \text{if } x \preceq y, \\ \frac{1}{2}t, & \text{otherwise.} \end{cases}$$

At first we prove that T is an α-proximal admissible mapping. For this assume that

$$\begin{cases} \alpha(x, y, t) \geq t, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

So

$$\begin{cases} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Now, since T is proximally order-preserving so, $u \preceq v$. That is, $\alpha(u, v, t) \geq t$ which implies that T is α-proximal admissible. Condition (ii) implies that T is α-ψ-proximal contractive mapping. Further by (iv) we have

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{and} \quad \alpha(x_0, x_1, t) \geq t.$$

Therefore all conditions of Theorem 1 hold and T has a best proximity point. □

Theorem 6 *Let* A *and* B *be nonempty subsets of a partially ordered complete triangular intuitionistic fuzzy metric space* $(X, M, N, *, \diamond, \preceq)$ *such that* $A_0(t)$ *is nonempty for all* $t > 0$. *Let* $T : A \to B$ *be a non-self-mapping satisfying the following assertions*:

(i) T *is proximally order-preserving and* $T(A_0(t)) \subseteq B_0(t)$ *for all* $t > 0$;

(ii) (4.1) *holds for all* $t > 0$;
(iii) *for any sequence* $\{y_n\}$ *in* $B_0(t)$ *and* $x \in A$ *satisfying* $M(x, y_n, t) \to M(A, B, t)$ *as* $n \to +\infty$, *then* $x \in A_0(t)$ *for all* $t > 0$;
(iv) *there exist elements* x_0 *and* x_1 *in* $A_0(t)$ *such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all } t > 0 \text{ and } x_0 \preceq x_1;$$

(v) *if* $\{x_n\}$ *is an increasing sequence in* X *such that* $x_n \to x$ *as* $n \to +\infty$, *then* $x_n \preceq x$ *for all* n.

Then there exists $x^* \in A$ *such that* $M(x^*, Tx^*, t) = M(A, B, t)$, *for all* $t > 0$, *that is,* T *has a best proximity point* $x^* \in A$.

Proof Define $\alpha : A \times A \times (0, \infty) \to [0, +\infty)$ as in Theorem 5. Also, assume $\alpha(x_n, x_{n+1}, t) \geq t$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by (v) we get $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x, t) \geq t$ for all $n \in \mathbb{N}$ and all $t > 0$. All other conditions can be proved as in the proof of Theorem 5. Thus all conditions of Theorem 2 hold and T has a best proximity point. □

Similarly from Theorems 3 and 4 we can deduce the following results.

Theorem 7 *Let* A *and* B *be nonempty subsets of a partially ordered complete triangular intuitionistic fuzzy metric space* $(X, M, N, *, \diamond, \preceq)$ *such that* $A_0(t)$ *is nonempty for all* $t > 0$. *Let* $T : A \to B$ *be a t-uniformly continuous non-self-mapping. Also suppose that the following assertions hold true*:
(i) T *is proximally order-preserving and* $T(A_0(t)) \subseteq B_0(t)$ *for all* $t > 0$;
(ii) *for* $x, y, u, v \in A$,

$$\left.\begin{array}{l} x \preceq y, \\ M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\}$$
$$\Longrightarrow \quad \frac{1}{M(u, v, t)} - 1 \leq \left(\frac{\frac{1}{M(x,v,t)} - 1 + \frac{1}{M(y,u,t)} - 1}{\frac{1}{M(x,v,t)} - 1 + \frac{1}{M(y,u,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x, y, t)} - 1 \right) \tag{4.2}$$

holds for all $t > 0$;
(iii) *for any sequence* $\{y_n\}$ *in* $B_0(t)$ *and* $x \in A$ *satisfying* $M(x, y_n, t) \to M(A, B, t)$ *as* $n \to +\infty$, *then* $x \in A_0(t)$ *for all* $t > 0$;
(iv) *there exist elements* x_0 *and* x_1 *in* $A_0(t)$ *such that*

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{for all } t > 0 \text{ and } x_0 \preceq x_1.$$

Then there exists $x^* \in A$ *such that* $M(x^*, Tx^*, t) = M(A, B, t)$, *for all* $t > 0$, *that is,* T *has a best proximity point* $x^* \in A$.

