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# Some new results on cyclic relatively nonexpansive mappings in convex metric spaces

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**Abstract**

In this article, we prove a best proximity point theorem for generalized cyclic contractions in convex metric spaces. Then we investigate the structure of minimal sets of cyclic relatively nonexpansive mappings in the setting of convex metric spaces. In this way, we obtain an extension of the Goebel-Karlovitz lemma, which is a key lemma in fixed point theory.

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**Keywords:** best proximity point; cyclic relatively nonexpansive mapping; convex metric space

## 1 Introduction

Let $(X,d)$ be a metric space, and let $A$, $B$ be subsets of $X$. A mapping $T: A\cup B\to A\cup B$ is said to be *cyclic* provided that $T(A)\subseteq B$ and $T(B)\subseteq A$. We begin by recalling the following extension of the Banach contraction principle.

**Theorem 1.1** ([1]) *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X,d)$. Suppose that $T$ is a cyclic mapping such that*

$$d(Tx,Ty)\leq \alpha d(x,y),$$

*for some $\alpha\in(0,1)$ and for all $x\in A$, $y\in B$. Then $T$ has a unique fixed point in $A\cap B$.*

In [2] Eldred and Veeramani introduced the class of cyclic contractions. Before stating the definition we recall that

$$\operatorname{dist}(A,B):=\inf\bigl\{d(x,y):x\in A,y\in B\bigr\}$$

denotes the distance between the subsets $A$ and $B$ of $X$.

**Definition 1.2** Let $A$ and $B$ be nonempty subsets of a metric space $X$. A mapping $T: A\cup B\to A\cup B$ is said to be a cyclic contraction if $T$ is cyclic and

$$d(Tx,Ty)\leq \alpha d(x,y)+(1-\alpha)\operatorname{dist}(A,B) \tag{1}$$

for some $\alpha\in(0,1)$ and for all $x\in A$, $y\in B$.

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Let $T$ be a cyclic mapping. A point $x \in A \cup B$ is said to be a *best proximity point* for $T$ provided that $d(x, Tx) = \operatorname{dist}(A, B)$.

For a uniformly convex Banach space $X$, Eldred and Veeramani proved the following theorem.

**Theorem 1.3** ([2]) *Let $A$ and $B$ be nonempty, closed, and convex subsets of a uniformly convex Banach space $X$ and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = \operatorname{dist}(A, B)$.*

An interesting extension of Theorem 1.3 can be found in [3, 4].

Recently, Suzuki *et al.* in [5] introduced the notion of the property UC, which is a kind of geometric property for subsets of a metric space $X$.

**Definition 1.4** ([5]) Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. Then $(A, B)$ is said to satisfy property UC if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $\lim_n d(x_n, y_n) = \operatorname{dist}(A, B)$ and $\lim_n d(z_n, y_n) = \operatorname{dist}(A, B)$, then we have $\lim_n d(x_n, z_n) = 0$.

We mention that if $A$ and $B$ are nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex, then $(A, B)$ satisfies the property UC. Other examples of pairs having the property UC can be found in [5].

The next theorem guarantees the existence, uniqueness, and convergence of a best proximity point for *cyclic contractions* in metric spaces by using the notion of the property UC.

**Theorem 1.5** ([5]) *Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$ such that $(A, B)$ satisfies the property* UC. *Assume that $A$ is complete. Let $T : A \cup B \to A \cup B$ be a generalized cyclic contraction, that is, there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \leq r \max\bigl\{d(x, y), d(x, Tx), d(y, Ty)\bigr\} + (1 - r) \operatorname{dist}(A, B) \tag{2}$$

*for all $x \in A$ and $y \in B$. Then $T$ has a unique best proximity point $z$ in $A$, and for every $x \in A$ the sequence $\{T^{2n}x\}$ converges to $z$.*

We mention that in [6] the authors proved Theorem 1.5 without using property UC and obtained the existence and not convergence of best proximity points for generalized cyclic contractions in Banach spaces (for more information one can refer to [7]).

We also recall that the weaker notion of the property UC was introduced in [8], called the *WUC property*, in order to study of the existence, uniqueness, and convergence of a best proximity point for cyclic contraction mappings.

Let $(A, B)$ be a nonempty pair of subsets of a metric space $(X, d)$. A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic relatively nonexpansive* if $T$ is cyclic and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$. It is clear that every nonexpansive mapping is relatively nonexpansive.

Eldred *et al.* [9] established the existence of best proximity points for cyclic relatively nonexpansive mappings by using a geometric notion of *proximal normal structure* in the setting of Banach spaces. For related results, we refer the reader to [10–18].

In this article, motivated by Theorem 1.5, we establish a best proximity point theorem for generalized cyclic contraction mappings in convex metric spaces. We also study the structure of minimal sets for cyclic relatively nonexpansive mappings. In this way, we obtain an extension of *the Goebel-Karlovitz lemma* which plays an important role in fixed point theory.

## 2 Preliminaries

The notion of convexity in metric spaces was introduced by Takahashi as follows.

