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Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds

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Abstract

In this paper, we study warped product submanifolds of nearly trans-Sasakian manifolds. The non-existence of warped product semi-slant submanifolds of type $N_\theta \times {}_f N_T$ is shown, whereas some characterization and new geometric obstructions are obtained for the warped products of type $N_T \times_f N_\theta$. We establish two general inequalities for the squared norm of the second fundamental form. The first inequality generalizes derived inequalities for some contact metric manifolds (Kadri *et al.* in J. Korean Math. Soc. 42:1101-1110, 2005; Munteanu in Publ. Math. (Debr.) 66:75-120, 2005; Mustafa *et al.* in Taiwan. J. Math. 17:1473-1486, 2013; Uddin and Khan in J. Inequal. Appl. 2012:304, 2012), while by a new technique, the second inequality is constructed to express the relation between extrinsic invariant (second fundamental form) and intrinsic invariant (scalar curvatures). The equality cases are also discussed.

MSC: 53C40; 53C42; 53C15

Keywords: warped products; almost contact manifold; nearly trans-Sasakian manifold; semi-slant submanifold; scalar curvature; isometric immersion; minimal immersion; N_T-minimal immersion

1 Introduction

In a natural way, warped products appeared in differential geometry generalizing the class of Riemannian product manifolds to a much larger one, called warped product manifolds, which are applied in general relativity to model the standard space time, especially in the neighborhood of massive stars and black holes [1, 2]. These manifolds were introduced by Bishop and O'Neill [3]. They defined warped products as follows: Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and let $f > 0$ be a differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then their warped product manifold $M = N_1 \times_f N_2$ is the Riemannian manifold $N_1 \times N_2 = (N_1 \times N_2, g)$ equipped with the Riemannian structure such that

$$\|X\|^2 = \left\|\pi_{1\star}(X)\right\|^2 + (f \circ \pi_1)^2 \left\|\pi_{2\star}(X)\right\|^2$$

for any vector field X tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = N_1 \times N_2$ is said to be *trivial* or simply *Riemannian product* if the warping function f is constant. For the survey on warped products as Riemannian submanifolds, we refer to [4, 5].

A $(2m + 1)$-dimensional C^∞ manifold $(\bar{M}, g, \phi, \xi, \eta)$ is said to have an *almost contact structure* if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g satisfying [6]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1, \tag{1.1}$$

$$\eta(X) = g(X, \xi), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$

where X and Y are vector fields on $\bar{M}$ [7]. We shall use the symbol $\Gamma(T\bar{M})$ to denote the Lie algebra of vector fields on the manifold $\bar{M}$.

In the classification of almost contact structures, Chinea and Gonzalez [8] divided these structures into twelve well-known classes; one of the class that appears in this classification is denoted by $C_1 \oplus C_5 \oplus C_6$. According to their classification, an almost contact metric manifold is a nearly trans-Sasakian manifold if it belongs to this class. Another line of thought was developed by Gherghe [9] who introduced nearly trans-Sasakian structure of type (α, β), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian one. In this sense an almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *nearly trans-Sasakian structure* if

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X &= \alpha\bigl(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\bigr) \\ &\quad - \beta\bigl(\eta(Y)\phi X + \eta(X)\phi Y\bigr)\end{aligned} \tag{1.3}$$

for any $X, Y \in \Gamma(T\bar{M})$. Moreover, nearly trans-Sasakian of type (α, β) is nearly-Sasakian, or nearly Kenmotsu, or nearly cosymplectic accordingly as $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$.

Kim *et al.* [10] initiated the study of semi-invariant submanifolds of nearly trans-Sasakian manifolds and obtained many results on the extrinsic geometric aspects of these submanifolds, whereas the slant submanifolds were studied in the setting of nearly trans-Sasakian manifolds by Al-Solamy and Khan [11]. Recently, we have initiated the study of CR-warped product in nearly trans-Sasakian manifolds [12]. In the present paper, we consider a warped product of proper slant and invariant submanifolds of nearly trans-Sasakian manifolds, called warped product semi-slant submanifolds. The paper is organized as follows. Section 2 is devoted to providing the basic definitions and formulas which are useful to the next section. In Section 3, general and special non-existence results are proved for warped products. In the case of existence of warped products, the necessary lemmas for the two inequalities and some geometric obstructions are obtained. In Section 4, a general inequality which generalizes the obtained inequalities in [12–15] is established. In Section 5, we develop a new technique to construct a general inequality for the second fundamental form in terms of the scalar curvatures of submanifolds and the warping function.

2 Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$, where ∇ is the induced Riemannian connection on M, N is a vector field normal to $\bar{M}$, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of the second fundamental form. They are related as

$$g(A_N X, Y) = g\big(h(X, Y), N\big), \tag{2.3}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M. For any $X \in \Gamma(TM)$, we decompose ϕX as follows:

$$\phi X = PX + FX, \tag{2.4}$$

where PX and FX are the tangential and normal components of ϕX, respectively.

For a submanifold M of an almost contact manifold $\bar{M}$, if F is identically zero then M is *invariant*, and if P is identically zero then M is *anti-invariant*.

For the orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_xM, the *mean curvature vector* $\vec{H}(x)$ is given by

$$\vec{H}(x) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $n = \dim(M)$. The submanifold M is *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in \Gamma(TM)$, then M is *totally umbilical*.