Theorem 8 *If in the above theorem, in place of t-uniform continuity of* T, *we assume that for any increasing sequence* $\{x_n\}$ *in* X *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $x_n \preceq x$ *for all* $n \in \mathbb{N}$. *Then there exists* $x^* \in A$ *such that* $M(x^*, Tx^*, t) = M(A, B, t)$, *for all* $t > 0$, *that is,* T *has a best proximity point* $x^* \in A$.

5 Application to fixed point theory

In this section we deduce new fixed point results in intuitionistic fuzzy metric space and ordered intuitionistic fuzzy metric space. Moreover, we derive certain recent fixed point results as corollaries to our best proximity results.

First we introduce the following concepts.

Definition 13 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space, $T:X\to X$ and $\alpha:X\times X\times(0,\infty)\to[0,\infty)$. We say, T is an α-admissible mapping if

$$x,y\in X,\quad \alpha(x,y,t)\geq t\quad\Longrightarrow\quad \alpha(Tx,Ty,t)\geq t$$

for all $t>0$.

Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space, $T:X\to X$ be a self-mapping. We define $\mathcal{M}^T(x,y,t)$ and $\mathcal{N}^T(x,y,t)$ as follows:

$$\begin{aligned}\mathcal{M}^T(x,y,t)=\max\biggl\{&\frac{1}{M(x,y,t)},\frac{1}{2}\biggl[\frac{1}{M(x,Tx,t)}+\frac{1}{M(y,Ty,t)}\biggr],\\&\frac{1}{2}\biggl[\frac{1}{M(x,Ty,t)}+\frac{1}{M(y,Tx,t)}-1\biggr]\biggr\}\end{aligned}$$

and

$$\mathcal{N}^T(x,y,t)=\max\bigl\{M(x,Tx,t),M(y,Ty,t),M(x,Ty,t),M(y,Tx,t)\bigr\}.$$

Definition 14 Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Let $T:X\to X$ be a self-mapping and $\alpha:X\times X\times(0,\infty)\to[0,\infty)$ be a function. We say T is an α-ψ-contractive mapping if

$$x,y\in X,\quad \alpha(x,y,t)\geq t\quad\Longrightarrow\quad \frac{1}{M(Tx,Ty,t)}-1\leq\psi\bigl(\mathcal{M}^T(x,y,t)-\mathcal{N}^T(x,y,t)\bigr)\qquad(5.1)$$

holds for all $t>0$, where $\psi\in\Psi$.

Theorem 9 *Let $(X,M,N,*,\diamond)$ be a complete triangular intuitionistic fuzzy metric space. Let $T:X\to X$ be a t-uniformly continuous self-mapping. Also suppose that the following assertions hold*:

(i) *T is an α-admissible mapping*;
(ii) *T is α-ψ-contractive mapping*;
(iii) *there exists x_0 in X such that $\alpha(x_0,Tx_0,t)\geq t$.*

Then T has a fixed point.

(iv) *Moreover, if $x,y\in\mathrm{Fix}(T)$ implies $\alpha(x,y,t)\geq t$, then T has a unique fixed point.*

Theorem 10 *Let $(X,M,N,*,\diamond)$ be a complete triangular intuitionistic fuzzy metric space. Let $T:X\to X$ be a self-mapping. Also suppose that the following assertions hold*:

(i) *T is an α-admissible mapping*;
(ii) *T is α-ψ-contractive mapping*;
(iii) *there exists x_0 in X such that $\alpha(x_0,Tx_0,t)\geq t$*;

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq t$ *for all* n *and all* $t > 0$ *with* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x, t) \geq t$ *for all* $n \in \mathbb{N}$ *and all* $t > 0$.

