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Weighted boundedness of multilinear singular integral operator with general kernels for the extreme cases

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Abstract
We prove the weighted boundedness properties for the multilinear operator associated to the singular integral operator with general kernels for the extreme cases.
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1 Introduction and preliminaries

As for the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–9]). Let T be the Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$, a classical result of Coifman *et al.* (see [5]) stated that the commutator $[b,T](f) = T(bf) - bT(f)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [10], the authors obtain the boundedness properties of the commutators for the extreme values of p (that is, $p = 1$ and $p = \infty$). Note that $[b,T]$ is not bounded for the end point boundedness. The purpose of this paper is to introduce some multilinear operator associated to the singular integral operator with general kernels (see [11]) and prove the weighted boundedness properties of the multilinear operators for the extreme cases.

First, let us introduce some preliminaries (see [6, 9]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable functions b and a weight function w (that is, a non-negative locally integrable function), let $w(Q) = \int_Q w(x)\,dx$, $w_Q = |Q|^{-1}\int_Q w(x)\,dx$, the weighted sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q \left|b(y) - b_Q\right| w(y)\,dy.$$

We say that b belongs to $BMO(w)$ if $b^{\#}$ belongs to $L^\infty(w)$, and we define $\|b\|_{BMO(w)} = \|b^{\#}\|_{L^\infty(w)}$. If $w = 1$, we denote $BMO(w) = BMO(R^n)$. It has been known that (see [9])

$$\|b - b_{2^k Q}\|_{BMO} \le Ck\|b\|_{BMO}.$$

We also define the central *BMO* space by $CMO(R^n)$, which is the space of those functions $f \in L_{\mathrm{loc}}(R^n)$ such that

$$\|f\|_{CMO} = \sup_{r>1} \left|Q(0,r)\right|^{-1} \int_Q \left|f(x) - f_Q\right| dx < \infty.$$

It is well known that (see [6, 9])

$$\|f\|_{CMO} \approx \sup_{r>1} \inf_{c\in C} \left|Q(0,r)\right|^{-1} \int_Q \left|f(x) - c\right| dx.$$

Definition 1 Let $1 < p < \infty$ and w be a non-negative weight functions on R^n. We shall call $B_p(w)$ the space of those functions f on R^n such that

$$\|f\|_{B_p(w)} = \sup_{r>1} \left[w\bigl(Q(0,r)\bigr)\right]^{-1/p} \|f\chi_{Q(0,r)}\|_{L^p(w)} < \infty.$$

The A_p weight is defined by (see [6])

$$A_p = \left\{0 < w \in L^1_{\mathrm{loc}}\bigl(R^n\bigr) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty\right\},$$
$$1 < p < \infty,$$

and

$$A_1 = \left\{0 < w \in L^1_{\mathrm{loc}}\bigl(R^n\bigr) : \sup_{Q\ni x} \frac{1}{|Q|} \int_Q w(y)\, dy \le Cw(x), \text{a.e.}\right\}.$$

2 Theorems

In this paper, we will study the following multilinear singular integral operator (see [11]).

Definition 2 Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and has a kernel K, that is, there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x,y) f(y)\, dy$$

for every bounded and compactly supported function f, where K satisfies

$$\left|K(x,y)\right| \le C|x-y|^{-n},$$
$$\int_{2|y-z|<|x-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right) dx \le C,$$

and there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \ge 1$,

$$\left(\int_{2^k|z-y|\le|x-y|<2^{k+1}|z-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right)^q dy\right)^{1/q}$$
$$\le C_k \bigl(2^k|z-y|\bigr)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let m_j be the positive integers ($j = 1,\dots,l$), $m_1 + \cdots + m_l = m$ and b_j be the functions on R^n ($j = 1,\dots,l$). Set, for $1 \le j \le l$,

$$R_{m_j+1}(b_j;x,y) = b_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T_b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m} K(x,y)f(y)\,dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 2 (see [7, 8]). Also note that when $m = 0$, T_b is just a multilinear commutator of T and b (see [1–3]). It is well known that a multilinear operator, as a non-trivial extension of a commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [1–3]). In this paper, we will study the weighted boundedness properties of the multilinear operators T_b for the extreme cases (see [12–15]).

We shall prove the following theorems in Section 3.

