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Iterative algorithms for finding the zeroes of sums of operators

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Abstract

Let H_1, H_2 be real Hilbert spaces, $C \subseteq H_1$ be a nonempty closed convex set, and $0 \notin C$. Let $A : H_1 \to H_2$, $B : H_1 \to H_2$ be two bounded linear operators. We consider the problem to find $x \in C$ such that $Ax = -Bx$ $(0 = Ax + Bx)$. Recently, Eckstein and Svaiter presented some splitting methods for finding a zero of the sum of monotone operator A and B. However, the algorithms are largely dependent on the maximal monotonicity of A and B. In this paper, we describe some algorithms for finding a zero of the sum of A and B which ignore the conditions of the maximal monotonicity of A and B.

Keywords: split equality problem; iterative algorithms; converge strongly

1 Introduction and preliminaries

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ be a nonempty closed convex set and $0 \notin C$. Let $A : H_1 \to H_2$, $B : H_1 \to H_2$ be two bounded linear operators. We consider the interesting problem of finding $x \in C$ such that

$$Ax = -Bx \quad (\text{or } 0 = Ax + Bx). \tag{1.1}$$

For convenience, we denote the problem by $\mathcal{P}$.

For $\mathcal{P}$ it is generally difficult to find zeroes of A and B separately. To overcome this difficulty, Eckstein and Svaiter [1] present the splitting methods for finding a zero of the sum of monotone operator A and B. Three basic families of splitting methods for this problem were identified in [1]:

(i) The *Douglas/Peaceman-Rachford* family, whose iteration is given by

$$\begin{aligned} y_k &= \left[2(I + \xi B)^{-1} + I\right]x_k, \\ z_k &= \left[2(I + \xi A)^{-1} + I\right]y_k, \\ x_{k+1} &= (1 - \rho_k)x_k + \rho_k z_k, \end{aligned}$$

where $\xi > 0$ is a fixed scalar, and $\{\rho_k\} \subseteq (0, 1]$ is a sequence of relaxation parameters.

(ii) The *double backward* splitting method, with iteration given by

$$y_k = (I + \lambda_k B)^{-1}x_k,$$

$$x_{k+1} = (I + \lambda_k A)^{-1} y_k,$$

where $\{\lambda_k\} \subseteq (0, \infty)$ a sequence of regularization parameters.

(iii) The *forward-backward* splitting method, with iteration given by

$$y_k \in (I - \lambda_k A)^{-1} x_k,$$
$$x_{k+1} = (I + \lambda_k B)^{-1} y_k,$$

where $\{\lambda_k\} \subseteq (0, \infty)$ a sequence of regularization parameters.

Convergence results for the scheme (i), in the case in which $\{\rho_k\}$ is contained in a compact subset of $(0,1)$, can be found in [2]; the convergence analysis of the double backward scheme given by (ii), which can be found in [3] and [4]; the standard convergence analysis for (iii) one can see [5]. However, the convergence results are largely dependent on the maximal monotonicity of A and B. It is therefore the aim of this paper to construct new algorithms for problem $\mathcal{P}$ which ignore the conditions of the maximal monotonicity of A and B.

The paper is organized as follows. In Section 2, we define the concept of the minimal norm solution of the problem $\mathcal{P}$ (1.1). Using Tychonov regularization, we obtain a net of solutions for some minimization problem approximating such minimal norm solution (see Theorem 2.4). In Section 3, we introduce an algorithm and prove the strong convergence of the algorithm, more importantly, its limit is the minimum-norm solution of the problem $\mathcal{P}$ (1.1) (see Theorem 3.2). In Section 4, we introduce KM-CQ-like iterative algorithm which converge strongly to a solution of the problem $\mathcal{P}$ (1.1) (see Theorem 4.3).

Throughout the rest of this paper, I denotes the identity operator on Hilbert space H, $\operatorname{Fix}(T)$ the set of the fixed points of an operator T and ∇f the gradient of the functional $f : H \to R$. An operator T on a Hilbert space H is *nonexpansive* if, for each x and y in H, $\|Tx - Ty\| \leq \|x - y\|$. T is said to be *averaged*, if there exist $0 < \alpha < 1$ and a nonexpansive operator N such that $T = (1 - \alpha)I + \alpha N$.

We know that the projection P_C from H onto a nonempty closed convex subset C of H is a typical example of a nonexpansive and averaged mapping, which is defined by

$$P_C(w) = \arg\min_{x \in C} \|x - w\|.$$

It is well known that $P_C(w)$ is characterized by the inequality

$$\langle w - P_C(w), x - P_C(w) \rangle \leq 0, \quad \forall x \in C.$$

We now collect some elementary facts which will be used in the proofs of our main results.

Lemma 1.1 [6, 7] *Let X be a Banach space, C a closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 1.2 [8] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with*

$\sum_{n=1}^{\infty} u_n < \infty$, *and* $\{t_n\}$ *a sequence of real numbers with* $\limsup_n t_n \leq 0$. *Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n t_n + u_n, \quad \forall n \in N.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 1.3 [9] *Let* $\{w_n\}$, $\{z_n\}$ *be bounded sequences in a Banach space and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *which satisfies the following condition*: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $w_{n+1} = (1 - \beta_n)w_n + \beta_n z_n$ *and* $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|w_{n+1} - w_n\| \leq 0$, *then* $\lim_{n\to\infty} \|z_n - w_n\| = 0$.

Lemma 1.4 [10] *Let f be a convex and differentiable functional and let C be a closed convex subset of H. Then $x \in C$ is a solution of the problem*

$$\min_{x\in C} f(x)$$

if and only if $x \in C$ satisfies the following optimality condition:

$$\langle \nabla f(x), \nu - x \rangle \geq 0, \quad \forall \nu \in C.$$

Moreover, if f is, in addition, strictly convex and coercive, then the minimization problem has a unique solution.