Let (M, g) be a submanifold of a Riemannian manifold $\bar{M}$ equipped with a Riemannian metric g. The *equation of Gauss* is given by

$$\begin{aligned} R(X, Y, Z, W) = {} & \bar{R}(X, Y, Z, W) + g\big(h(X, W), h(Y, Z)\big) \\ & - g\big(h(X, Z), h(Y, W)\big) \end{aligned} \tag{2.5}$$

for all $X, Y, Z, W \in \Gamma(TM)$, where $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M, respectively, and h is the second fundamental form.

Definition 2.1 [16] An immersion $\varphi : N_1 \times_f N_2 \to \bar{M}$ is called N_i*-totally geodesic* if the partial second fundamental form h_i vanishes identically. It is called N_i*-minimal* if the partial mean curvature vector $\vec{H}_i$ vanishes for $i = 1, 2$.

The *scalar curvature* $\tau(x)$ of M is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j), \tag{2.6}$$

where $K(e_i \wedge e_j)$ is the *sectional curvature* of the plane section spanned by e_i and e_j at $x \in M$. Let Π_k be a k-plane section of T_xM, and let $\{e_1, \ldots, e_k\}$ be any orthonormal basis of Π_k. The scalar curvature $\tau(\Pi_k)$ of Π_k is given by [16]

$$\tau(\Pi_k) = \sum_{1\leq i<j\leq k} K(e_i \wedge e_j).$$

The scalar curvature of $\tau(x)$ of M at x is identical with the scalar curvature of the tangent space T_xM of M at x, that is, $\tau(x) = \tau(T_xM)$. Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_x(\Pi_k)$ of Π_k at x under the exponential map at x. If Π_2 is a 2-plane section, $\tau(\Pi_2)$ is simply the sectional curvature $K(\Pi_2)$ of Π_2, [4, 16, 17].

Now, let us put

$$h_{ij}^r = g\big(h(e_i, e_j), e_r\big), \tag{2.7}$$

where $i, j \in \{1, \dots, n\}$ and $r \in \{n+1, \dots, 2m+1\}$. Then, in view of the equation of Gauss, we have

$$K(e_i \wedge e_j) = \bar{K}(e_i \wedge e_j) + \sum_{r=n+1}^{2m+1} \big(h_{ii}^r h_{jj}^r - \big(h_{ij}^r\big)^2\big), \tag{2.8}$$

where $K(e_i \wedge e_j)$ and $\bar{K}(e_i \wedge e_j)$ denote the sectional curvature of the plane section spanned by e_i and e_j at x in the submanifold M and in the ambient manifold $\bar{M}$, respectively. Taking the summation over the orthonormal frame of the tangent space of M in the above equation, we obtain

$$2\tau(x) = 2\bar{\tau}(T_xM) + n^2\|H\|^2 - \|h\|^2, \tag{2.9}$$

where $\bar{\tau}(T_xM) = \sum_{1\leq i<j\leq n} \bar{K}(e_i \wedge e_j)$ denotes the scalar curvature of the n-plane section T_xM for each $x \in M$ in the ambient manifold $\bar{M}$.

There are different classes of submanifolds which we introduce briefly such as slant submanifolds, CR-submanifolds and semi-slant submanifolds. We shall always consider ξ to be tangent to the submanifold M. For a slant submanifold M, there is a non-zero vector X tangent to M at x such that X is not proportional to ξ_x. We denote by $0 \leq \theta(X) \leq \pi/2$ the angle between ϕX and T_xM called the Wirtinger angle. If the Wirtinger angle $\theta(X)$ is constant for all $X \in T_xM - \langle\xi_x\rangle$ and $x \in M$, then M is said to be a *slant submanifold* and the angle $\theta(X)$ is called the *slant angle* of M [18]. Obviously, if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

We recall the following result for a slant submanifold of an almost contact metric manifold.

Theorem 2.1 [18] *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.10}$$

Furthermore, if θ is a slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of equation (2.10)

$$g(PX, PY) = \cos^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big), \tag{2.11}$$

$$g(FX, FY) = \sin^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big) \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$.

The idea of semi-slant submanifolds of almost Hermitian manifolds was given by Papaghuic [19]. In fact, semi-slant submanifolds were defined on the line of CR-submanifolds. These submanifolds are defined and investigated by Cabrerizo *et al.* for almost contact manifolds [20]. They defined these submanifolds as follows.

Definition 2.2 [20, 21] A submanifold M of an almost contact manifold $\bar{M}$ is said to be a semi-slant submanifold if there exist two orthogonal distributions D and D_θ such that:

(i) $TM = D \oplus D_\theta \oplus \langle\xi\rangle$.
(ii) D is invariant, *i.e.*, $\phi D \subseteq D$.
(iii) D_θ is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$.

In the above definition, if $\theta = \pi/2$ then M is contact CR-submanifold of $\bar{M}$. If ν is the invariant subspace of the normal bundle $T^\perp M$, then in case of semi-slant submanifolds, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = FD_\theta \oplus \nu. \tag{2.13}$$

For the differential function ψ on M, the gradient $\operatorname{grad}\psi$ and the Laplacian $\Delta\psi$ of ψ are defined respectively by

$$g(\operatorname{grad}\psi, X) = X\psi, \tag{2.14}$$

$$\Delta\psi = \sum_{i=1}^{n}\big((\nabla_{e_i}e_i)\psi - e_ie_i\psi\big) \tag{2.15}$$

for any vector field X tangent to M, where ∇ denotes the Riemannian connection on M.