**Definition 2.1** ([19]) Let $(X,d)$ be a metric space and $I := [0,1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on $X$ provided that, for each $(x,y;\lambda) \in X \times X \times I$ and $u \in X$,

$$d\bigl(u, \mathcal{W}(x,y;\lambda)\bigr) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

A metric space $(X,d)$ together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X,d,\mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Fréchet space is not necessarily a convex metric space. Other examples of convex metric spaces which are not embedded in any Banach space can be found in [19].

Here, we recall some notations and definitions of [6, 19].

**Definition 2.2** ([19]) A subset $K$ of a convex metric space $(X,d,\mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x,y;\lambda) \in K$ for all $x,y \in K$ and $\lambda \in I$.

**Proposition 2.3** ([19]) *Let $(X,d,\mathcal{W})$ be a convex metric space and let $B(x;r)$ denote the closed ball centered at $x \in X$ with radius $r \geq 0$. Then $B(x;r)$ is a convex subset of $X$.*

**Proposition 2.4** ([19]) *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of $X$, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of $X$.*

**Definition 2.5** ([19]) A convex metric space $(X,d,\mathcal{W})$ is said to have property (C) if every bounded decreasing net of nonempty, closed, and convex subsets of $X$ has a nonempty intersection.

For example every weakly compact convex subset of a Banach space has property (C). The next example ensures that condition (C) is natural as well in the metrical setting.

**Example 2.1** ([20]) Let $\mathcal{H}$ be a Hilbert space and let $X$ be a nonempty closed subset of $\{x \in \mathcal{H} : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then $\frac{\alpha x + \beta y}{\|\alpha x + \beta y\|} \in X$ and $\operatorname{diam}(X) \leq \frac{\sqrt{2}}{2}$, where $\operatorname{diam}(X) := \sup\{d(x,y) : x,y \in X\}$. Let $d(x,y) := \cos^{-1}(\langle x,y\rangle)$ for all $x,y \in X$, where $\langle \cdot,\cdot \rangle$ is the inner product of $\mathcal{H}$. If we define the convex structure $\mathcal{W} : X \times X \times I \to X$ with $\mathcal{W}(x,y,\lambda) := \frac{\lambda x + (1-\lambda)y}{\|\lambda x + (1-\lambda)y\|}$, then $(X,d)$ is a complete convex metric space which has the property (C) (for more information see Example 2 of [20]).

Let $A$ and $B$ be two nonempty subsets of a convex metric space $(X,d,\mathcal{W})$. We shall say that a pair $(A,B)$ in a convex metric space $(X,d,\mathcal{W})$ satisfies a property if both $A$ and $B$

satisfy that property. For instance, $(A,B)$ is convex if and only if both $A$ and $B$ are convex; $(A,B) \subseteq (C,D) \Leftrightarrow A \subseteq C$, and $B \subseteq D$. We shall also adopt the following notations:

$$\delta_x(A) := \sup\{d(x,y) : y \in A\} \quad \text{for all } x \in X,$$
$$\delta(A,B) := \sup\{d(x,y) : x \in A, y \in B\},$$
$$\operatorname{diam}(A) := \delta(A,A).$$

The *closed and convex hull* of a set $A$ will be denoted by $\overline{\operatorname{con}}(A)$ and is defined by

$$\overline{\operatorname{con}}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The pair $(x,y) \in A \times B$ is said to be *proximal* in $(A,B)$ if $d(x,y) = \operatorname{dist}(A,B)$. Moreover, we set

$$A_0 := \{x \in A : d(x,y') = \operatorname{dist}(A,B), \text{ for some } y' \in B\},$$
$$B_0 := \{y \in B : d(x',y) = \operatorname{dist}(A,B), \text{ for some } x' \in A\}.$$

Note that if $(A,B)$ is a nonempty, weakly compact, and convex pair of subsets of a Banach space $X$, then so is the pair $(A_0,B_0)$, and it is easy to see that $\operatorname{dist}(A,B) = \operatorname{dist}(A_0,B_0)$.

**Definition 2.6** A pair of sets $(A,B)$ is said to be proximal if $A = A_0$ and $B = B_0$.

The following result follows from the proof of Theorem 2.1 in [9].

**Lemma 2.7** *Let $(A,B)$ be a nonempty weakly compact convex pair of a Banach space $X$ and $T : A \cup B \to A \cup B$ a cyclic relatively nonexpansive mapping. Then there exists $(K_1,K_2) \subseteq (A_0,B_0) \subseteq (A,B)$ which is minimal with respect to being nonempty, closed, convex, and a $T$-invariant pair of subsets of $(A,B)$ such that*

$$\operatorname{dist}(K_1,K_2) = \operatorname{dist}(A,B).$$

*Moreover, the pair $(K_1,K_2)$ is proximal.*

**Definition 2.8** ([21]) Let $(A,B)$ be a nonempty pair of subsets of a metric space $(X,d)$. We say that the pair $(A,B)$ is proximal compactness provided that every net $(\{x_\alpha\},\{y_\alpha\})$ of $A \times B$ satisfying the condition that $d(x_\alpha,y_\alpha) \to \operatorname{dist}(A,B)$, has a convergent subnet in $A \times B$. Also, we say that $A$ is semi-compactness if $(A,A)$ is proximal compactness.