Lemma 1.5 [11] *Let A and B be averaged operators and suppose that* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B)$ *is nonempty. Then* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B) = \mathrm{Fix}(AB) = \mathrm{Fix}(BA)$.

2 The minimum-norm solution of the problem $\mathcal{P}$

In this section, we propose the concept of the minimal norm solution of $\mathcal{P}$ (1.1). Then, using Tychonov regularization, we obtain the minimal norm solution by a net of solution for some minimization problem.

We use Γ to denote the solution set of $\mathcal{P}$, *i.e.*,

$$\Gamma = \{x \in H_1, Ax = -Bx, x \in C\}$$

and assume consistency of $\mathcal{P}$. Hence Γ is closed, convex, and nonempty.

Let $H = H_1 \times H_1$, $M = \{(x, x), x \in H_1\} \subseteq H$, P be the linear operator from H_1 onto M, and P has the matrix form

$$P = \begin{bmatrix} I \\ I \end{bmatrix},$$

that is to say, $P(x) = (x, x)$, $\forall x \in H_1$.

Define $G : H \to H_2$ by $G((x, y)) = Ax + By$, $\forall (x, y) \in H_2$. Then G has the matrix form $G = [A, B]$, and $GP = A + B$, $PG^*GP = A^*A + A^*B + B^*A + B^*B$.

The problem can now be reformulated as finding $x \in C$ with $GPx = 0$, or solving the following minimization problem:

$$\min_{x\in C} f(x) = \frac{1}{2}\|GPx\|^2, \tag{2.1}$$

which is ill-posed. A classical way is the well-known Tychonov regularization, which approximates a solution of problem (2.1) by the unique minimizer of the regularized problem:

$$\min_{x\in C} f_\alpha(x) = \frac{1}{2}\|GPx\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{2.2}$$

where $\alpha > 0$ is the regularization parameter. Denote by x_α the unique solution of (2.2).

Proposition 2.1 *For $\alpha > 0$, the solution x_α of* (2.2) *is uniquely defined. x_α is characterized by the inequality*

$$\langle P^*G^*GPx_\alpha + \alpha x_\alpha, x - x_\alpha\rangle \geq 0, \quad \forall x \in C.$$

Proof Obviously, $f(x) = \frac{1}{2}\|GPx\|^2$ is convex and differentiable with gradient $\nabla f(x) = P^*G^*GPx$. Recall that $f_\alpha(x) = f(x) + \frac{1}{2}\alpha\|x\|^2$, we see that f_α is strictly convex and differentiable with gradient

$$\nabla f_\alpha(x) = P^*G^*GPx + \alpha x.$$

According Lemma 1.4, x_α is characterized by the inequality

$$\langle P^*G^*GPx_\alpha + \alpha x_\alpha, x - x_\alpha\rangle \geq 0, \quad \forall x \in C. \tag{2.3}$$

□

Definition 2.2 An element $\tilde{x} \in \Gamma$ is said to be the *minimal norm solution* of SEP (1.1) if $\|\tilde{x}\| = \inf_{x\in\Gamma}\|x\|$.

The following proposition collects some useful properties of $\{x_\alpha\}$ the unique solution of (2.2).

Proposition 2.3 *Let x_α be given as the unique solution of* (2.2). *Then we have*:

(i) $\|x_\alpha\|$ *is decreasing for* $\alpha \in (0,\infty)$.

(ii) $\alpha \mapsto x_\alpha$ *defines a continuous curve from* $(0,\infty)$ *to* H_1.

Proof Let $\alpha > \beta > 0$, since x_α and x_β are the unique minimizers of f_α and f_β, respectively, we get

$$\frac{1}{2}\|GPx_\alpha\|^2 + \frac{1}{2}\alpha\|x_\alpha\|^2 \leq \frac{1}{2}\|GPx_\beta\|^2 + \frac{1}{2}\alpha\|x_\beta\|^2,$$
$$\frac{1}{2}\|GPx_\beta\|^2 + \frac{1}{2}\beta\|x_\beta\|^2 \leq \frac{1}{2}\|GPx_\alpha\|^2 + \frac{1}{2}\beta\|x_\alpha\|^2.$$

It follows that $\|x_\alpha\| \leq \|x_\beta\|$. Thus $\|x_\alpha\|$ is decreasing for $\alpha \in (0,\infty)$.

According to Proposition 2.1, we get

$$\langle P^*G^*GPx_\alpha + \alpha x_\alpha, x_\beta - x_\alpha\rangle \geq 0,$$

and

$$\langle P^*G^*GPx_\beta + \beta x_\beta, x_\alpha - x_\beta\rangle \geq 0.$$

It follows that

$$\langle x_\alpha - x_\beta, \alpha x_\alpha - \beta x_\beta\rangle \leq \langle x_\alpha - x_\beta, P^*G^*GP(x_\beta - x_\alpha)\rangle \leq 0.$$

Thus

$$\alpha\|x_\alpha - x_\beta\| \leq (\alpha - \beta)\langle x_\alpha - x_\beta, x_\beta\rangle.$$

It turns out that

$$\|x_\alpha - x_\beta\|^2 \leq \frac{|\alpha - \beta|}{\alpha}\|x_\beta\|.$$

Hence, $\alpha \mapsto x_\alpha$ is a continuous curve from $(0, \infty)$ to H_1. □

Theorem 2.4 *Let x_α be the unique solution of* (2.2). *Then x_α converges strongly to the minimum-norm solution $\tilde{x}$ of $\mathcal{P}$* (1.1) *with $\alpha \to 0$.*

Proof For any $0 < \alpha < \infty$, x_α is given as (2.2), we get

$$\frac{1}{2}\|GPx_\alpha\|^2 + \frac{1}{2}\alpha\|x_\alpha\|^2 \leq \frac{1}{2}\|GP\tilde{x}\|^2 + \frac{1}{2}\alpha\|\tilde{x}\|^2.$$

Since $\tilde{x} \in \Gamma$ is a solution for $\mathcal{P}$,

$$\frac{1}{2}\|GPx_\alpha\|^2 + \frac{1}{2}\alpha\|x_\alpha\|^2 \leq \frac{1}{2}\alpha\|\tilde{x}\|^2.$$

It follows that $\|x_\alpha\| \leq \|\tilde{x}\|$ for all $\alpha > 0$. Thus $\{x_\alpha\}$ is a bounded net in H_1.