Theorem 1 *Let T be the singular integral operator as Definition 2, and we have the sequence $\{k^m C_k\} \in l^1$, $w \in A_1$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\dots,l$. Then T_b is bounded from $L^\infty(w)$ to $BMO(w)$.*

Theorem 2 *Let T be the singular integral operator as Definition 2, and we have the sequence $\{k^m C_k\} \in l^1$, $1 < p < \infty$, $w \in A_1$, and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1,\dots,l$. Then T_b is bounded from $B_p(w)$ to $CMO(w)$.*

3 Proofs of theorems

We begin with two preliminaries lemmas.

Lemma 1 (see [3]) *Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$\bigl|R_m(b;x,y)\bigr| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} \bigl|D^\alpha b(z)\bigr|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2 (see [11]) *Let T be the singular integral operator as Definition 2, and we have the sequence $\{C_k\} \in l^1$. Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$.*

Proof of Theorem 1 It is only for us to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)} \int_Q \bigl|T_b(f)(x) - C_Q\bigr| w(x)\,dx \le C\|f\|_{L^\infty(w)}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_m(b_j;x,y) =$

$R_m(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
T_b(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y)f(y)\,dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x,y)f_1(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y)f_1(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,y)f_1(y)\,dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,y)f_2(y)\,dy \\
&= T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right) \\
&\quad - T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
&\quad - T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + T\left(\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_2\right),
\end{aligned}$$

then

$$\begin{aligned}
\left|T_b(f)(x) - T_{\tilde{b}}(f_2)(x_0)\right| &\le \left|T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)\right| \\
&\quad + \left|T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right)\right| \\
&\quad + \left|T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right| \\
&\quad + \left|T\left(\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right| \\
&\quad + \left|T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0)\right| \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q \left|T_b(f)(x) - T_{\tilde{b}}(f_2)(x_0)\right| w(x)\,dx \\
&\quad\le \frac{1}{w(Q)}\int_Q I_1(x)w(x)\,dx + \frac{1}{w(Q)}\int_Q I_2(x)w(x)\,dx + \frac{1}{w(Q)}\int_Q I_3(x)w(x)\,dx \\
&\qquad + \frac{1}{w(Q)}\int_Q I_4(x)w(x)\,dx + \frac{1}{w(Q)}\int_Q I_5(x)w(x)\,dx \\
&\quad= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4, and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{b}_j; x, y) \le C|x-y|^m \sum_{|\alpha_j|=m} \left\|D^{\alpha_j} b_j\right\|_{BMO},$$

thus, by the $L^p(w)$-boundedness of T for $1 < p < \infty$ (Lemma 2) and Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j} b_j\right\|_{BMO}\right)\frac{1}{w(Q)}\int_Q \left|T(f_1)(x)\right| w(x)\,dx \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j} b_j\right\|_{BMO}\right)\left(\frac{1}{w(Q)}\int_{R^n}\left|T(f_1)(x)\right|^p w(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j} b_j\right\|_{BMO}\right)\left(\frac{1}{w(Q)}\int_{R^n}\left|f_1(x)\right|^p w(x)\,dx\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j} b_j\right\|_{BMO}\right)\left(\frac{w(\tilde{Q})}{w(Q)}\right)^{1/p}\|f\|_{L^\infty(w)} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j} b_j\right\|_{BMO}\right)\|f\|_{L^\infty(w)}.
\end{aligned}$$

For I_2, since $w \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^{p_0}\,dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q w(x)\,dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [13]), thus, by the L^p-boundedness of T for $p > 1$, we get

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2} b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{w(Q)}\int_Q \left|T\left(D^{\alpha_1}\tilde{b}_1 f_1\right)(x)\right| w(x)\,dx \\
&\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2} b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{b}_1 f_1\right)(x)\right|^p w(x)\,dx\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{w(Q)} \int_{R^n} \left|D^{\alpha_1} \tilde{b}_1(x) f_1(x)\right|^p w(x)\, dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} \left|D^{\alpha_1} b_1(x) - \left(D^{\alpha_1} b_1\right)_{\tilde{Q}}\right|^{pp_0'}\, dx \right)^{1/pp_0'} \\
&\quad \times w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{p_0}\, dx \right)^{1/pp_0} \|f\|_{L^\infty(w)} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)\, dx \right)^{1/p} \|f\|_{L^\infty(w)} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