3 Warped product submanifolds

In this section, we study warped product submanifolds of nearly trans-Sasakian manifolds. We recall the following results on warped products for later use.

Lemma 3.1 *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f. Then*

(i) $\nabla_X Y \in \Gamma(TN_1)$,
(ii) $\nabla_X Z = \nabla_Z X = (X\ln f)Z$,
(iii) $\nabla_Z W = \nabla_Z^{N_2} W - (g(Z, W)/f)\operatorname{grad} f$

for any $X, Y \in \Gamma(TN_1)$ and $Z, W \in \Gamma(TN_2)$, where ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2, respectively, and grad$f$ *is the gradient of f.*

Corollary 3.1 *On a warped product manifold $M = N_1 \times_f N_2$, we have*:

(i) *N_1 is totally geodesic in M,*
(ii) *N_2 is totally umbilical in M.*

In the following, we prove the non-existence of warped products of the form $M = N_1 \times_f N_2$ in a nearly trans-Sasakian manifold such that ξ is tangent to N_2.

Theorem 3.1 *Let $\bar{M}$ be a nearly trans-Sasakian manifold which is not nearly Sasakian, and let $M = N_1 \times_f N_2$ be a warped product submanifold of $\bar{M}$ such that ξ is tangent to N_2, then M is simply a Riemannian product of N_1 and N_2, where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$.*

Proof For any $X \in \Gamma(TN_1)$, we have $(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta\phi X$. Since for a contact metric manifold $\bar{M}$, $(\bar{\nabla}_\xi \phi)X = 0$ [22], hence we get

$$\phi\bar{\nabla}_X\xi = \alpha X + \beta\phi X. \tag{3.1}$$

Taking the inner product with ϕX in (3.1) and using Lemma 3.1(ii) and the fact that ξ is tangent to N_2, we get $\beta\|X\|^2 = 0$. This means that the first factor of the warped product vanishes, which proves the theorem completely. □

In view of the above theorem, we get a non-existence result about the warped product semi-slant submanifolds in a nearly trans-Sasakian manifold, *i.e.*, there do not exist warped product semi-slant submanifolds $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ of a nearly trans-Sasakian manifold when the characteristic vector field ξ is a tangent to the second factor. Now, we show that the warped products of type $N_\theta \times_f N_T$ are also Riemannian products if ξ is tangent to the first factor.

Theorem 3.2 *There do not exist warped product semi-slant submanifolds of type $M = N_\theta \times_f N_T$ of a nearly trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_θ, unless $\bar{M}$ is nearly β-Kenmotsu.*

Proof Consider an arbitrary vector X tangent to N_T, then making use of (1.3) it follows $(\bar{\nabla}_X \phi)\xi + (\bar{\nabla}_\xi \phi)X = -\alpha X - \beta\phi X$. Since $(\bar{\nabla}_\xi \phi)X = 0$, for any $X \in \Gamma(T\bar{M})$, thus this relation can be simplified as

$$-\phi\bar{\nabla}_X\xi = -\alpha X - \beta\phi X. \tag{3.2}$$

Taking the inner product with X in (3.2), we get

$$g(\bar{\nabla}_X\xi, \phi X) = -\alpha\|X\|^2. \tag{3.3}$$

By orthogonality of the vector fields X and ϕX and by Lemma 3.1(ii), the left-hand side of (3.3) vanishes identically, hence we reach $\alpha\|X\|^2 = 0$, this means that the first factor of the warped product $N_\theta \times_f N_T$ vanishes, which proves the theorem. □

From the above discussion, we conclude that there do not exist warped product semi-slant submanifolds of type $N_\theta \times_f N_T$ in a nearly trans-Sasakian manifold $\bar{M}$ in both the cases either ξ is tangent to the first factor or to the second. Also, the warped product $N_T \times_f N_\theta$ is just a Riemannian product when the characteristic vector field ξ is tangent to N_θ. Now, we discuss the warped product submanifolds $N_T \times_f N_\theta$ such that ξ is tangent to N_T.

First, we prove a key lemma characterizing geometric properties of the warped product submanifolds $N_T \times_f N_\theta$ of a nearly trans-Sasakian manifold $\bar{M}$.

Lemma 3.2 *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_T. Then the following relations hold*:

(i) $\xi \ln f = \beta$,

(ii) $g(h(X,Y), FZ) = 0$,

(iii) $g(h(\xi, Z), FW) = -\alpha g(Z, W)$,

(iv) $g(h(X,Z), FZ) = -\{(\phi X \ln f) + \alpha\eta(X)\}\|Z\|^2$,

(v) $g(h(X,Z), FPZ) = -g(h(X,PZ), FZ) = \frac{1}{3}\cos^2\theta\{(X\ln f) - \beta\eta(X)\}\|Z\|^2$,

(vi) $g(h(X,X),\zeta) = -g(h(\phi X, \phi X),\zeta)$

for any $X, Y \in \Gamma(TN_T)$ and for any $Z, W \in \Gamma(TN_\theta)$ and $\zeta \in \Gamma(\nu)$.