All we need to prove is that for any sequence $\{\alpha_n\}$ such that $\alpha_n \to 0$, $\{x_{\alpha_n}\}$ contains a subsequence converging strongly to $\tilde{x}$. For convenience, we set $x_n = x_{\alpha_n}$.

In fact $\{x_n\}$ is bounded, by passing to a subsequence if necessary, we may assume that $\{x_n\}$ converges weakly to a point $\hat{x} \in S$. Due to Proposition 2.1, we get

$$\langle P^*G^*GPx_n + \alpha_n x_n, \tilde{x} - x_n\rangle \geq 0.$$

It turns out that

$$\langle GPx_n, GP\tilde{x} - GPx_n\rangle \geq \alpha_n\langle x_n, x_n - \tilde{x}\rangle.$$

Since $\tilde{x} \in \Gamma$, it follows that

$$\langle GPx_n, -GPx_n\rangle \geq \alpha_n\langle x_n, x_n - \tilde{x}\rangle.$$

Noting that $\|x_n\| \leq \|\tilde{x}\|$, we have

$$\|GPx_n\| \leq 2\alpha_n\|\tilde{x}\| \to 0.$$

Moreover, note that $\{x_n\}$ converges weakly to a point $\hat{x} \in C$, thus $\{GPx_n\}$ converges weakly to $GP\hat{x}$. It follows that $GP\hat{x} = 0$, *i.e.* $\hat{x} \in \Gamma$.

Finally, we prove that $\hat{x} = \tilde{x}$ and this finishes the proof.

Recall that $\{x_n\}$ converges weakly to $\hat{x}$ and $\|x_n\| \leq \|\tilde{x}\|$, one can deduce that

$$\|\hat{x}\| \leq \liminf_n \|x_n\| \leq \|\tilde{x}\| = \min\bigl\{\|x\| : x \in \Gamma\bigr\}.$$

This shows that $\hat{x}$ is also a point in Γ with minimum-norm. By the uniqueness of minimum-norm element, we get $\hat{x} = \tilde{x}$. □

Finally, we will introduce another method to get the minimum-norm solution of the problem $\mathcal{P}$.

Lemma 2.5 *Let $T = I - \gamma P^*G^*GP$, where $0 < \gamma < 2/\rho(P^*G^*GP)$ with $\rho(P^*G^*GP)$ being the spectral radius of the self-adjoint operator P^*G^*GP on H_1. Then we have the following*:

(1) $\|T\| \leq 1$ (*i.e.* T *is nonexpansive*) *and averaged*;

(2) $\mathrm{Fix}(T) = \{x \in H_1, Ax = -Bx\}$, $\mathrm{Fix}(P_CT) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(T) = \Gamma$;

(3) $x \in \mathrm{Fix}(P_CT)$ *if and only if x is a solution of the variational inequality* $\langle P^*G^*GPx, v - x\rangle \geq 0$, $\forall v \in C$.

Proof (1) It is easily proved that $\|T\| \leq 1$, we only need to prove that $T = I - \gamma P^*G^*GP$ is averaged. Indeed, choose $0 < \beta < 1$, such that $\gamma/(1-\beta) < 2/\rho(P^*G^*GP)$, then $T = I - \gamma P^*G^*GP = \beta I + (1-\beta)V$, where $V = I - \gamma/(1-\beta)P^*G^*GP$ is a nonexpansive mapping. That is to say T is averaged.

(2) If $x \in \{x \in H_1, Ax = -Bx\}$, it is obviously that $x \in \mathrm{Fix}(T)$. Conversely, assume that $x \in \mathrm{Fix}(T)$, we have $x = x - \gamma P^*G^*GPx$, hence $\gamma P^*G^*GPx = 0$ then $\|GPx\|^2 = \langle P^*G^*GPx, x\rangle = 0$, we get $x \in \{x \in H_1, Ax = -Bx\}$. We have $\mathrm{Fix}(T) = \{x \in H_1, Ax = -Bx\}$.

Now we prove $\mathrm{Fix}(P_CT) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(T) = \Gamma$. By $\mathrm{Fix}(T) = \{x \in H_1, Ax = -Bx\}$, $\mathrm{Fix}(P_C) \cap \mathrm{Fix}(T) = \Gamma$ is obviously. On the other hand, since $\mathrm{Fix}(P_C) \cap \mathrm{Fix}(T) = \Gamma \neq \emptyset$, and both P_C and T are averaged, from Lemma 1.5, we have $\mathrm{Fix}(P_CT) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(T)$.

(3)

$$\begin{aligned} \bigl\langle P^*G^*GPx, v - x\bigr\rangle \geq 0, \quad \forall v \in C \quad &\Leftrightarrow \quad \bigl\langle x - \bigl(x - \gamma P^*G^*GPx\bigr), v - x\bigr\rangle \geq 0, \quad \forall v \in S \\ &\Leftrightarrow \quad w = P_C\bigl(w - \gamma P^*G^*GPx\bigr) \\ &\Leftrightarrow \quad w \in \mathrm{Fix}(P_CT). \end{aligned}$$

□

Remark 2.6 Choose a constant γ satisfying that $0 < \gamma < 2/\rho(P^*G^*GP)$. For $\alpha \in (0, \frac{2-\gamma\|P^*G^*GP\|}{2\gamma})$, we define a mapping

$$W_\alpha(x) := P_C\bigl[(1-\alpha\gamma)I - \gamma P^*G^*GP\bigr]x.$$

It is clear that W_α is a contractive. Hence, W_α has a unique fixed point x_α, we have

$$x_\alpha = P_C\bigl[(1-\alpha\gamma)I - \gamma P^*G^*GP\bigr]x_\alpha. \tag{2.4}$$