For I_3, similar to the proof of I_2, we get

$$
I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
$$

Similarly, for I_4, choose $1 < r_1, r_2 < \infty$ such that $1/r_1 + 1/r_2 + 1/p_0 = 1$, we obtain, by Hölder's inequality and the reverse of Hölder's inequality,

$$
\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{w(Q)} \int_Q \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right| w(x)\, dx \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{w(Q)} \int_{R^n} \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right|^p w(x)\, dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} w(Q)^{-1/p} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x) \right|^p w(x)\, dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| D^{\alpha_1} \tilde{b}_1(x) \right|^{pr_1}\, dx \right)^{1/pr_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| D^{\alpha_2} \tilde{b}_2(x) \right|^{pr_2}\, dx \right)^{1/pr_2} \\
&\quad \times w(Q)^{-1/p} |Q|^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{p_0}\, dx \right)^{1/pp_0} \|f\|_{L^\infty(w)} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

For I_5, we write

$$
\begin{aligned}
&T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0) \\
&\quad = \int_{R^n} \left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f_2(y)\, dy \\
&\qquad + \int_{R^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0, y) f_2(y)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{R^n}\bigl(R_{m_2}(\tilde{b}_2;x,y)-R_{m_2}(\tilde{b}_2;x_0,y)\bigr)\frac{R_{m_1}(\tilde{b}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)\,dy \\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\biggl[\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)-\frac{R_{m_2}(\tilde{b}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\biggr] \\
&\times D^{\alpha_1}\tilde{b}_1(y)f_2(y)\,dy \\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\biggl[\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y)-\frac{R_{m_1}(\tilde{b}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)\biggr] \\
&\times D^{\alpha_2}\tilde{b}_2(y)f_2(y)\,dy \\
&+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\biggl[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\biggr] \\
&\times D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)\,dy \\
={}&I_5^{(1)}(x)+I_5^{(2)}(x)+I_5^{(3)}(x)+I_5^{(4)}(x)+I_5^{(5)}(x)+I_5^{(6)}(x).
\end{aligned}
$$

By Lemma 1 and the inequality (see [9])

$$|b_{Q_1}-b_{Q_2}|\le C\log\bigl(|Q_2|/|Q_1|\bigr)\|b\|_{BMO}\quad\text{for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
\bigl|R_{m_j}(\tilde{b}_j;x,y)\bigr| &\le C|x-y|^{m_j}\sum_{|\alpha|=m_j}\bigl(\|D^\alpha b_j\|_{BMO}+\bigl|(D^\alpha b_j)_{\tilde{Q}(x,y)}-(D^\alpha b_j)_{\tilde{Q}}\bigr|\bigr) \\
&\le Ck|x-y|^{m_j}\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{BMO}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by the conditions on K,

$$
\begin{aligned}
\bigl|I_5^{(1)}(x)\bigr| &\le \int_{R^n}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right|\bigl|K(x,y)\bigr|\prod_{j=1}^{2}\bigl|R_{m_j}(\tilde{b}_j;x,y)\bigr|\bigl|f_2(y)\bigr|\,dy \\
&\quad+\int_{R^n}\bigl|K(x,y)-K(x_0,y)\bigr||x_0-y|^{-m}\prod_{j=1}^{2}\bigl|R_{m_j}(\tilde{b}_j;x,y)\bigr|\bigl|f_2(y)\bigr|\,dy \\
&\le \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x_0-y|^m}\right|\bigl|K(x,y)\bigr|\prod_{j=1}^{2}\bigl|R_{m_j}(\tilde{b}_j;x,y)\bigr|\bigl|f(y)\bigr|\,dy \\
&\quad+\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\bigl|K(x,y)-K(x_0,y)\bigr||x_0-y|^{-m}\prod_{j=1}^{2}\bigl|R_{m_j}(\tilde{b}_j;x,y)\bigr|\bigl|f(y)\bigr|\,dy \\
&\le C\prod_{j=1}^{2}\biggl(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\biggr)\sum_{k=0}^{\infty}k^2\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1}}\bigl|f(y)\bigr|\,dy \\
&\quad+C\prod_{j=1}^{2}\biggl(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\biggr)\sum_{k=0}^{\infty}k^2\biggl(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\bigl|f(y)\bigr|^{q'}\,dy\biggr)^{1/q'}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left| K(x,y) - K(x_0,y) \right|^q dy \right)^{1/q} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 \left(2^{-k} + C_k \right) \|f\|_{L^\infty(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