Proof The first three parts can be proved by the same way as we have proved for contact CR-warped products in [12]. Now, as we consider ξ is tangent to N_T, then for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, we have

$$(\bar{\nabla}_X\phi)Z + (\bar{\nabla}_Z\phi)X = -\alpha\eta(X)Z - \beta\eta(X)\phi Z.$$

Taking the inner product with Z, we obtain

$$g\big((\bar{\nabla}_X\phi)Z + (\bar{\nabla}_Z\phi)X, Z\big) = -\alpha\eta(X)\|Z\|^2. \tag{3.4}$$

Also, we have

$$\begin{aligned}(\bar{\nabla}_X\phi)Z &= \bar{\nabla}_X\phi Z - \phi\bar{\nabla}_X Z\\ &= \nabla_X PZ + h(X,PZ) - A_{FZ}X + \nabla^\perp_X FZ - \phi\nabla_X Z - \phi h(X,Z).\end{aligned}$$

Taking the inner product with Z and using Lemma 3.1(ii), we obtain

$$g\big((\bar{\nabla}_X\phi)Z, Z\big) = 0. \tag{3.5}$$

Similarly, we can obtain

$$g\big((\bar{\nabla}_Z\phi)X, Z\big) = (\phi X\ln f)\|Z\|^2 + g\big(h(X,Z), FZ\big). \tag{3.6}$$

Then from (3.4), (3.5) and (3.6) we obtain part (iv) of the lemma. Now, from the structure equation (1.3) and Lemma 3.1(ii), we have

$$g\big((\bar{\nabla}_X\phi)PZ + (\bar{\nabla}_{PZ}\phi)X, Z\big) = \beta\eta(X)\cos^2\theta\|Z\|^2 \tag{3.7}$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$ such that ξ is tangent to N_T. Again, by Lemma 3.1(ii) and the Gauss-Weingarten formulas, we obtain

$$g\big((\bar{\nabla}_X\phi)PZ, Z\big) = g\big(h(X,PZ), FZ\big) - g\big(h(X,Z), FPZ\big) \tag{3.8}$$

and

$$g\big((\bar{\nabla}_{PZ}\phi)X, Z\big) = g\big(h(X,PZ), FZ\big) + (X\ln f)\cos^2\theta\|Z\|^2. \tag{3.9}$$

Thus from (3.7), (3.8) and (3.9) we derive

$$2g\big(h(X,PZ),FZ\big) - g\big(h(X,Z),FPZ\big) = \big\{\beta\eta(X) - (X\ln f)\big\}\cos^2\theta\|Z\|^2. \tag{3.10}$$

Interchanging Z by PZ in (3.10), we obtain

$$-2g\big(h(X,Z),FPZ\big) + g\big(h(X,PZ),FZ\big) = \big\{\beta\eta(X) - (X\ln f)\big\}\cos^2\theta\|Z\|^2. \tag{3.11}$$

Then, by (3.10) and (3.11), we get

$$g\big(h(X,PZ),FZ\big) = -g\big(h(X,Z),FPZ\big), \tag{3.12}$$

which is the first equality of the fifth part of the lemma. The second equality of (v) follows from (3.10) and (3.12). For the last part of the lemma, for any $X \in \Gamma(TN_T)$, we have $\bar{\nabla}_X\phi X - \phi\bar{\nabla}_X X = \alpha\|X\|^2\xi - \eta(X)X - \beta\eta(X)\phi X$. By means of (2.1), this relation reduces to

$$\nabla_X\phi X + h(\phi X,X) - \phi\nabla_X X - \phi h(X,X) = \alpha\|X\|^2\xi - \alpha\eta(X)X - \beta\eta(X)\phi X.$$

Taking the inner product in the above equation with $\phi\zeta$, for any vector $\zeta \in \Gamma(\nu)$, we deduce that

$$g\big(h(\phi X,X),\phi\zeta\big) - g\big(h(X,X),\zeta\big) = 0. \tag{3.13}$$

Interchanging X by ϕX in the above equation and making use of (1.1) and the fact that ν is an invariant normal subbundle of $T^\perp M$, we have

$$-g\big(h(X,\phi X),\phi\zeta\big) + \eta(X)g\big(h(\xi,\phi X),\phi\zeta\big) = g\big(h(\phi X,\phi X),\zeta\big). \tag{3.14}$$

Now, by means of (1.3), we derive

$$h(\phi X,\xi) - 2\phi h(X,\xi) - \phi\nabla_X\xi = \alpha\big(\eta(X)\xi - X\big) - \beta\phi X. \tag{3.15}$$

Taking the inner product with $\phi\zeta$ in (3.15), we obtain

$$g\big(h(\phi X,\xi),\phi\zeta\big) - 2g\big(h(X,\xi),\zeta\big) = 0.$$

Interchanging ζ by $\phi\zeta$ in the first step and X by ϕX in the second one, taking in consideration that $h(\xi,\xi) = 0$, we obtain the following couple of tensorial relations:

$$g\big(h(\phi X,\xi),\zeta\big) + 2g\big(h(X,\xi),\phi\zeta\big) = 0 \tag{3.16}$$

and

$$g\big(h(X,\xi),\phi\zeta\big) + 2g\big(h(\phi X,\xi),\zeta\big) = 0. \tag{3.17}$$