Theorem 2.7 *Let x_α be given as* (2.4). *Then x_α converges strongly to the minimum-norm solution $\tilde{x}$ of the problem $\mathcal{P}$* (1.1) *when $\alpha \to 0$.*

Proof Choose $\check{x} \in \Gamma$, noting that $\alpha \in (0, \frac{2-\gamma\|P^*G^*GP\|}{2\gamma})$, $I - \frac{\gamma}{(1-\alpha\gamma)}P^*G^*GP$ is nonexpansive, it turns out that

$$\begin{aligned}
\|x_\alpha - \check{x}\| &= \big\|P_C\big[(1-\alpha\gamma)I - \gamma P^*G^*GP\big]x_\alpha - P_C\big[\check{x} - \gamma P^*G^*GP\check{x}\big]\big\| \\
&\leq \big\|\big[(1-\alpha\gamma)I - \gamma P^*G^*GP\big]x_\alpha - \big[\check{x} - \gamma P^*G^*GP\check{x}\big]\big\| \\
&= \bigg\|(1-\alpha\gamma)\bigg[x_\alpha - \frac{\gamma}{1-\alpha\gamma}P^*G^*GPx_\alpha\bigg] \\
&\quad - (1-\alpha\gamma)\bigg[\check{x} - \frac{\gamma}{1-\alpha\gamma}P^*G^*GP\check{x}\bigg] - \alpha\gamma\check{x}\bigg\| \\
&\leq (1-\alpha\gamma)\bigg\|\bigg(x_\alpha - \frac{\gamma}{1-\alpha\gamma}P^*G^*GPx_\alpha\bigg) - \bigg(\check{x} - \frac{\gamma}{1-\alpha\gamma}P^*G^*GP\check{x}\bigg)\bigg\| + \alpha\gamma\|\check{x}\| \\
&\leq (1-\alpha\gamma)\|x_\alpha - \check{x}\| + \alpha\gamma\|\check{x}\|.
\end{aligned}$$

That is,

$$\|x_\alpha - \check{x}\| \leq \|\check{x}\|.$$

Hence $\{x_\alpha\}$ is bounded.

Taking into account of (2.4), we have

$$\big\|x_\alpha - P_C\big[I - \gamma P^*G^*GP\big]x_\alpha\big\| \leq \alpha\|\gamma x_\alpha\| \to 0.$$

We assert that $\{x_\alpha\}$ is relatively norm compact as $\alpha \to 0^+$. In fact, assume that $\{\alpha_n\} \subseteq (0, \frac{2-\gamma\|P^*G^*GP\|}{2\gamma})$ and $\alpha_n \to 0^+$ as $n \to \infty$. For convenience, we put $x_n := x_{\alpha_n}$, we get

$$\big\|x_n - P_C\big[I - \gamma P^*G^*GP\big]x_n\big\| \leq \alpha_n\|\gamma x_n\| \to 0.$$

Since P_C is nonexpansive, one concludes that

$$\begin{aligned}
\|x_\alpha - \check{x}\|^2 &= \big\|P_C\big[(1-\alpha\gamma)I - \gamma P^*G^*GP\big]x_\alpha - P_C\big[\check{x} - \gamma P^*G^*GP\check{x}\big]\big\|^2 \\
&\leq \big\langle\big[(1-\alpha\gamma)I - \gamma P^*G^*GP\big]x_\alpha - \big[\check{x} - \gamma P^*G^*GP\check{x}\big], x_\alpha - \check{x}\big\rangle \\
&= \bigg\langle(1-\alpha\gamma)\bigg[x_\alpha - \frac{\gamma}{1-\alpha\gamma}P^*G^*GPx_\alpha\bigg] \\
&\quad - (1-\alpha\gamma)\bigg[\check{x} - \frac{\gamma}{1-\alpha\gamma}P^*G^*GP\check{x}\bigg], x_\alpha - \check{x}\bigg\rangle - \alpha\gamma\langle\check{x}, x_\alpha - \check{x}\rangle \\
&\leq (1-\alpha\gamma)\|x_\alpha - \check{x}\|^2 - \alpha\gamma\langle\check{x}, x_\alpha - \check{x}\rangle.
\end{aligned}$$

That is ,

$$\|x_\alpha - \check{x}\|^2 \leq \langle-\check{x}, x_\alpha - \check{x}\rangle.$$

Thus,

$$\|x_n - \check{x}\|^2 \leq \langle-\check{x}, x_n - \check{x}\rangle, \quad \forall\check{x} \in \Gamma.$$

Due to $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to $\tilde{x}$. Noting that

$$\left\|x_n - P_C\left[I - \gamma P^* G^* GP\right]x_n\right\| \le \alpha_n \|\gamma x_n\| \to 0,$$

and applying Lemma 1.1, we obtain $\tilde{x} \in \mathrm{Fix}(P_C[I - \gamma P^* G^* GP]) = \Gamma$.

Since

$$\|x_n - \check{x}\|^2 \le \langle -\check{x}, x_n - \check{x}\rangle, \quad \forall \check{x} \in \Gamma,$$

it concludes that

$$\|x_n - \tilde{x}\|^2 \le \langle -\tilde{x}, x_n - \tilde{x}\rangle.$$

Hence, if $\{x_n\}$ converges weakly to $\tilde{x}$, then $\{x_n\}$ converges strongly to $\tilde{x}$. That is to say $\{x_\alpha\}$ is relatively norm compact as $\alpha \to 0^+$.