For $I_5^{(2)}(x)$, by the formula (see [3]):

$$
R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|} \left(D^\beta \tilde{b}_j; x, x_0 \right) (x-y)^\beta
$$

and Lemma 1, we have

$$
\left| R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) \right| \le C \sum_{|\beta|<m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} \left\| D^\alpha b_j \right\|_{BMO},
$$

thus

$$
\begin{aligned}
\left| I_5^{(2)}(x) \right| &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \frac{|x-x_0|}{|x_0-y|^{n+1}} \left| f(y) \right| dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \sum_{k=1}^{\infty} k 2^{-k} \|f\|_{L^\infty(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

Similarly,

$$
\left| I_5^{(3)}(x) \right| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{BMO} \right) \|f\|_{L^\infty(w)}.
$$

For $I_5^{(4)}(x)$, similar to the proof of $I_5^{(1)}(x)$ and $I_5^{(2)}(x)$, we get

$$
\begin{aligned}
\left| I_5^{(4)}(x) \right| &\le C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y) \right| \frac{|(x_0-y)^{\alpha_1} K(x_0,y)|}{|x_0-y|^m} \\
&\quad \times \left| D^{\alpha_1} \tilde{b}_1(y) \right| \left| f_2(y) \right| dy \\
&\quad + C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| \frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| \left| K(x,y) \right| \left| R_{m_2}(\tilde{b}_2; x, y) \right| \\
&\quad \times \left| D^{\alpha_1} \tilde{b}_1(y) \right| \left| f_2(y) \right| dy \\
&\quad + C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| K(x,y) - K(x_0,y) \right| \left| \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| \left| R_{m_2}(\tilde{b}_2; x, y) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f_2(y)\right| dy \\
&\le C \sum_{|\alpha_2|=m_2} \left\|D^{\alpha_2} b_2\right\|_{BMO} \sum_{k=1}^{\infty} k2^{-k} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \left|D^{\alpha_1}\tilde{b}_1(y)\right| dy\right) \|f\|_{L^\infty(w)} \\
&\quad + C \sum_{|\alpha_2|=m_2} \left\|D^{\alpha_2} b_2\right\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|K(x,y)-K(x_0,y)\right|^q dy\right)^{1/q} \\
&\quad \times \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|D^{\alpha_1}\tilde{b}_1(y)\right|^{q'} dy\right)^{1/q'} \|f\|_{L^\infty(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \sum_{k=1}^{\infty} k^2\left(2^{-k}+C_k\right) \|f\|_{L^\infty(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

Similarly,

$$
\left|I_5^{(5)}(x)\right| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \|f\|_{L^\infty(w)}.
$$

For $I_5^{(6)}(x)$, taking $1 < r_1, r_2 < \infty$ such that $1/r_1 + 1/r_2 = 1$, then

$$
\begin{aligned}
\left|I_5^{(6)}(x)\right| &\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left| \frac{(x-y)^{\alpha_1+\alpha_2} K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} K(x_0,y)}{|x_0-y|^m} \right| \\
&\quad \times \left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f_2(y)\right| dy \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} \left(2^{-k}+C_k\right) \|f\|_{L^\infty(w)} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \left|D^{\alpha_1}\tilde{b}_1(y)\right|^{r_1} dy\right)^{1/r_1} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \left|D^{\alpha_2}\tilde{b}_2(y)\right|^{r_2} dy\right)^{1/r_2} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \sum_{k=1}^{\infty} k^2\left(2^{-k}+C_k\right) \|f\|_{L^\infty(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \|f\|_{L^\infty(w)}.
\end{aligned}
$$