From (3.16) and (3.17) we deduce that

$$g\big(h(X,\xi),\phi\zeta\big) = g\big(h(\phi X,\xi),\zeta\big). \tag{3.18}$$

In view of (3.17) and (3.18), we get $g(h(X,\xi),\phi\zeta)=0$. Again, interchanging X by ϕX in this relation yields

$$g\big(h(\phi X,\xi),\phi\zeta\big)=0. \tag{3.19}$$

Then, by (3.14) and (3.19), we reach

$$-g\big(h(\phi X,X),\phi\zeta\big)-g\big(h(\phi X,\phi X),\zeta\big)=0. \tag{3.20}$$

Thus from (3.13) and (3.20) we get the assertion. □

4 An inequality for warped product submanifolds $N_T \times_f N_\theta$

In the setting of almost contact structures, many authors have proved general inequalities in terms of the squared norm of the second fundamental form and the gradient of the warping function in various structures [12–15]. In fact, all these inequalities are the extension of the original inequality constructed by Chen in the almost Hermitian setting [23]. However, no one proved this relation for warped product semi-slant submanifolds. For this reason, our inequality generalizes the inequalities obtained for CR-warped products in the almost contact setting. Another reason is that a nearly trans-Sasakian structure includes all almost contact structures as a special case.

From now on, we shall follow the following orthonormal basis frame of the ambient manifold $\bar{M}$ for the warped product semi-slant submanifold $M=N_T\times_f N_\theta$ such that ξ is tangent to N_T. We shall denote by D and D_θ the tangent spaces of N_T and N_θ, respectively, instead of TN_T and TN_θ. We set $\{e_1,\ldots,e_s,e_{s+1}=\phi e_1,\ldots,e_{(n_1-1=2s)}=\phi e_s,e_{(n_1=2s+1)}=\xi,e_{n_1+1}=e_1^\star,\ldots,e_{n_1+q}=e_q^\star,e_{n_1+q+1}=e_{q+1}^\star=\sec\theta Pe_1^\star,\ldots,e_{(n=n_1+n_2)}=e_{(n_2=2q)}^\star=\sec\theta Pe_q^\star,e_{n+1}=\csc\theta Fe_1^\star,\ldots,e_{n+n_2}=\csc\theta Fe_{n_2}^\star,e_{n+n_2+1}=\bar{e}_1,\ldots,e_{2m+1}=\bar{e}_{2l}\}$ as a basis frame of $T\bar{M}$, then $\{e_1,\ldots,e_s,e_{s+1}=\phi e_1,\ldots,e_{n_1-1}=\phi e_s,e_{n_1}=\xi,e_{n_1+1}=e_1^\star,\ldots,e_{n_1+q}=e_q^\star,e_{n_1+q+1}=e_{q+1}^\star=\sec\theta Pe_1^\star,\ldots,e_{(n=n_1+n_2)}=e_{(n_2=2q)}^\star=\sec\theta Pe_q^\star\}$ are the basis of TM such that $e_1,\ldots,e_s$, $e_{s+1}=\phi e_1,\ldots,e_{n_1-1}=\phi e_s,e_{n_1}=\xi$ are tangent to D and $e_1^\star,\ldots,e_q^\star,e_{q+1}^\star=\sec\theta Pe_1^\star,\ldots$, $e_{(n_2=2q)}^\star=\sec\theta Pe_q^\star$ are tangent to D_θ, hence $\{e_{n+1}=\csc\theta Fe_1^\star,\ldots,e_{n+n_2}=\csc\theta Fe_{n_2}^\star,e_{n+n_2+1}=\bar{e}_1,\ldots,e_{2m+1}=\bar{e}_{2l}\}$ are the basis of the normal bundle $T^\perp M$ such that $e_{n+1}=\csc\theta Fe_1^\star,\ldots$, $e_{n+n_2}=\csc\theta Fe_{n_2}^\star$ are tangent to FD_θ and $e_{n+n_2+1}=\bar{e}_1,\ldots,e_{2m+1}=\bar{e}_{2l}$ are tangent to the invariant normal subbundle ν with dimension $2l$. We use this frame in the following theorem.

Theorem 4.1 *Let $M=N_T\times_f N_\theta$ be a warped product semi-slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_T, where N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$ with real dimensions $2s+1$ and $2q$, respectively. Then*

(i) *The second fundamental form h of M satisfies the following inequality*:

$$\|h\|^2\geq 2q\left[\left\{\frac{2}{9}\cot^2\theta+2\csc^2\theta\right\}\big(\|\operatorname{grad}(\ln f)\|^2-\beta^2\big)+\alpha^2\right]. \tag{4.1}$$

(ii) *If the equality sign in* (i) *holds identically, then N_T and N_θ are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.*

Proof In view of the adopted frame and the definition of the second fundamental form, it is straightforward to get the following expansion:

$$\begin{aligned}
\|h\|^2 &= \sum_{i,j=1}^{n} g\big(h(e_i,e_j),h(e_i,e_j)\big) = \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n} g\big(h(e_i,e_j),e_r\big)^2 \\
&= \sum_{r=n+1}^{n+n_2}\sum_{i,j=1}^{n} g\big(h(e_i,e_j),e_r\big)^2 + \sum_{r=n+n_2+1}^{2m+1}\sum_{i,j=1}^{n} g\big(h(e_i,e_j),e_r\big)^2 \\
&\geq \sum_{r=n+1}^{n+n_2}\sum_{i,j=1}^{n} g\big(h(e_i,e_j),e_r\big)^2 = \sum_{l=2s+2}^{n}\sum_{i,j=1}^{n} g\big(h(e_i,e_j),\phi e_l\big)^2.
\end{aligned}$$