Then T *has a fixed point*.

(v) *Moreover, if* $x, y \in \operatorname{Fix}(T)$ *implies* $\alpha(x, y, t) \geq t$, *then* T *has a unique fixed point*.

Theorem 11 *Let* $(X, M, N, *, \diamond)$ *be a complete triangular intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a t-uniformly continuous self-mapping. Also suppose that the following assertions hold*:

(i) T *is an* α-*admissible mapping*;

(ii)

$$x, y \in X, \quad \alpha(x, y, t) \geq t$$
$$\Longrightarrow \quad \frac{1}{M(Tx, Ty, t)} - 1 \leq \left(\frac{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1}{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for all $t > 0$;

(iii) *there exists* x_0 *in* X *such that* $\alpha(x_0, Tx_0, t) \geq t$.

Then T *has a fixed point*.

(iv) *Moreover, if* $x, y \in \operatorname{Fix}(T)$ *implies* $\alpha(x, y, t) \geq t$, *then* T *has a unique fixed point*.

Theorem 12 *Let* $(X, M, N, *, \diamond)$ *be a complete triangular intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a self-mapping. Also suppose that the following assertions hold*:

(i) T *is an* α-*admissible mapping*;

(ii)

$$x, y \in X, \quad \alpha(x, y, t) \geq t$$
$$\Longrightarrow \quad \frac{1}{M(Tx, Ty, t)} - 1 \leq \left(\frac{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1}{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for all $t > 0$;

(iii) *there exist elements* x_0 *in* X *such that* $\alpha(x_0, Tx_0, t) \geq t$;

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq t$ *for all* n *and all* $t > 0$ *with* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x, t) \geq t$ *for all* n *and all* $t > 0$.

Then T *has a fixed point*.

(v) *Moreover, if* $x, y \in \operatorname{Fix}(T)$ *implies* $\alpha(x, y, t) \geq t$, *then* T *has a unique fixed point*.

By taking $\alpha(x, y, t) = t$ for all $x, y \in X$ and all $t > 0$, we obtain the following corrected version of Theorem 2.2 in [27].

Corollary 1 (Theorem 2.2 of [27]) *Let* $(X, M, N, *, \diamond)$ *be a complete triangular intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a t-uniformly continuous mapping satisfying*

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \left(\frac{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1}{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x, y, t)} - 1 \right)$$

holds for all $x, y \in X$ *and all* $t > 0$. *Then* T *has a fixed point*.

Theorem 13 *Let $(X, M, N, *, \diamond, \preceq)$ be a partially ordered complete triangular intuitionistic fuzzy metric space. Let $T : X \to X$ be a t-uniformly continuous self-mapping. Also assume the following assertions hold true*:

(i) *T is an increasing mapping*;

(ii) *assume*

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq \left(\frac{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1}{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x,y,t)} - 1 \right)$$

holds for all $x, y \in X$ with $x \preceq y$ and $t > 0$;

(iii) *there exists x_0 in X such that $x_0 \preceq Tx_0$*.

Then T has a fixed point.

Theorem 14 *Let $(X, M, N, *, \diamond, \preceq)$ be a partially ordered complete triangular intuitionistic fuzzy metric space. Let $T : X \to X$ be a self-mapping. Also assume the following assertions hold true*:

(i) *T is an increasing mapping*;

(ii) *assume*

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq \left(\frac{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1}{\frac{1}{M(x,Ty,t)} - 1 + \frac{1}{M(y,Tx,t)} - 1 + \frac{1}{t}} \right) \left(\frac{1}{M(x,y,t)} - 1 \right)$$

holds for all $x, y \in X$ with $x \preceq y$ and $t > 0$;

(iii) *there exist elements x_0 in X such that $x_0 \preceq Tx_0$*;

(iv) *if $\{x_n\}$ be an increasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$*.

Then T has a fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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