Moreover, again using

$$\|x_n - \check{x}\|^2 \le \langle -\check{x}, x_n - \check{x}\rangle, \quad \forall \check{x} \in \Gamma,$$

let $n \to \infty$, we have

$$\|\tilde{x} - \check{x}\|^2 \le \langle -\check{x}, \tilde{x} - \check{x}\rangle, \quad \forall \check{x} \in \Gamma.$$

This implies that

$$\langle -\check{x}, \check{x} - \tilde{x}\rangle \le 0, \quad \forall \check{x} \in \Gamma.$$

This is equivalent to

$$\langle -\tilde{x}, \check{x} - \tilde{x}\rangle \le 0, \quad \forall \check{x} \in \Gamma.$$

It turns out that $\tilde{x} \in P_C(0)$. Consequently, each cluster point of x_α is equals $\tilde{x}$. Thus $x_\alpha \to \tilde{x}(\alpha \to 0)$ the minimum-norm solution of the problem $\mathcal{P}$. □

3 Iterative algorithm for the minimum-norm solution of the problem $\mathcal{P}$

In this section, we introduce the following algorithm and prove the strong convergence of the algorithm, more importantly, its limit is the minimum-norm solution of the problem $\mathcal{P}$.

Algorithm 3.1 For an arbitrary point $x_0 \in H_1$ the sequence $\{x_n\}$ is generated by the iterative algorithm

$$x_{n+1} = P_C\left\{(1 - \alpha_n)\left[I - \gamma P^* G^* GP\right]x_n\right\}, \tag{3.1}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_n \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_n |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Now, we prove the strong convergence of the iterative algorithm.

Theorem 3.2 *The sequence* $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to the minimum-norm solution* $\tilde{x}$ *of the problem* $\mathcal{P}$ (1.1).

Proof Let R_n and R be defined by

$$\begin{aligned} R_n x &:= P_C\{(1-\alpha_n)[I - \gamma P^* G^* GP]\}x = P_C[(1-\alpha_n)Tx], \\ Rx &:= P_C(I - \gamma P^* G^* GP)x = P_C(Tx), \end{aligned}$$

where $T = I - \gamma P^* G^* GP$, by Lemma 2.5 , it is easy to see that R_n is a contraction with contractive constant $1 - \alpha_n$. Algorithm (3.1) can be written as $x_{n+1} = R_n x_n$.

For any $\hat{x} \in \Gamma$, we have

$$\begin{aligned} \|R_n\hat{x} - \hat{x}\| &= \|P_C[(1-\alpha_n)T\hat{x}] - \hat{x}\| \\ &= \|P_C[(1-\alpha_n)T\hat{x}] - P_S(T\hat{x})\| \\ &\le \|(1-\alpha_n)T\hat{x} - T\hat{x}\| \\ &= \alpha_n\|T\hat{x}\| \le \alpha_n\|\hat{x}\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_{n+1} - \hat{x}\| &= \|R_n x_n - \hat{x}\| \le \|R_n x_n - R_n\hat{x}\| + \|R_n\hat{x} - \hat{x}\| \\ &\le \|P_C[(1-\alpha_n)T\hat{x}] - P_S(T\hat{x})\| \\ &\le (1-\alpha_n)\|x_n - \hat{x}\| + \alpha_n\|\hat{x}\| \\ &\le \max\{\|x_n - \hat{x}\|, \|\hat{x}\|\}. \end{aligned}$$

It follows that $\|x_n - \hat{x}\| \le \max\{\|x_0 - \hat{x}\|, \|\hat{x}\|\}$. So $\{x_n\}$ is bounded.

Next we prove that $\lim_n \|x_{n+1} - x_n\| = 0$.

Indeed,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|R_n x_n - R_{n-1}x_{n-1}\| \\ &\le \|R_n x_n - R_n x_{n-1}\| + \|R_n x_{n-1} - R_{n-1}x_{n-1}\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| + \|R_n x_{n-1} - R_{n-1}x_{n-1}\|. \end{aligned}$$

Notice that

$$\begin{aligned} \|R_n x_{n-1} - R_{n-1}x_{n-1}\| &= \|P_C[(1-\alpha_n)Tx_{n-1}] - P_C[(1-\alpha_{n-1})Tx_{n-1}]\| \\ &\le \|(1-\alpha_n)Tx_{n-1} - (1-\alpha_{n-1})Tx_{n-1}\| \end{aligned}$$

$$\begin{aligned}&= |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}| \|x_{n-1}\|.\end{aligned}$$

Hence

$$\|x_{n+1} - x_n\| \le (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\|.$$

By virtue of the assumptions (1)-(3) and Lemma 1.2, we have

$$\lim_n \|x_{n+1} - x_n\| = 0.$$

Therefore,

$$\begin{aligned}\|x_n - Rx_n\| &\le \|x_{n+1} - x_n\| + \|R_n x_n - Rx_n\| \\ &\le \|x_{n+1} - x_n\| + \big\|(1 - \alpha_n)Tx_n - Tx_n\big\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n \|x_n\| \to 0.\end{aligned}$$

By the demiclosedness principle ensures that each weak limit point of $\{x_n\}$ is a fixed point of the nonexpansive mapping $R = P_C T$, that is, a point of the solution set Γ of SEP (1.1).

Finally, we will prove that $\lim_n \|x_{n+1} - \tilde{x}\| = 0$.