Using the orthonormal frame of D and D_θ gives

$$\begin{aligned}
\|h\|^2 \geq{}& \sum_{l=2s+2}^{n}\sum_{i,j=1}^{2s+1} g\big(h(e_i,e_j),\phi e_l\big)^2 + 2\sum_{j,l=2s+2}^{n}\sum_{i=1}^{2s+1} g\big(h(e_i,e_j),\phi e_l\big)^2 \\
&+ \sum_{i,j,l=2s+2}^{n} g\big(h(e_i,e_j),\phi e_l\big)^2.
\end{aligned} \tag{4.2}$$

By Lemma 3.2(ii), the first term of the right-hand side in (4.2) is identically zero, so let us compute the next term

$$\|h\|^2 \geq 2\sum_{j,l=2s+2}^{n}\sum_{i=1}^{2s} g\big(h(e_i,e_j),\phi e_l\big)^2 + 2\sum_{j,l=2s+2}^{n} g\big(h(\xi,e_j),\phi e_l\big)^2. \tag{4.3}$$

Making use of Lemma 3.2(iii), the second term of the right-hand side in (4.3) can be evaluated, while by means of the orthonormal frame the first term is expanded to give four terms; as a result (4.3) takes the following form:

$$\begin{aligned}
\|h\|^2 \geq{}& 2\csc^2\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} g\big(h(e_i,e_j),Fe_j^\star\big)^2 \\
&+ 2\csc^2\theta\sec^2\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} g\big(h\big(e_i,Pe_j^\star\big),Fe_j^\star\big)^2 \\
&+ 2\csc^2\theta\sec^2\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} g\big(h(e_i,e_j),FPe_j^\star\big)^2 \\
&+ 2\csc^2\theta\sec^4\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} g\big(h\big(e_i,Pe_j^\star\big),FPe_j^\star\big)^2 + 2\sum_{j,l=2s+2}^{n}\big(-\alpha g(e_j,e_l)\big)^2.
\end{aligned} \tag{4.4}$$

Using Lemma 3.2(iii)-(v), we derive

$$\begin{aligned}
\|h\|^2 \geq{}& 2\csc^2\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} \big((\phi e_i \ln f) + \alpha\eta(e_i)\big)^2 \|e_j\|^4 \\
&+ \frac{2}{9}\cos^2\theta\csc^2\theta \sum_{j=1}^{q}\sum_{i=1}^{2s} \big((e_i \ln f) - \beta\eta(e_i)\big)^2 \|e_j\|^4
\end{aligned}$$

$$+ \frac{2}{9}\cos^2\theta\csc^2\theta\sum_{j=1}^{q}\sum_{i=1}^{2s}\big((e_i\ln f) - \beta\eta(e_i)\big)^2\|e_j\|^4$$
$$+ 2\csc^2\theta\sum_{j=1}^{q}\sum_{i=1}^{2s}\big((\phi e_i\ln f) + \alpha\eta(e_i)^2\big)\|e_j\|^4 + 2q\alpha^2. \tag{4.5}$$

In view of the assumed orthonormal frame, the 1-form $\eta(e_i)$ is identically zero for all $i \in \{1,\dots,2s\}$, hence we reach

$$\|h\|^2 \geq 4\csc^2\theta\sum_{j=1}^{q}\sum_{i=1}^{2s}(\phi e_i\ln f)^2\|e_j\|^4$$
$$+ \frac{4}{9}\csc^2\theta\cos^2\theta\sum_{j=1}^{q}\sum_{i=1}^{2s}(e_i\ln f)^2\|e_j\|^4 + 2q\alpha^2. \tag{4.6}$$

Then from (2.14) and Lemma 3.2(i) the above inequality takes the form

$$\|h\|^2 \geq 2q\left[\left\{\frac{2}{9}\cot^2\theta + 2\csc^2\theta\right\}\big(\|\nabla\ln f\|^2 - \beta^2\big) + \alpha^2\right],$$

which is the inequality (i). Now, assume that the equality sign in (4.1) holds identically, then from (4.2), (4.3) and Lemma 3.2(ii) we deduce that

$$h(D,D) = 0, \qquad h(D_\theta,D_\theta) = 0, \qquad h(D,D_\theta) \subset FD_\theta. \tag{4.7}$$

Hence, combining statement of Corollary 3.1(i) with the first condition in (4.7) shows that N_T is totally geodesic in $\bar{M}$. On the other hand, if we denote by h^θ the second fundamental form of N_θ in M, then we get

$$g\big(h^\theta(Z,W),X\big) = g(\nabla_Z W,X) = -(X\ln f)g(Z,W) = -g(Z,W)g(\nabla\ln f,X),$$

which is equivalent to

$$h^\theta(Z,W) = -\nabla\ln f g(Z,W). \tag{4.8}$$

This means that N_θ is totally umbilical in M, thus the second condition of (4.7) with (4.8) and Corollary 3.1(ii) imply that N_θ is totally umbilical in $\bar{M}$. Also, all three conditions of (4.7) give the minimality of M. □

Note In inequality (5.1), if $\alpha = 0$ and $\beta = 1$, then it reduces to

$$\|h\|^2 \geq 2q\left[\left\{\frac{2}{9}\cot^2\theta + 2\csc^2\theta\right\}\big(\|\nabla\ln f\|^2 - 1\big)\right],$$

which is the inequality for nearly Kenmotsu manifolds. Also, if $\alpha = 1$ and $\beta = 0$, then the inequality reduces for the nearly Sasakian manifolds. The equality cases can also be discussed.