Thus

$$
I_5 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j} b_j\right\|_{BMO}\right) \|f\|_{L^\infty(w)}.
$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 It is only for us to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q\big|T_b(f)(x)-C_Q\big|w(x)\,dx\le C\|f\|_{B_p(w)}$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(0,d)$ with $d>1$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{b}_j(x)=b_j(x)-\sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(b_j;x,y)=R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j=D^\alpha b_j-(D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. Similar to the proof of Theorem 1, we write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\big|T_b(f)(x)-T_{\tilde{b}}(f_2)(0)\big|w(x)\,dx\\
&\quad\le\frac{1}{w(Q)}\int_Q\left|T\left(\frac{\prod_{j=1}^2R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)\right|w(x)\,dx\\
&\qquad+\frac{1}{w(Q)}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|w(x)\,dx\\
&\qquad+\frac{1}{w(Q)}\int_Q\left|T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|w(x)\,dx\\
&\qquad+\frac{1}{w(Q)}\int_Q\left|T\left(\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|w(x)\,dx\\
&\qquad+\frac{1}{w(Q)}\int_Q\big|T_{\tilde{b}}(f_2)(x)-T_{\tilde{b}}(f_2)(0)\big|w(x)\,dx\\
&\quad=L_1+L_2+L_3+L_4+L_5.
\end{aligned}$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}
L_1&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left(\frac{1}{w(Q)}\int_{R^n}\big|T(f_1)(x)\big|^pw(x)\,dx\right)^{1/p}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)w(\tilde{Q})^{-1/p}\|f\chi_{\tilde{Q}}\|_{L^p(w)}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}$$

For L_2, taking $r,s,t>1$ such that $r<p$, $t=pp_0/(p-r)$, and $1/s+1/(p/r)+1/t=1$, then, by the reverse of Hölder's inequality,

$$\begin{aligned}
L_2&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{w(Q)}\int_{R^n}\big|T\big(D^{\alpha_1}\tilde{b}_1f_1\big)(x)\big|^rw(x)\,dx\right)^{1/r}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}w(Q)^{-1/r}\sum_{|\alpha_1|=m_1}\left(\int_{R^n}\big|D^{\alpha_1}\tilde{b}_1(x)f_1(x)\big|^rw(x)\,dx\right)^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} w(Q)^{-1/r} \sum_{|\alpha_1|=m_1} \left(\int_{\tilde{Q}} \left|D^{\alpha} \tilde{b}_1(x)\right|^{rs} dx \right)^{1/rs} \\
&\quad \times \left(\int_{\tilde{Q}} \left|f(x)\right|^p w(x)\, dx \right)^{1/p} \left(\int_{\tilde{Q}} w(x)^{(1-r/p)t}\, dx \right)^{1/rt} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/r} |Q|^{1/rs} \|f\chi_{\tilde{Q}}\|_{L^p(w)} |Q|^{1/rt} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{p_0}\, dx \right)^{1/rt} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/r} |Q|^{1/rs} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)\, dx \right)^{p_0/rt} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}, \\
L_3 &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
\end{aligned}
$$

For L_4, taking $r, s_1, s_2, t > 1$ such that $r < p$, $t = pp_0/(p-r)$, and $1/s_1 + 1/s_2 + 1/(p/r) + 1/t = 1$, then, by the reverse of Hölder's inequality,

$$
\begin{aligned}
L_4 &\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{w(Q)} \int_{R^n} \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right|^r w(x)\, dx \right)^{1/r} \\
&\le C w(Q)^{-1/r} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x) \right|^r w(x)\, dx \right)^{1/r} \\
&\le C w(Q)^{-1/r} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\int_{\tilde{Q}} \left| D^{\alpha} \tilde{b}_1(x) \right|^{rs_1} dx \right)^{1/rs_1} \left(\int_{\tilde{Q}} \left| D^{\alpha} \tilde{b}_2(x) \right|^{rs_2} dx \right)^{1/rs_2} \\
&\quad \times \left(\int_{\tilde{Q}} \left|f(x)\right|^p w(x)\, dx \right)^{1/p} \left(\int_{\tilde{Q}} w(x)^{(1-r/p)t}\, dx \right)^{1/rt} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(Q)^{-1/r} |Q|^{1/rs_1 + 1/rs_2 + 1/rt} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)\, dx \right)^{p_0/rt} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) w(\tilde{Q})^{-1/p} \|f\chi_{\tilde{Q}}\|_{L^p(w)} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p(w)}.
\end{aligned}
$$