It is clear that if $(A,B)$ is a compact pair in a metric space $(X,d)$ then $(A,B)$ is proximal compactness.

**Definition 2.9** Let $(A,B)$ be a nonempty pair of sets in a Banach space $X$. A point $p$ in $A$ ($q$ in $B$) is said to be a diametral point with respect to $B$ (w.r.t. $A$) if $\delta_p(B) = \delta(A,B)$ ($\delta_q(A) = \delta(A,B)$). A pair $(p,q)$ in $A \times B$ is diametral if both points $p$ and $q$ are diametral.

## 3 Main results

In this section, we study the structure of minimal sets of cyclic relatively nonexpansive mappings in the setting of convex metric spaces.

### 3.1 Generalized cyclic contractions in convex metric spaces

We begin our main results of this paper with the following existence theorem.

**Theorem 3.1** *Let* $(A,B)$ *be a nonempty, bounded, closed, and convex pair in a convex metric space* $(X,d,\mathcal{W})$. *Suppose that* $T : A\cup B \to A\cup B$ *is a generalized cyclic contraction. If* $X$ *has the property* (C) *then* $T$ *has a best proximity pair.*

*Proof* Let $\Sigma$ denote the set of all nonempty, bounded, closed, and convex pairs $(E,F)$ which are subsets of $(A,B)$ such that $T$ is cyclic on $E\cup F$. Note that $(A,B)\in\Sigma$. Also, $\Sigma$ is partially ordered by reverse inclusion, that is, $(E_1,F_1)\leq(E_2,F_2)\Leftrightarrow(E_2,F_2)\subseteq(E_1,F_1)$. By the fact that $X$ has the property (C), every increasing chain in $\Sigma$ is bounded above. So, by using Zorn's lemma we obtain a maximal element, say $(C,D)\in\Sigma$. We note that $(\overline{\mathrm{con}}(T(D)),\overline{\mathrm{con}}(T(C)))$ is a nonempty, bounded, closed, and convex pair in $X$ and $(\overline{\mathrm{con}}(T(D)),\overline{\mathrm{con}}(T(C)))\subseteq(C,D)$. Furthermore,

$$T\bigl(\overline{\mathrm{con}}\bigl(T(D)\bigr)\bigr)\subseteq T(C)\subseteq\overline{\mathrm{con}}\bigl(T(C)\bigr),$$

and also

$$T\bigl(\overline{\mathrm{con}}\bigl(T(C)\bigr)\bigr)\subseteq\overline{\mathrm{con}}\bigl(T(D)\bigr),$$

that is, $T$ is cyclic on $\overline{\mathrm{con}}(T(D))\cup\overline{\mathrm{con}}(T(C))$. It now follows from the maximality of $(C,D)$ that

$$\overline{\mathrm{con}}\bigl(T(D)\bigr)=C,\qquad\overline{\mathrm{con}}\bigl(T(C)\bigr)=D.$$

Let $x\in C$, then $D\subseteq B(x;\delta_x(D))$. Now, if $y\in D$ we have

$$\begin{aligned}d(Tx,Ty)&\leq r\max\bigl\{d(x,y),d(x,Tx),d(Ty,y)\bigr\}+(1-r)\operatorname{dist}(A,B)\\&\leq r\delta(C,D)+(1-r)\operatorname{dist}(A,B).\end{aligned}$$

Therefore, for all $y\in D$ we have

$$Ty\in B\bigl(Tx;r\delta(C,D)+(1-r)\operatorname{dist}(A,B)\bigr),$$

and then

$$T(D)\subseteq B\bigl(Tx;r\delta(C,D)+(1-r)\operatorname{dist}(A,B)\bigr).$$

Thus,

$$C=\overline{\mathrm{con}}\bigl(T(D)\bigr)\subseteq B\bigl(Tx;r\delta(C,D)+(1-r)\operatorname{dist}(A,B)\bigr),$$

from which one concludes that

$$d(z, Tx) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B), \quad \text{for all } z \in C.$$

So,

$$\delta_{Tx}(C) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B). \tag{3}$$

Similarly, if $y \in D$ we obtain

$$\delta_{Ty}(D) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B). \tag{4}$$

Put

$$E := \{x \in C : \delta_x(D) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B)\},$$
$$F := \{y \in D : \delta_y(C) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B)\}.$$

Note that $T(D) \subseteq E$ and $T(C) \subseteq F$ and we have

$$E = \bigcap_{y \in D} B\bigl(y; r\delta(C, D) + (1 - r)\operatorname{dist}(A, B)\bigr) \cap C,$$
$$F = \bigcap_{x \in C} B\bigl(x; r\delta(C, D) + (1 - r)\operatorname{dist}(A, B)\bigr) \cap D.$$