Choose $0 < \beta < 1$, such that $\gamma/(1-\beta) < 2/\rho(P^*G^*GP)$, then $T = I - \gamma P^*G^*GP = \beta I + (1-\beta)V$, where $V = I - \gamma/(1-\beta)P^*G^*GP$ is a nonexpansive mapping. Taking $z \in \Gamma$, we deduce that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\|P_C\big[(1-\alpha_n)Tx_n\big] - z\big\|^2 \\ &\le \big\|(1-\alpha_n)Tx_n - z\big\|^2 \\ &\le (1-\alpha_n)\|Tx_n - z\|^2 + \alpha_n\|z\|^2 \\ &\le \big\|\beta(x_n - z) + (1-\beta)(Vx_n - z)\big\|^2 + \alpha_n\|z\|^2 \\ &\le \beta\big\|(x_n - z)\big\|^2 + (1-\beta)\big\|(Vx_n - z)\big\|^2 - \beta(1-\beta)\|x_n - Vx_n\|^2 + \alpha_n\|z\|^2 \\ &\le \big\|(x_n - z)\big\|^2 - \beta(1-\beta)\|x_n - Vx_n\|^2 + \alpha_n\|z\|^2.\end{aligned}$$

Then

$$\begin{aligned}\beta(1-\beta)\|x_n - Vx_n\| &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|z\|^2 \\ &\le \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\big(\|x_n - z\| - \|x_{n+1} - z\|\big)\alpha_n\|z\|^2 \\ &\le \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\big(\|x_n - x_{n+1}\|\big)\alpha_n\|z\|^2 \to 0.\end{aligned}$$

Note that $T = I - \gamma P^*G^*GP = \beta I + (1-\beta)V$, it follows that $\lim_n \|Tx_n - x_n\| = 0$.

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\limsup_n \langle x_n - \tilde{x}, -\tilde{x}\rangle = \lim_k \langle x_{n_k} - \tilde{x}, -\tilde{x}\rangle$.

By virtue of the boundedness of x_n, we may further assume with no loss of generality that x_{n_k} converges weakly to a point $\check{x}$. Since $\|Rx_n - x_n\| \to 0$, using the demiclosedness principle, $\check{x} \in \mathrm{Fix}(R) = \mathrm{Fix}(P_C T) = \Gamma$. Noticing that $\tilde{x}$ is the projection of the origin onto Γ, we get

$$\limsup_n \langle x_n - \tilde{x}, -\tilde{x}\rangle = \lim_k \langle x_{n_k} - \tilde{x}, -\tilde{x}\rangle = \langle \check{x} - \tilde{x}, -\tilde{x}\rangle \le 0.$$

Finally, we compute

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \bigl\|P_C\bigl[(1-\alpha_n)Tx_n\bigr] - \tilde{x}\bigr\|^2 \\
&= \bigl\|P_C\bigl[(1-\alpha_n)Tx_n\bigr] - P_C T\tilde{x}\bigr\|^2 \\
&\le \bigl\|(1-\alpha_n)Tx_n - T\tilde{x}\bigr\|^2 \\
&= \bigl\|(1-\alpha_n)Tx_n - \tilde{x}\bigr\|^2 \\
&= \bigl\|(1-\alpha_n)(Tx_n - \tilde{x}) + \alpha_n(-\tilde{x})\bigr\|^2 \\
&= (1-\alpha_n)^2\bigl\|(Tx_n - \tilde{x})\bigr\|^2 + \alpha_n^2\|\tilde{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle Tx_n - \tilde{x}, -\tilde{x}\rangle \\
&\le (1-\alpha_n)\bigl\|(Tx_n - \tilde{x})\bigr\|^2 + \alpha_n\bigl[\alpha_n\|\tilde{x}\|^2 + 2(1-\alpha_n)\langle Tx_n - \tilde{x}, -\tilde{x}\rangle\bigr].
\end{aligned}$$

Since, $\limsup_n \langle x_n - \tilde{x}, -\tilde{x}\rangle \le 0$, $\|x_n - Tx_n\| \to 0$, we know that $\limsup_n(\alpha_n\|\tilde{x}\|^2 + 2(1-\alpha_n)\langle Tx_n - \tilde{x}, -\tilde{x}\rangle) \le 0$, by Lemma 1.2, we conclude that $\lim_n \|x_{n+1} - \tilde{x}\| = 0$. This completes the proof. □

4 KM-CQ-like iterative algorithm for the problem $\mathcal{P}$

In this section, we establish a KM-CQ-like algorithm converges strongly to a solution of the problem $\mathcal{P}$.

Algorithm 4.1 For an arbitrary initial point x_0, sequence $\{x_n\}$ is generated by the iteration:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\bigl[(1-\alpha_n)\bigl(I - \gamma P^*G^*GP\bigr)\bigr]x_n, \tag{4.1}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Lemma 4.2 *If $z \in \mathrm{Fix}(T) = \mathrm{Fix}(I - \gamma P^*G^*GP)$, then for any x we have $\|Tx - z\|^2 \le \|x - z\|^2 - \beta(1-\beta)\|Vx - x\|^2$, where β and V are the same in Lemma* 2.5(1).

Proof By Lemma 2.5(1), we know that $T = \beta I + (1-\beta)V$, where $0 < \beta < 1$ and V is a nonexpansive. It is clear that $z \in \mathrm{Fix}(T) = \mathrm{Fix}(V)$, and

$$\begin{aligned}
\|Tx - z\|^2 &= \bigl\|\beta x + (1-\beta)Vx - z\bigr\|^2 \\
&\le \beta\|x - z\|^2 + (1-\beta)\|Vx - z\|^2 - \beta(1-\beta)\|Vx - x\|^2 \\
&\le \beta\|x - z\|^2 + (1-\beta)\|x - z\|^2 - \beta(1-\beta)\|Vx - x\|^2 \\
&= \|x - z\|^2 - \beta(1-\beta)\|Vx - x\|^2.
\end{aligned}$$

□

Theorem 4.3 *The sequence* $\{x_n\}$ *generated by algorithm* (4.1) *converges strongly to a solution of the problem* $\mathcal{P}$.