For L_5, similar to the proof of the proof of I_5 in Theorem 1, we have

$$\begin{aligned}
&T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(0) \\
&= \int_{R^n}\left(\frac{K(x,y)}{|x-y|^m} - \frac{K(0,y)}{|y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j;x,y)f_2(y)\,dy \\
&\quad + \int_{R^n}\big(R_{m_1}(\tilde{b}_1;x,y) - R_{m_1}(\tilde{b}_1;0,y)\big)\frac{R_{m_2}(\tilde{b}_2;x,y)}{|y|^m}K(0,y)f_2(y)\,dy \\
&\quad + \int_{R^n}\big(R_{m_2}(\tilde{b}_2;x,y) - R_{m_2}(\tilde{b}_2;0,y)\big)\frac{R_{m_1}(\tilde{b}_1;x_0,y)}{|y|^m}K(0,y)f_2(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y) - \frac{R_{m_2}(\tilde{b}_2;0,y)(-y)^{\alpha_1}}{|y|^m}K(0,y)\right] \\
&\quad \times D^{\alpha_1}\tilde{b}_1(y)f_2(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y) - \frac{R_{m_1}(\tilde{b}_1;0,y)(-y)^{\alpha_2}}{|y|^m}K(0,y)\right] \\
&\quad \times D^{\alpha_2}\tilde{b}_2(y)f_2(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y) - \frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m}K(0,y)\right] \\
&\quad \times D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)\,dy \\
&= L_5^{(1)}(x) + L_5^{(2)}(x) + L_5^{(3)}(x) + L_5^{(4)}(x) + L_5^{(5)}(x) + L_5^{(6)}(x).
\end{aligned}$$

For $L_5^{(1)}(x)$, taking $1 < r < \infty$ such that $1/p + 1/q + 1/r = 1$, by $w \in A_1 \subset A_{p/r+1}$, we get

$$\begin{aligned}
\big|L_5^{(1)}(x)\big| &\le C\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|\frac{1}{|x-y|^m} - \frac{1}{|y|^m}\right|\big|K(x,y)\big|\prod_{j=1}^{2}\big|R_{m_j}(\tilde{b}_j;x,y)\big|\big|f(y)\big|\,dy \\
&\quad + \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|K(x,y) - K(0,y)\big||y|^{-m} \\
&\quad \times \prod_{j=1}^{2}\big|R_{m_j}(\tilde{b}_j;x,y)\big|\big|f(y)\big|w(y)^{1/p}w(y)^{-1/p}\,dy \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2\frac{d}{(2^k d)^{n+1}}\big|f(y)\big|\,dy \\
&\quad + C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}b_j\|_{BMO}\right)\sum_{k=0}^{\infty}k^2\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\big|K(x,y) - K(0,y)\big|^q\,dy\right)^{1/q} \\
&\quad \times \left(\int_{2^{k+1}\tilde{Q}}\big|f(y)\big|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-r/p}\,dy\right)^{1/r} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-k}w\big(2^k\tilde{Q}\big)^{-1/p}\left(\int_{2^k\tilde{Q}}\big|f(y)\big|^p w(y)\,dy\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-1/(p-1)}\,dy\right)^{(p-1)/p} \\
&+ C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2C_k w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|^p w(y)\,dy\right)^{1/p} \\
&\times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)\,dy\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} w(y)^{-r/p}\,dy\right)^{1/r} \\
\le{}& C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\|f\|_{B_p(w)}.
\end{aligned}
$$

Similarly, we get, for $1 < r_1, r_2, r_3, r_4, s < \infty$ with $1/p + 1/r_1 + 1/s = 1$, $1/p + 1/q + 1/r_2 + 1/s = 1$, and $1/p + 1/q + 1/r_3 + 1/r_4 + 1/s = 1$,