Further, if $x \in E$ then by (3), $Tx \in F$, *i.e.* $T(E) \subseteq F$ and also, by (4), $T(F) \subseteq E$. This proves that $T$ is cyclic on $E \cup F$. Maximality of $(C, D)$ implies that $E = C$ and $F = D$. We now conclude that

$$\delta_x(D) \le r\delta(C, D) + (1 - r)\operatorname{dist}(A, B), \quad \text{for all } x \in C.$$

So,

$$\delta(C, D) = \operatorname{dist}(A, B).$$

Now, for each pair $(p, q) \in C \times D$ we must have

$$d(p, Tp) = d(Tq, q) = \operatorname{dist}(A, B),$$

which completes the proof of the theorem. □

**Remark 3.1** Note that Theorem 3.1 holds once the maximal sets $K_1$ and $K_2$ have been fixed and the cyclic mapping $T : A \cup B \to A \cup B$ satisfies the condition that there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \le r\delta(K_1, K_2) + (1 - r)\operatorname{dist}(A, B), \tag{5}$$

for all $(x, y) \in A \times B$.

The next corollary, obtained from Theorem 3.1, immediately follows.

**Corollary 3.2** *Let $(A,B)$ be a nonempty, bounded, closed, and convex pair in a reflexive Banach space $X$. Suppose that $T : A \cup B \to A \cup B$ is a generalized cyclic contraction. Then $T$ has a best proximity point.*

Let us illustrate Theorem 3.1 with the following example.

**Example 3.1** Let $X := [-1,1]$ and define a metric $d$ on $X$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{|x|,|y|\}, & \text{if } x \neq y. \end{cases}$$

Define $\mathcal{W} : X \times X \times I \to X$ with

$$\mathcal{W}(x,y,\lambda) = \lambda \min\{|x|,|y|\},$$

for each $x,y \in X$ and $\lambda \in I$ (see [22]). Then $\mathcal{W}$ is a convex stricture on $X$. In this order, let $x,y \in X$ and $\lambda \in I$. We may assume that $|x| \leq |y|$. Then for each $u \in X$ we have

$$\begin{aligned} d(u,\mathcal{W}(x,y,\lambda)) &= \max\{|u|,\lambda \min\{|x|,|y|\}\} \\ &= \max\{|u|,\lambda|x|\} \leq \max\{|u|,|x|\} \\ &= \lambda \max\{|u|,|x|\} + (1-\lambda)\max\{|u|,|x|\} \\ &\leq \lambda \max\{|u|,|x|\} + (1-\lambda)\max\{|u|,|y|\} \\ &= \lambda d(u,x) + (1-\lambda)d(u,y). \end{aligned}$$

This implies that $(X,d,\mathcal{W})$ is a convex metric space. Now, let $E$ be a nonempty convex subset of $X$. Then $\mathcal{W}(x,y,\lambda) \in E$ for each $x,y \in E$ and $\lambda \in I$. If $\lambda = 0$, then we conclude that $0 \in E$. Therefore, the convex metric space $(X,d,\mathcal{W})$ must have the property (C). Suppose that $A := [0,1]$ and $B := \{-1,0\}$. Then $(A,B)$ is a bounded, closed, and convex pair of subsets of $X$. Let $T : A \cup B \to A \cup B$ be a mapping defined by

$$Tx = \begin{cases} 0, & \text{if } x \in A, \\ \frac{1}{2}, & \text{if } x = -1. \end{cases}$$

Clearly, $T$ is cyclic on $A \cup B$. On the other hand, $T$ is generalized cyclic contraction for each $r \in [\frac{1}{2},1)$. It now follows from Theorem 3.1 that $T$ has a best proximity point which is a fixed point in this case.

### 3.2 Extension of Goebel-Karlovitz lemma

The following result is another version of Lemma 2.7 in the setting of convex metric spaces.

**Lemma 3.3** *Let $(A,B)$ be a nonempty, bounded, closed, and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that $A_0$ is nonempty and $(A,B)$ is proximal compactness. Assume that*

*$T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If $X$ has the property* (C) *then there exists a pair $(K_1, K_2) \subseteq (A, B)$ which is minimal with respect to being nonempty, closed, convex, and a $T$-invariant pair of subsets of $(A, B)$ such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

*Proof* Let $\Sigma$ denote the set of all nonempty, closed, and convex pairs $(E, F)$ which are subsets of $(A, B)$ such that $T$ is cyclic on $E \cup F$ and $d(x, y) = \operatorname{dist}(A, B)$ for some $(x, y) \in E \times F$. Note that $(A, B) \in \Sigma$ by the fact that $A_0$ is nonempty. Also, $\Sigma$ is partially ordered by reverse inclusion. Assume that $\{(E_\alpha, F_\alpha)\}_\alpha$ is a increasing chain in $\Sigma$. Set $E := \bigcap E_\alpha$ and $F := \bigcap F_\alpha$. Since $X$ has the property (C), we conclude that $(E, F)$ is a nonempty pair. Also, by Proposition 2.4, $(E, F)$ is a closed and convex pair. Moreover,

$$T(E) = T\left(\bigcap E_\alpha\right) \subseteq \bigcap T(E_\alpha) \subseteq \bigcap F_\alpha = F.$$