Proof For any solution $\hat{x}$ of the problem $\mathcal{P}$, according to Lemma 2.5, $\hat{x} \in \operatorname{Fix}(P_C T) = \operatorname{Fix}(P_C) \cap \operatorname{Fix}(T)$, where $T = I - \gamma P^* G^* G P$, and

$$\begin{aligned}
\|x_{n+1} - \hat{x}\| &= \big\|(1-\beta_n)x_n + \beta_n P_C\big[(1-\alpha_n)T\big]x_n - \hat{x}\big\| \\
&= \big\|(1-\beta_n)(x_n - \hat{x}) + \beta_n\big(P_C\big[(1-\alpha_n)T\big]x_n - \hat{x}\big)\big\| \\
&\le (1-\beta_n)\|x_n - \hat{x}\| + \beta_n\big\|P_C\big[(1-\alpha_n)T\big]x_n - \hat{x}\big\| \\
&\le (1-\beta_n)\|x_n - \hat{x}\| \\
&\quad + \beta_n\big\|P_C\big[(1-\alpha_n)T\big]x_n - P_C\big[(1-\alpha_n)T\big]\hat{x}\big\| \\
&\quad + \beta_n\big\|P_C\big[(1-\alpha_n)T\big]\hat{x} - \hat{x}\big\| \\
&\le (1-\beta_n)\|x_n - \hat{x}\| + \beta_n(1-\alpha_n)\|x_n - \hat{x}\| + \beta_n\alpha_n\|\hat{x}\| \\
&= (1-\beta_n\alpha_n)\|x_n - \hat{x}\| + \beta_n\alpha_n\|\hat{x}\| \\
&\le \max\big\{\|x_n - \hat{x}\|, \|\hat{x}\|\big\}.
\end{aligned}$$

One can deduce that

$$\|x_n - \hat{x}\| \le \max\big\{\|x_0 - \hat{x}\|, \|\hat{x}\|\big\}.$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$. Moreover,

$$\begin{aligned}
\big\|P_C\big[(1-\alpha_n)T\big]x_n - \hat{x}\big\| &\le \big\|(1-\alpha_n)Tx_n - \hat{x}\big\| \\
&= \big\|(1-\alpha_n)[Tx_n - \hat{x}] - \alpha_n\hat{x}\big\| \\
&\le (1-\alpha_n)\|x_n - \hat{x}\| + \alpha_n\|\hat{x}\| \\
&\le \max\big\{\|x_n - \hat{x}\|, \|\hat{x}\|\big\}.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we see that $\{Tx_n\}$, $(1-\alpha_n)Tx_n$, and $\{P_C[(1-\alpha_n)T]x_n\}$ are also bounded.

Let $z_n = P_C[(1-\alpha_n)T]x_n$, and $M > 0$ such that $M = \sup_{n\ge1}\{Tx_n\}$. Noting that

$$\begin{aligned}
\big\|P_C\big[(1-\alpha_{n+1})T\big]x_n - P_C\big[(1-\alpha_n)T\big]x_n\big\| &\le \big\|(1-\alpha_{n+1})Tx_n - (1-\alpha_n)Tx_n\big\| \\
&= \big\|(\alpha_n - \alpha_{n+1})Tx_n\big\| \\
&\le M|\alpha_n - \alpha_{n+1}|.
\end{aligned}$$

One concludes that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \big\|P_C\big[(1-\alpha_{n+1})T\big]x_{n+1} - P_C\big[(1-\alpha_n)T\big]x_n\big\| \\
&\le \big\|P_C\big[(1-\alpha_{n+1})T\big]x_{n+1} - P_C\big[(1-\alpha_{n+1})T\big]x_n\big\| \\
&\quad + \big\|P_C\big[(1-\alpha_{n+1})T\big]x_n - P_C\big[(1-\alpha_n)T\big]x_n\big\|
\end{aligned}$$

$$\begin{aligned}&\le (1-\alpha_{n+1})\|x_{n+1}-x_n\| + \big\|P_C\big[(1-\alpha_{n+1})T\big]x_n - P_C\big[(1-\alpha_n)T\big]x_n\big\| \\ &\le (1-\alpha_{n+1})\|x_{n+1}-x_n\| + M|\alpha_n-\alpha_{n+1}|.\end{aligned}$$

Since $0<\alpha_n<1$ and $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n|=0$, we have

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le M|\alpha_n-\alpha_{n+1}|,$$

and

$$\limsup_{n\to\infty}\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le 0.$$

Applying Lemma 1.3, we get

$$\lim_{n\to\infty}\big\|P_C\big[(1-\alpha_n)T\big]x_n - x_n\big\| = \lim_{n\to\infty}\|z_n-x_n\| = 0.$$

Hence,

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|(1-\beta_n)x_n + \beta_n P_C\big[(1-\alpha_n)T\big]x_n - x_n\big\| \\ &= \beta_n\big\|P_C\big[(1-\alpha_n)T\big]x_n - x_n\big\| \to 0.\end{aligned}$$

Let R_n and R be defined by

$$\begin{aligned}&R_n x := P_C\big\{(1-\alpha_n)\big[I-\gamma P^*G^*GP\big]\big\}x = P_C\big[(1-\alpha_n)Tx\big], \\ &Rx := P_C\big(I-\gamma P^*G^*GP\big)x = P_C(Tx).\end{aligned}$$

Noting that

$$\begin{aligned}\|x_n-Rx_n\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-Rx_n\| \\ &= \|x_n-x_{n+1}\| + \big\|(1-\beta_n)x_n + \beta_n R_n x_n - Rx_n\big\| \\ &\le \|x_n-x_{n+1}\| + (1-\beta_n)\|x_n-Rx_n\| + \beta_n\|R_nx_n-Rx_n\|.\end{aligned}$$

So, we have

$$\begin{aligned}\|x_n-Rx_n\| &\le \|x_n-x_{n+1}\|/\beta_n + \|R_nx_n-Rx_n\| \\ &= \|x_n-x_{n+1}\|/\beta_n + \big\|P_C\big[(1-\alpha_n)T\big]x_n - P_CTx_n\big\| \\ &\le \|x_n-x_{n+1}\|/\beta_n + \big\|(1-\alpha_n)Tx_n - Tx_n\big\| \\ &\le \|x_n-x_{n+1}\|/\beta_n + M\alpha_n.\end{aligned}$$

By assumption, we have

$$\lim_{n\to\infty}\|x_n-Rx_n\| = 0.$$

Furthermore, $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to a point $\check{x}$. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to $\check{x}$.