$$
\begin{aligned}
&\left|L_5^{(2)}(x) + L_5^{(3)}(x) + L_5^{(4)}(x) + L_5^{(5)}(x) + L_5^{(6)}(x)\right| \\
&\le C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{BMO}\right)\sum_{|\alpha|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{d}{(2^k d)^{n+1}}\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f(y)\right|dy \\
&\quad + C\left(\sum_{|\alpha|=m_1}\|D^{\alpha}b_1\|_{BMO}\right)\sum_{|\alpha|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{d}{(2^k d)^{n+1}}\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f(y)\right|dy \\
&\quad + C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{BMO}\right)\sum_{|\alpha|=m_1}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|K(x,y)-K(0,y)\right| \\
&\quad \times\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f(y)\right|dy \\
&\quad + C\left(\sum_{|\alpha|=m_1}\|D^{\alpha}b_1\|_{BMO}\right)\sum_{|\alpha|=m_2}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|K(x,y)-K(0,y)\right| \\
&\quad \times\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f(y)\right|dy \\
&\quad + C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|K(x,y)-K(0,y)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f(y)\right|dy \\
&\le C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{BMO}\right)\sum_{|\alpha|=m_1}\sum_{k=0}^{\infty}k\left(\int_{2^{k+1}\tilde{Q}}\left|D^{\alpha_1}\tilde{b}_1(y)\right|^{r_1}dy\right)^{1/r_1} \\
&\quad \times\left(\int_{2^{k+1}\tilde{Q}}\left|f(y)\right|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p}\,dy\right)^{1/s} \\
&\quad + C\left(\sum_{|\alpha|=m_1}\|D^{\alpha}b_1\|_{BMO}\right)\sum_{|\alpha|=m_2}\sum_{k=0}^{\infty}k\left(\int_{2^{k+1}\tilde{Q}}\left|D^{\alpha_2}\tilde{b}_2(y)\right|^{r_1}dy\right)^{1/r_1} \\
&\quad \times\left(\int_{2^{k+1}\tilde{Q}}\left|f(y)\right|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p}\,dy\right)^{1/s} \\
&\quad + C\left(\sum_{|\alpha|=m_2}\|D^{\alpha}b_2\|_{BMO}\right)\sum_{|\alpha|=m_1}\sum_{k=0}^{\infty}k\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|K(x,y)-K(0,y)\right|^q dy\right)^{1/q} \\
&\quad \times\left(\int_{2^{k+1}\tilde{Q}}\left|D^{\alpha_1}\tilde{b}_1(y)\right|^{r_2}dy\right)^{1/r_2}\left(\int_{2^{k+1}\tilde{Q}}\left|f(y)\right|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p}\,dy\right)^{1/s}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\left(\sum_{|\alpha|=m_1} \|D^{\alpha} b_1\|_{BMO}\right) \sum_{|\alpha|=m_2} \sum_{k=0}^{\infty} k\left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|K(x,y)-K(0,y)\right|^q\,dy\right)^{1/q} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} \left|D^{\alpha_2}\tilde{b}_2(y)\right|^{r_2}\,dy\right)^{1/r_2} \left(\int_{2^{k+1}\tilde{Q}} \left|f(y)\right|^p w(y)\,dy\right)^{1/p} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p}\,dy\right)^{1/s} \\
&+ C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|K(x,y)-K(0,y)\right|^q\,dy\right)^{1/q} \left(\int_{2^{k+1}\tilde{Q}} w(y)^{-s/p}\,dy\right)^{1/s} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} \left|D^{\alpha_1}\tilde{b}_1(y)\right|^{r_3}\,dy\right)^{1/r_3} \left(\int_{2^{k+1}\tilde{Q}} \left|D^{\alpha_2}\tilde{b}_2(y)\right|^{r_4}\,dy\right)^{1/r_4} \\
&\times \left(\int_{2^{k+1}\tilde{Q}} \left|f(y)\right|^p w(y)\,dy\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO}\right) \sum_{k=1}^{\infty} k^2\left(2^{-k}+C_k\right) w\left(2^k\tilde{Q}\right)^{-1/p} \left(\int_{2^k\tilde{Q}} \left|f(y)\right|^p w(y)\,dy\right)^{1/p} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO}\right) \|f\|_{B_p(w)}.
\end{aligned}
$$

Thus

$$
L_5 \le C\prod_{j=1}^{2}\left(\sum_{\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO}\right) \|f\|_{B_p(w)}.
$$

This finishes the proof of Theorem 2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper. MZ carried out the ideas and methods of studies, YG participated in the design of the study and performed the statistical analysis, the sequence alignment and drafted the manuscript. The authors read and approved the final manuscript.

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