Similarly we can see that $T(F) \subseteq E$, that is, $T$ is cyclic on $E \cup F$. Now, let $(x_\alpha, y_\alpha) \in E_\alpha \times F_\alpha$ be such that $d(x_\alpha, y_\alpha) = \operatorname{dist}(A, B)$. Since $(A, B)$ is proximal compactness, $(x_\alpha, y_\alpha)$ has a convergent subsequence, say $(x_{\alpha_i}, y_{\alpha_i})$, such that $x_{\alpha_i} \to x \in A$ and $y_{\alpha_i} \to y \in B$. Thus,

$$d(x, y) = \lim_i d(x_{\alpha_i}, y_{\alpha_i}) = \operatorname{dist}(A, B).$$

Therefore, there exists an element $(x, y) \in E \times F$ such that $d(x, y) = \operatorname{dist}(A, B)$. Hence, every increasing chain in $\Sigma$ is bounded above with respect to a reverse inclusion relation. Then by using Zorn's lemma we can get a maximal element, say $(K_1, K_2)$, which is minimal with respect to set inclusion and so, is minimal with respect to being nonempty, closed, convex, and a $T$-invariant pair of subsets of $(A, B)$ such that

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B). \qquad \square$$

**Lemma 3.4** *Let $(A, B)$ be a nonempty, bounded, closed, and convex pair of a convex metric space $(X, d, \mathcal{W})$ such that $A_0$ is nonempty, $X$ has the property* (C) *and $(A, B)$ is proximal compactness. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Suppose that $(K_1, K_2) \subseteq (A, B)$ is a minimal, closed, convex pair which is $T$-invariant such that* $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$. *Then each pair $(p, q) \in K_1 \times K_2$ with $d(p, q) = \operatorname{dist}(A, B)$ contains a diametral point* (*with respect to* $(K_1, K_2)$).

*Proof* By the fact that $T$ is cyclic, a similar argument to Theorem 3.1 implies that $T$ is also cyclic on $\overline{\operatorname{con}}(T(K_1)) \cup \overline{\operatorname{con}}(T(K_2))$. Let $(x_0, y_0) \in K_1 \times K_2$ be such that $d(x_0, y_0) = \operatorname{dist}(A, B)$. The relatively nonexpansiveness of $T$ implies that

$$\begin{aligned}\operatorname{dist}(A, B) &\le \operatorname{dist}\bigl(\overline{\operatorname{con}}\bigl(T(K_2)\bigr), \overline{\operatorname{con}}\bigl(T(K_1)\bigr)\bigr) \\ &\le d(Ty_0, Tx_0) \le d(x_0, y_0) = \operatorname{dist}(A, B).\end{aligned}$$

So, $\operatorname{dist}(\overline{\operatorname{con}}(T(K_2)), \overline{\operatorname{con}}(T(K_1))) = \operatorname{dist}(A, B)$. Now, by the minimality of $(K_1, K_2)$, we must have

$$\overline{\operatorname{con}}\bigl(T(K_1)\bigr) = K_2, \qquad \overline{\operatorname{con}}\bigl(T(K_2)\bigr) = K_1.$$

Assume that $(p,q) \in K_1 \times K_2$ such that $d(p,q) = \operatorname{dist}(A,B)$ and suppose there is no diametral point in $(p,q)$, that is,

$$\max\{\delta_p(K_2), \delta_q(K_1)\} < \delta(K_1,K_2).$$

Put $r_1 := \delta_p(K_2)$ and $r_2 := \delta_q(K_1)$. Let $r := \max\{r_1,r_2\} < \delta(K_1,K_2)$ and define

$$C_r(K_2) := K_1 \cap \left(\bigcap_{x\in K_2} B(x;r)\right), \qquad C_r(K_1) := K_2 \cap \left(\bigcap_{x\in K_1} B(x;r)\right).$$

Note that $(C_r(K_2), C_r(K_1))$ is a nonempty, closed, and convex pair in $X$ by Propositions 2.3 and 2.4, and since $(p,q) \in (C_r(K_2), C_r(K_1))$,

$$\operatorname{dist}(C_r(K_2), C_r(K_1)) = \operatorname{dist}(A,B).$$

It is not difficult to see that, for $(x,y) \in K_1 \times K_2$,

$$(x,y) \in (C_r(K_2), C_r(K_1)) \quad \Leftrightarrow \quad K_2 \subseteq B(x;r), \qquad K_1 \subseteq B(y;r).$$