Remark 1 Theorem 3.1 in [13], Theorem 3.4 in [14] and Theorem 3.2 in [15] are the special cases of the above inequality.

Remark 2 The above inequality generalizes Theorem 4.1 in [12].

5 Another inequality for warped products

Let $\varphi : M = N_1 \times_f N_2 \longrightarrow \bar{M}$ be an isometric immersion of the warped product $N_1 \times_f N_2$ into the Riemannian manifold $\bar{M}$ of constant sectional curvature c. Denote by n_1, n_2, n the dimensions of $N_1, N_2, N_1 \times_f N_2$, respectively. Then for unit vector fields X, Z tangent to N_1, N_2, respectively, we have

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{(\nabla_X X)f - X^2 f\}. \tag{5.1}$$

If we choose the local orthonormal frame $e_1, \dots, e_n$ such that $e_1, \dots, e_{n_1}$ are tangent to N_1 and $e_{n_1+1}, \dots, e_n$ are tangent to N_2, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n_1} K(e_i \wedge e_j) \tag{5.2}$$

for each $j = n_1 + 1, \dots, n$.

In this section, our aim is to develop a new method which is giving a useful formula for the squared norm of the mean curvature vector $\vec{H}$ under φ. Geometrically, this formula declares the N_T-minimality of φ.

We know that

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} \left(h_{11}^r + \cdots + h_{nn}^r\right)^2.$$

Taking in consideration that $(n = n_1 + n_2)$, where n_1 and n_2 are the dimensions of N_T and N_θ, respectively, we obtain

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} \left(h_{11}^r + \cdots + h_{n_1 n_1}^r + h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2.$$

Moreover, for every $r \in \{n+1, \dots, 2m+1\}$, using the frame of D and the fact that $h(\xi, \xi) = 0$, then n_1 coefficients of the right-hand side can be decomposed as follows:

$$\begin{aligned}
&\left(h_{11}^r + \cdots + h_{n_1 n_1}^r + h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2 \\
&\quad = \left(h_{11}^r + \cdots + h_{ss}^r + h_{s+1 s+1}^r + \cdots + h_{2s2s}^r + h_{\xi\xi}^r + h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2 \\
&\quad = \left(h_{11}^r + \cdots + h_{ss}^r + h_{s+1 s+1}^r + \cdots + h_{2s2s}^r + h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2.
\end{aligned} \tag{5.3}$$

From (2.7) we know that e_r belongs to the normal bundle $TM^\perp$ for every $r \in \{n+1, \dots, 2m+1\}$. Then in view of (2.13) we have two cases: either it belongs to FD_θ or to ν.

Case (i). If $e_r \in \Gamma(FD_\theta)$, then from Lemma 3.2(ii) we know that $g(h(X,X), FZ) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D_\theta)$; consequently (5.3) reduces to

$$\left(h_{11}^r + \cdots + h_{n_1 n_1}^r + h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2 = \left(h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r\right)^2. \tag{5.4}$$

Case (ii). If $e_r \in \Gamma(\nu)$, then by means of Lemma 3.2(vi), we can make an expansion of (5.3) as follows:

$$\begin{aligned}
&\left(h_{11}^r + \cdots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2 \\
&\quad = \left(g\left(h(e_1,e_1),e_r\right) + \cdots + g\left(h(e_s,e_s),e_r\right) + g\left(h(\phi e_1,\phi e_1),e_r\right)\right. \\
&\qquad \left. + \cdots + g\left(h(\phi e_s,\phi e_s),e_r\right) + h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2 \\
&\quad = \left(g\left(h(e_1,e_1),e_r\right) + \cdots + g\left(h(e_s,e_s),e_r\right) - g\left(h(e_1,e_1),e_r\right) - \cdots\right. \\
&\qquad \left. - g\left(h(e_s,e_s),e_r\right) + h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2 \\
&\quad = \left(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2.
\end{aligned} \tag{5.5}$$

Then from (5.4) and (5.5) we can deduce that

$$\left(h_{11}^r + \cdots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2 = \left(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2$$

for every normal vector e_r belongs to the normal bundle $T^{\perp}M$. In other words,

$$\sum_{r=n+1}^{2m+1} \left(h_{11}^r + \cdots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2 = \sum_{r=n+1}^{2m+1} \left(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2.$$

By the end of this discussion, we can state the following lemma.

Lemma 5.1 *Let $\varphi : M = N_T \times_f N_\theta \longrightarrow \bar{M}$ be an isometric immersion from a warped product semi-slant submanifold into a nearly trans-Sasakian manifold $\bar{M}$. Then we have*

$$\|\vec{H}\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} \left(h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r\right)^2,$$

i.e., φ is an N_T-minimal immersion, where $\vec{H}$ is the mean curvature vector and n_1, n_2, n and $(2m+1)$ are the dimensions of N_T, N_θ, M and $\bar{M}$, respectively.

From the Gauss equation and the above key Lemma 5.1, we are able to state and prove the following general inequality.