Since $\|Rx_n - x_n\| \to 0$, using the demiclosedness principle we know that $\check{x} \in \mathrm{Fix}(R) = \mathrm{Fix}(P_C T) = \mathrm{Fix}(P_C) \cap \mathrm{Fix}(T) = \Gamma$.

Finally, we will prove that $\lim_n \|x_{n+1} - \check{x}\| = 0$. In fact,

$$\begin{aligned} \|x_{n+1} - \check{x}\|^2 &= \big\|(1-\beta_n)x_n + \beta_n P_C\big[(1-\alpha_n)T\big]x_n - P_C T\check{x}\big\|^2 \\ &\le (1-\beta_n)\|x_n - \check{x}\|^2 + \beta_n \big\|P_C\big[(1-\alpha_n)T\big]x_n - P_C T\check{x}\big\|^2 \\ &\le (1-\beta_n)\|x_n - \check{x}\|^2 + \beta_n \big\|(1-\alpha_n)Tx_n - \check{x}\big\|^2 \\ &= (1-\beta_n)\|x_n - \check{x}\|^2 + \beta_n \big\|(1-\alpha_n)(Tx_n - \check{x}) + \alpha_n\check{x}\big\|^2 \\ &= (1-\beta_n)\|x_n - \check{x}\|^2 + \beta_n\big[(1-\alpha_n)^2\|Tx_n - \check{x}\|^2 + \alpha_n^2\|\check{x}\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle Tx_n - \check{x}, -\check{x}\rangle\big] \\ &\le (1-\beta_n)\|x_n - \check{x}\|^2 + \beta_n\big[(1-\alpha_n)\|x_n - \check{x}\|^2 + \alpha_n^2\|\check{x}\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\langle Tx_n - \check{x}, -\check{x}\rangle\big] \\ &= (1-\alpha_n\beta_n)\|x_n - \check{x}\|^2 + \alpha_n\beta_n\big[2(1-\alpha_n)\langle Tx_n - \check{x}, -\check{x}\rangle + \alpha_n\|\check{x}\|^2\big]. \end{aligned}$$

Using Lemma 1.2, we only need to prove that

$$\limsup_{n\to\infty}\langle Tx_n - \check{x}, -\check{x}\rangle \le 0.$$

Applying Lemma 2.5, T is averaged, that is $T = \beta I + (1-\beta)V$, where $0 < \beta < 1$ and V is nonexpansive. Hence, for $z \in \mathrm{Fix}(P_C T)$, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \big\|(1-\beta_n)x_n + \beta_n P_C\big[(1-\alpha_n)T\big]x_n - z\big\|^2 \\ &\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big\|(1-\alpha_n)Tx_n - z\big\|^2 \\ &= (1-\beta_n)\|x_n - z\|^2 + \beta_n\big\|(1-\alpha_n)(Tx_n - z) - \alpha_n z\big\|^2 \\ &\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big[(1-\alpha_n)\|Tx_n - z\|^2 + \alpha_n\|z\|^2\big] \\ &\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big[\|Tx_n - z\|^2 + \alpha_n\|z\|^2\big]. \end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le (1-\beta_n)\|x_n - z\|^2 \\ &\quad + \beta_n\big[\|x_n - z\|^2 - \beta(1-\beta)\|Vx_n - x_n\|^2 + \alpha_n\|z\|^2\big] \\ &\le \|x_n - z\|^2 - \beta_n\beta(1-\beta)\|Vx_n - x_n\|^2 + \beta_n\alpha_n\|z\|^2. \end{aligned}$$

Let $N > 0$ such that $\|x_n - z\| \le N$ for all n, then it concludes that

$$\begin{aligned} \beta_n\beta(1-\beta)\|Vx_n - x_n\|^2 &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \beta_n\alpha_n\|z\|^2 \\ &\le 2N\big|\|x_n - z\| - \|x_{n+1} - z\|\big| + \beta_n\alpha_n\|z\|^2 \\ &\le 2N\|x_n - x_{n+1}\| + \beta_n\alpha_n\|z\|^2. \end{aligned}$$

Hence,

$$\beta(1-\beta)\|Vx_n - x_n\|^2 \le \frac{2N\|x_n - x_{n+1}\|}{\beta_n} + \alpha_n\|z\|^2.$$

Since $\|x_n - x_{n+1}\| \to 0$, we get

$$\|Vx_n - x_n\| \to 0.$$

Therefore,

$$\|Tx_n - x_n\| \to 0.$$

It follows that

$$\limsup_{n\to\infty}\langle Tx_n - \check{x}, -\check{x}\rangle = \limsup_{n\to\infty}\langle x_n - \check{x}, -\check{x}\rangle.$$

Since $\{x_n\}$ converges weakly to $\check{x}$, it follows that

$$\limsup_{n\to\infty}\langle Tx_n - \check{x}, -\check{x}\rangle \le 0. \qquad \square$$

Similar to the proof of Theorem 4.3, we can get the result that the following iterative algorithm converges strongly to a solution of the problem $\mathcal{P}$ also. Since the proof is similar to Theorem 4.3, we omit it.

Algorithm 4.4 For an arbitrary initial point x_0, sequence $\{x_n\}$ is generated by the iteration:

$$x_{n+1} = (1-\beta_n)(1-\alpha_n)\bigl(I - \gamma P^* G^* GP\bigr)x_n + \beta_n P_C\bigl[(1-\alpha_n)\bigl(I - \gamma P^* G^* GP\bigr)\bigr]x_n, \tag{4.2}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Recently, Eckstein and Svaiter present some splitting methods for finding a zero of the sum of monotone operator *A* and *B*. However, the algorithms are largely dependent on the maximal monotonicity of *A* and *B*. In this paper, we describe some algorithms for finding a zero of the sum of *A* and *B* which ignore the conditions of the maximal monotonicity of *A* and *B*.

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