We now prove that $T$ is cyclic on $C_r(K_2) \cup C_r(K_1)$. Let $u \in C_r(K_2)$. We must verify that $Tu \in C_r(K_1)$, that is, $K_1 \subseteq B(Tu;r)$. By the relatively nonexpansiveness of $T$, for $v \in K_2$ we have

$$d(Tv,Tu) \le d(u,v) \le r,$$

then $Tv \in B(Tu;r)$, which implies that $T(K_2) \subseteq B(Tu;r)$. Therefore, $K_1 = \overline{\operatorname{cov}}(T(K_2)) \subseteq B(Tu;r)$ and hence, $Tu \in C_r(K_1)$. Thus, $T(C_r(K_2)) \subseteq C_r(K_1)$. Similarly, we can see that $T(C_r(K_1)) \subseteq C_r(K_2)$. Therefore, $T$ is cyclic on $C_r(K_1) \cup C_r(K_2)$. Now, the minimality of $(K_1,K_2)$ implies that $C_r(K_1) = K_2$ and $C_r(K_2) = K_1$. Hence, $K_2 \subseteq \bigcap_{x\in K_1} B(x;r)$ and so, for each $y \in K_2$, $\delta_y(K_1) \le r$. Therefore, we obtain

$$\delta(K_1,K_2) = \sup_{y\in K_2} \delta_y(K_1) \le r,$$

which is a contradiction. Thus, our assumption was wrong and either $p$ or $q$ must be a diametral point for $(K_1,K_2)$. □

**Definition 3.5** Let $(A,B)$ be a nonempty pair of subsets of a metric space $(X,d)$. Suppose that $T : A\cup B \to A\cup B$ is a cyclic mapping. Then a sequence $\{x_n\}$ in $A\cup B$ is said to be an approximate best proximity point sequence for $T$ if

$$\lim_{n\to\infty} d(x_n, Tx_n) = \operatorname{dist}(A,B).$$

Note that if $\operatorname{dist}(A,B) = 0$, then the sequence $\{x_n\}$ is said to be an approximate fixed point sequence for $T$.

The following lemma guarantees the existence of approximate best proximity sequences for cyclic relatively nonexpansive mappings.

**Lemma 3.6** *Let $(A,B)$ be a nonempty, bounded, closed, and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that $A_0$ is nonempty, $X$ has the property* (C) *and $(A,B)$ is proximal compactness. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Then there exists an approximate best proximity point sequence for $T$ in $A$.*

*Proof* It follows from Lemma 3.3 that there exists a pair $(K_1,K_2) \subseteq (A,B)$ which is minimal with respect to being nonempty, closed, convex, and a $T$-invariant pair of subsets of $(A,B)$ and there exists $(x^*,y^*) \in K_1 \times K_2$ such that

$$\operatorname{dist}(K_1,K_2) = d\bigl(x^*,y^*\bigr) = \operatorname{dist}(A,B).$$

For any $r \in (0,1)$ put $\alpha := -r^2 + 2r$. Then $\alpha \in (0,1)$. Define the mapping $T_r : A \cup B \to A \cup B$ as follows:

$$T_r(x) = \begin{cases} \mathcal{W}(Tx,y^*,r); & x \in A, \\ \mathcal{W}(Tx,x^*,r); & x \in B. \end{cases}$$

Since $T$ is cyclic and $(A,B)$ is a convex pair in convex metric space $(X,d,\mathcal{W})$, we conclude that $T_r$ is cyclic on $A \cup B$. Now, for each $(x,y) \in A \times B$ we have

$$\begin{aligned} d(T_rx, T_ry) &= d\bigl(\mathcal{W}\bigl(Tx,y^*,r\bigr),\mathcal{W}\bigl(Ty,x^*,r\bigr)\bigr) \\ &\le rd\bigl(\mathcal{W}\bigl(Tx,y^*,r\bigr),Ty\bigr) + (1-r)d\bigl(\mathcal{W}\bigl(Tx,y^*,r\bigr),x^*\bigr) \\ &\le r\bigl[rd(Ty,Tx) + (1-r)d\bigl(Ty,y^*\bigr)\bigr] + (1-r)\bigl[rd\bigl(Tx,x^*\bigr) + (1-r)d\bigl(y^*,x^*\bigr)\bigr] \\ &\le r^2d(x,y) + r(1-r)d\bigl(Ty,y^*\bigr) + r(1-r)d\bigl(x^*,Tx\bigr) + (1-r)^2d\bigl(x^*,y^*\bigr) \\ &\le r^2\delta(K_1,K_2) + 2r\delta(K_1,K_2) - 2r^2\delta(K_1,K_2) + (1-r)^2\operatorname{dist}(A,B) \\ &= \bigl(-r^2+2r\bigr)\delta(K_1,K_2) + \bigl(1-\bigl(-r^2+2r\bigr)\bigr)\operatorname{dist}(A,B) \\ &= \alpha\delta(K_1,K_2) + (1-\alpha)\operatorname{dist}(A,B). \end{aligned}$$

Hence, for each $r \in (0,1)$ we have

$$d(T_rx, T_ry) \le \alpha\delta(K_1,K_2) + (1-\alpha)\operatorname{dist}(A,B).$$