Theorem 5.1 *Let $\varphi : M = N_T \times_f N_\theta \longrightarrow \bar{M}$ be an isometric immersion from a warped product semi-slant submanifold into a nearly trans-Sasakian manifold $\bar{M}$ such that ξ is tangent to N_T. Then we have*

(i) $\frac{1}{2}\|h\|^2 \geq \bar{\tau}(TM) - \bar{\tau}(TN_T) - \bar{\tau}(TN_\theta) - \frac{n_2 \Delta f}{f}$, *where n_2 is the dimension of N_θ.*

(ii) *If the equality sign in* (i) *holds identically, then N_T and N_θ are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.*

Proof We start by recalling (2.9) as a consequence of (2.5) as

$$\|h\|^2 = -2\tau + 2\bar{\tau}(TM) + n^2\|H\|^2.$$

Making use of (2.6) in the above equation, we deduce

$$\|h\|^2 = -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i\wedge e_j) - 2\tau(TN_T) - 2\tau(TN_\theta) + 2\bar{\tau}(TM) + n^2\|H\|^2.$$

Then from Lemma 3.2 and relation (2.8) it follows

$$\begin{aligned}\|h\|^2 ={}& -\frac{2n_2\Delta f}{f} - 2\bar{\tau}(TN_T) - 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<k\le n_1}\left(h^r_{ii}h^r_{kk} - \left(h^r_{ik}\right)^2\right) - 2\bar{\tau}(TN_\theta)\\ & - 2\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j<t\le n}\left(h^r_{jj}h^r_{tt} - \left(h^r_{jt}\right)^2\right) + 2\bar{\tau}(TM) + n^2\|H\|^2.\end{aligned}$$

The above equation is equivalent to the following form:

$$\begin{aligned}\|h\|^2 ={}& -\frac{2n_2\Delta f}{f} - 2\bar{\tau}(TN_T) - \sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}\left(h^r_{ii}h^r_{kk} - \left(h^r_{ik}\right)^2\right) - 2\bar{\tau}(TN_\theta)\\ & + 2\bar{\tau}(TM) - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}\left(h^r_{jj}h^r_{tt} - \left(h^r_{jt}\right)^2\right) + n^2\|H\|^2.\end{aligned}$$

The above equation takes the following form when we add and subtract the same term on the right-hand side:

$$\begin{aligned}\|h\|^2 ={}& -\frac{2n_2\Delta f}{f} - 2\bar{\tau}(TN_T) - \sum_{r=n+1}^{2m+1}\left(\left(h^r_{11}\right)^2 + \cdots + \left(h^r_{n_1n_1}\right)^2\right)\\ & - \sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}\left(h^r_{ii}h^r_{kk} - \left(h^r_{ik}\right)^2\right) + \sum_{r=n+1}^{2m+1}\left(\left(h^r_{11}\right)^2 + \cdots + \left(h^r_{n_1n_1}\right)^2\right)\\ & - 2\bar{\tau}(TN_\theta) - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}\left(h^r_{jj}h^r_{tt} - \left(h^r_{jt}\right)^2\right) + 2\bar{\tau}(TM) + n^2\|H\|^2\\ ={}& -\frac{2n_2\Delta f}{f} - 2\bar{\tau}(TN_T) + \sum_{r=n+1}^{2m+1}\sum_{i,k=1}^{n_1}\left(h^r_{ik}\right)^2 - \sum_{r=n+1}^{2m+1}\left(h^r_{11} + \cdots + h^r_{n_1n_1}\right)^2\\ & - 2\bar{\tau}(TN_\theta) - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}\left(h^r_{jj}h^r_{tt} - \left(h^r_{jt}\right)^2\right) + 2\bar{\tau}(TM) + n^2\|H\|^2.\end{aligned}$$

Similarly, we can add and subtract the same term for the sixth term in the above equation; and finally, we derive

$$\begin{aligned}\|h\|^2 ={}& -\frac{2n_2\Delta f}{f} + 2\bar{\tau}(TM) - 2\bar{\tau}(TN_T) + \sum_{r=n+1}^{2m+1}\sum_{i,k=1}^{n_1}\left(h^r_{ik}\right)^2\\ & + \sum_{r=n+1}^{2m+1}\sum_{j,t=n_1+1}^{n}\left(h^r_{jt}\right)^2 - \sum_{r=n+1}^{2m+1}\left(h^r_{11} + \cdots + h^r_{n_1n_1}\right)^2 - 2\bar{\tau}(TN_\theta)\\ & - \sum_{r=n+1}^{2m+1}\left(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\right)^2 + n^2\|H\|^2.\end{aligned}$$

Taking account of Lemma 5.1, we get the inequality (i). For the equality case, from the last relation we get

$$\sum_{r=n+1}^{2m+1}\sum_{i,k=1}^{n_1} g\big(h(e_i,e_k),e_r\big)=0 \tag{5.6}$$

and

$$\sum_{r=n+1}^{2m+1}\sum_{j,t=n_1+1}^{n} g\big(h(e_j,e_t),e_r\big)=0. \tag{5.7}$$

From (5.6) and (5.7) we obtain that the immersion $\varphi: M \to \bar{M}$ is totally geodesic. Also, from Corollary 3.1 we know that the immersion $N_T \to M$ is totally geodesic and the immersion $N_\theta \to M$ is totally umbilical, hence the result (ii). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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