By using Remark 3.1, we conclude that the cyclic mapping $T_r$ has a best proximity point, say $p_r \in A$, for each $r \in (0,1)$. Thus,

$$\begin{aligned} \operatorname{dist}(A,B) &\le d(p_r, Tp_r) \\ &\le d\bigl(p_r, T_r(p_r)\bigr) + d\bigl(T_r(p_r), Tp_r\bigr) \\ &= \operatorname{dist}(A,B) + d\bigl(\mathcal{W}\bigl(Tp_r,y^*,r\bigr), Tp_r\bigr) \\ &\le \operatorname{dist}(A,B) + (1-r)d\bigl(Tp_r,y^*\bigr) \\ &\le \operatorname{dist}(A,B) + (1-r)\operatorname{diam}(B). \end{aligned}$$

If $r \to 1^-$ in the above relation, we obtain

$$d(p_r, Tp_r) \to \operatorname{dist}(A,B).$$

That is, there exists a sequence $\{x_n\}$ in $A$ such that $d(x_n, Tx_n) \to \operatorname{dist}(A,B)$, which completes the proof. □

Here, we state the main result of this paper.

**Theorem 3.7** *Let $(A,B)$ be a nonempty, bounded, closed, and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that $X$ has the property* (C). *Assume that $A_0$ is nonempty and $(A,B)$ is proximal compactness and satisfies the property UC. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Let $(K_1,K_2) \subseteq (A,B)$ be a minimal closed and convex pair which is $T$-invariant such that* $\operatorname{dist}(K_1,K_2) = \operatorname{dist}(A,B)$, *and $\{x_n\}$ a sequence in $K_1$ such that* $\lim_{n\to\infty} d(x_n, Tx_n) = \operatorname{dist}(A,B)$. *Then for all $(p,q) \in K_1 \times K_2$ with $d(p,q) = \operatorname{dist}(A,B)$ we have*

$$\max\Big\{\limsup_{n\to\infty} d(x_n,q), \limsup_{n\to\infty} d(p, Tx_n)\Big\} = \delta(K_1,K_2).$$

*Moreover, the set of best proximity points of $T$ is nonempty.*

*Proof* We see from Lemma 3.4 that each point $(p,q) \in K_1 \times K_2$ with $d(p,q) = \operatorname{dist}(A,B)$ contains a diametral point, that is,

$$\max\big\{\delta_p(K_2), \delta_q(K_1)\big\} = \delta(K_1,K_2).$$

Let $\{x_n\}$ be the sequence given by the statement, which exists because of Lemma 3.6, and assume that there exists $(u,v) \in K_1 \times K_2$ and $r < \delta(A,B)$ such that $d(u,v) = \operatorname{dist}(A,B)$ and

$$\limsup_{n\to\infty} d(u, Tx_n) \le r, \qquad \limsup_{n\to\infty} d(x_n, v) \le r.$$

Note that

$$d\big(T^2x_n, Tx_n\big) \le d(Tx_n, x_n) \to \operatorname{dist}(A,B),$$

as $n \to \infty$. Since $(A,B)$ has the property UC,

$$\lim_{n\to\infty} d\big(T^2x_n, x_n\big) = 0. \tag{6}$$

Set

$$C_1 := \Big\{y \in K_2 : \limsup_{n\to\infty} d(x_n, y) \le r\Big\}$$

and

$$C_2 := \Big\{x \in K_1 : \limsup_{n\to\infty} d(x, Tx_n) \le r\Big\}.$$

Note that $(C_1, C_2)$ is a nonempty, bounded, and closed pair in $X$. We show that $(C_1, C_2)$ is also convex. Let $y_1, y_2 \in C_1$ and $\lambda \in [0,1]$. We have

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(x_n, \mathcal{W}(y_1, y_2, \lambda)\bigr) \\
&\quad\le \limsup_{n\to\infty}\bigl[\lambda d(x_n, y_1) + (1-\lambda) d(x_n, y_2)\bigr] \\
&\quad\le \lambda r + (1-\lambda) r = r.
\end{aligned}$$

Thus, $C_1$ is convex. Similarly, we can see that $C_2$ is convex. Besides, by the fact that $(u, v) \in C_1 \times C_2$ we conclude that $\operatorname{dist}(C_2, C_1) = \operatorname{dist}(A, B)$. We now verify that $T$ is cyclic on $C_2 \cup C_1$. Suppose that $x \in C_2$. Then $Tx \in K_2$ and $\limsup_{n\to\infty} d(x, Tx_n) \le r$. It now follows from (6) that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, Tx) &\le \limsup_{n\to\infty}\bigl[d\bigl(x_n, T^2x_n\bigr) + d\bigl(T^2x_n, Tx\bigr)\bigr] \\
&\le \limsup_{n\to\infty} d(x, Tx_n) \le r,
\end{aligned}$$

that is, $Tx \in C_1$. Now, let $y \in C_1$. Then $Ty \in K_1$ and

$$\limsup_{n\to\infty} d(Ty, Tx_n) \le \limsup_{n\to\infty} d(x_n, y) \le r,$$

that is, $Ty \in C_2$. Hence, $T$ is cyclic on $C_2 \cup C_1$. From the minimality of $(K_1, K_2)$ one deduces that $K_1 = C_2$ and $K_2 = C_1$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = \operatorname{dist}(A, B)$ and $(A, B)$ is proximal compactness, we may assume that $x_n \to p \in K_1$ and $Tx_n \to q \in K_2$. Therefore, $d(p, q) = \operatorname{dist}(A, B)$ and we have

$$d(p, y) = \limsup_{n\to\infty} d(x_n, y) \le r \quad \text{and} \quad d(q, x) = \limsup_{n\to\infty} d(x, Tx_n) \le r,$$

for all $(x, y) \in K_1 \times K_2$. Therefore $\delta_p(K_2) \le r < \delta(A, B)$ and $\delta_q(K_1) \le r < \delta(A, B)$, which is a contradiction by the fact that the pair $(p, q)$ contains a diametral point. Hence,

$$\max\Bigl\{\limsup_{n\to\infty} d(x_n, q), \limsup_{n\to\infty} d(p, Tx_n)\Bigr\} = \delta(K_1, K_2).$$

On the other hand, $\limsup_{n\to\infty} d(x_n, q) = d(p, q) = \operatorname{dist}(A, B)$ and

$$\limsup_{n\to\infty} d(p, Tx_n) \le \limsup_{n\to\infty}\bigl[d(p, q) + d(q, Tx_n)\bigr] = \operatorname{dist}(A, B).$$

We now conclude that

$$\operatorname{dist}(A, B) = \max\Bigl\{\limsup_{n\to\infty} d(x_n, q), \limsup_{n\to\infty} d(p, Tx_n)\Bigr\} = \delta(K_1, K_2),$$

and so, for each pair $(x, y) \in K_1 \times K_2$ we have

$$d(x, Tx) = d(Ty, y) = \operatorname{dist}(A, B).$$

This completes the proof of theorem. □

The next corollary is an extension of the classical Goebel-Karlovitz lemma [23, 24] in convex metric spaces.

**Corollary 3.8** *Let $A$ be a nonempty, bounded, closed, and convex subset of a convex metric space $(X, d, \mathcal{W})$ such that $X$ has the property* (C). *Assume that $A$ is semi-compactness. Let $T : A \to A$ be a nonexpansive mapping. Suppose that $K_1 \subseteq A$ is a minimal closed and convex subset which is $T$-invariant and $\{x_n\}$ is an approximate fixed point sequence in $K_1$. Then for each $p \in K_1$ we have*

$$\limsup_{n\to\infty} d(p, Tx_n) = \operatorname{diam}(K).$$

*Moreover, the set of fixed points of $T$ is nonempty.*

*Proof* If we consider $A = B$ and $K_1 = K_2$ in Theorem 3.7, then $(A, B)$ has the property UC and the result follows by observing that $\delta(K_1, K_2) = \operatorname{diam}(K_1)$. □

The following theorem is another version of Theorem 3.7, in the setting of reflexive Banach spaces.

**Theorem 3.9** ([7]) *Let $(A, B)$ be a nonempty, bounded, closed, and convex pair of a Banach space $X$ such that $(A, B)$ satisfies property* UC. *Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. Let $(K_1, K_2) \subseteq (A, B)$ be a minimal closed and convex pair which is $T$-invariant such that* $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$, *and $\{x_n\}$ a sequence in $K_1$ such that* $\lim_{n\to\infty} \|x_n - Tx_n\| = \operatorname{dist}(A, B)$. *Then for all $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ we have*

$$\max\Big\{\limsup_{n\to\infty} \|x_n - q\|, \limsup_{n\to\infty} \|p - Tx_n\|\Big\} = \delta(K_1, K_2).$$

*Proof* At first, we note that every reflexive Banach space is a convex metric space which has the property (C). It is sufficient to consider a convex structure $\mathcal{W} : X \times X \times [0, 1] \to X$ with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Moreover, $A_0$ is nonempty. Indeed, if $(\{x_n\}, \{y_n\})$ is a sequence in $A \times B$ such that $\|x_n - y_n\| \to \operatorname{dist}(A, B)$, as $X$ is reflexive and $(A, B)$ is a bounded and closed pair in $X$, the sequence $(\{x_n\}, \{y_n\})$ has a subsequence $(\{x_{n_k}\}, \{y_{n_k}\})$ such that $x_{n_k} \rightharpoonup p \in A$ and $y_{n_k} \rightharpoonup q \in B$, where '$\rightharpoonup$' denotes the weak convergence. It now follows from the weak lower semicontinuity of the norm that

$$\|p - q\| \le \liminf_{k\to\infty} \|x_{n_k} - y_{n_k}\| = \operatorname{dist}(A, B),$$

that is, $A_0$ is nonempty. By a similar argument to Theorem 3.7, the result follows. □

**Remark 3.2** Note that in Theorem 3.9, we cannot deduce the existence of a best proximity point unless we add another condition. For instance, if the sequence $\{x_n\}$, considered in Theorem 3.9, converges to a point in $A$ then the best proximity point set of $T$ will be nonempty.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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