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Stability of homomorphisms on fuzzy Lie C^*-algebras via fixed point method

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Abstract
In this paper, first, we define fuzzy C^*-algebras and fuzzy Lie C^*-algebras; then, using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in fuzzy C^*-algebras and fuzzy Lie C^*-algebras for an m-variable additive functional equation.

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1 Introduction and preliminaries

The stability problem of functional equations originated with a question of Ulam [1] concerning the stability of group homomorphisms: let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. We recall a fundamental result in fixed-point theory. Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a *generalized metric* on Ω if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 1.1 [2] *Let (Ω, d) be a complete generalized metric space and let $J : \Omega \to \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.*

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms and derivations in fuzzy Lie C^*-algebras for the following additive functional equation [3]:

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right) \quad (m \in \mathbb{N}, m \geq 2). \tag{1.1}$$

We use the definition of fuzzy normed spaces given in [4–10] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting (see also [11–16]).

Definition 1.2 [4] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

Definition 1.3 [4] (1) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

(2) Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4, 10]).

Definition 1.4 [12] A *fuzzy normed algebra* $(X, \mu, *, *')$ is a fuzzy normed space $(X, N, *)$ with algebraic structure such that

(N_7) $N(xy, ts) \geq N(x,t) * N(y,s)$ for all $x, y \in X$ and all $t, s > 0$, in which $*'$ is a continuous t-norm.

Every normed algebra $(X, \|\cdot\|)$ defines a fuzzy normed algebra $(X, N, \min)$, where

$$N(x,t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ iff

$$\|xy\| \le \|x\|\|y\| + s\|y\| + t\|x\| \quad (x, y \in X; t, s > 0).$$

This space is called the induced fuzzy normed algebra.

Definition 1.5 (1) Let $(X, N, *)$ and $(Y, N, *)$ be fuzzy normed algebras. An $\mathbb{R}$-linear mapping $f : X \to Y$ is called a *homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in X$.

(2) An $\mathbb{R}$-linear mapping $f : X \to X$ is called a *derivation* if $f(xy) = f(x)y + xf(y)$ for all $x, y \in X$.

Definition 1.6 Let $(\mathcal{U}, N, *, *')$ be a fuzzy Banach algebra, then an involution on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

(i) $u^{**} = u$ for $u \in \mathcal{U}$;
(ii) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$;
(iii) $(uv)^* = v^* u^*$ for $u, v \in \mathcal{U}$.

If, in addition $N(u^* u, ts) = N(u, t) * N(u, s)$ for $u \in \mathcal{U}$ and $t > 0$, then $\mathcal{U}$ is a fuzzy C^*-algebra.

2 Stability of homomorphisms in fuzzy C^*-algebras

Throughout this section, assume that A is a fuzzy C^*-algebra with norm N_A and that B is a fuzzy C^*-algebra with norm N_B.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x_1, \dots, x_m) := \sum_{i=1}^{m} \mu f\left(m x_i + \sum_{j=1, j \neq i}^{m} x_j\right) + f\left(\mu \sum_{i=1}^{m} x_i\right) - 2f\left(\mu \sum_{i=1}^{m} m x_i\right)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x_1, \dots, x_m \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in fuzzy C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in fuzzy C^*-algebras for the functional equation $D_\mu f(x_1, \dots, x_m) = 0$.

Theorem 2.1 *Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^m \times (0, \infty) \to [0, 1]$, $\psi : A^2 \times (0, \infty) \to [0, 1]$ and $\eta : A \times (0, \infty) \to [0, 1]$ such that*

$$N_B\bigl(D_\mu f(x_1, \dots, x_m), t\bigr) \ge \varphi(x_1, \dots, x_m, t), \tag{2.1}$$

$$\lim_{j \to \infty} \varphi\bigl(m^j x_1, \dots, m^j x_m, m^j t\bigr) = 1, \tag{2.2}$$

$$N_B\bigl(f(xy) - f(x)f(y), t\bigr) \ge \psi(x, y, t), \tag{2.3}$$

$$\lim_{j \to \infty} \psi\bigl(m^j x, m^j y, m^{2j} t\bigr) = 1, \tag{2.4}$$

$$N_B\bigl(f(x^*) - f(x)^*, t\bigr) \ge \eta(x, t), \tag{2.5}$$

$$\lim_{j \to \infty} \eta\bigl(m^j x, m^j t\bigr) = 1 \tag{2.6}$$

for all $\mu \in \mathbb{T}^1$, *all* $x_1, \dots, x_m, x, y \in A$ *and* $t > 0$. *If there exists an* $L < 1$ *such that*

$$\varphi(mx, 0, \dots, 0, mLt) \geq \varphi(x, 0, \dots, 0, t) \tag{2.7}$$

for all $x \in A$ *and* $t > 0$, *then there exists a unique homomorphism* $H : A \to B$ *such that*

$$N_B\bigl(f(x) - H(x), t\bigr) \geq \varphi\bigl(x, 0, \dots, 0, (m - mL)t\bigr) \tag{2.8}$$

for all $x \in A$ *and* $t > 0$.

Proof Consider the set $X := \{g : A \to B\}$ and introduce the *generalized metric* on X:

$$d(g, h) = \inf\bigl\{C \in \mathbb{R}_+ : N_B\bigl(g(x) - h(x), Ct\bigr) \geq \varphi(x, 0, \dots, 0, t), \forall x \in A, t > 0\bigr\}.$$

It is easy to show that (X, d) is complete. Now, we consider the linear mapping $J : X \to X$ such that $Jg(x) := \frac{1}{m}g(mx)$ for all $x \in A$. By Theorem 3.1 of [17], $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in X$. Letting $\mu = 1$, $x = x_1$ and $x_2 = \cdots = x_m = 0$ in equation (2.1), we get

$$N_B\bigl(f(mx) - mf(x), t\bigr) \geq \varphi(x, 0, \dots, 0, t) \tag{2.9}$$

for all $x \in A$ and $t > 0$. Therefore

$$N_B\left(f(x) - \frac{1}{m}f(mx), t\right) \geq \varphi(x, 0, \dots, 0, mt)$$

for all $x \in A$ and $t > 0$. Hence $d(f, Jf) \leq \frac{1}{m}$. By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, *i.e.*,

$$H(mx) = mH(x) \tag{2.10}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \bigl\{g \in X : d(f, g) < \infty\bigr\}.$$

This implies that H is a unique mapping satisfying equation (2.10) such that there exists $C \in (0, \infty)$ satisfying

$$N_B\bigl(H(x) - f(x), Ct\bigr) \geq \varphi(x, 0, \dots, 0, t)$$

for all $x \in A$ and $t > 0$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^n} = H(x) \tag{2.11}$$

for all $x \in A$.

(3) $d(f,H) \le \frac{1}{1-L} d(f,Jf)$, which implies the inequality $d(f,H) \le \frac{1}{m-mL}$. This implies that the inequality (2.8) holds.

It follows from equations (2.1), (2.2), and (2.11) that

$$\begin{aligned} & N_B\left(\sum_{i=1}^{m} H\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + H\left(\sum_{i=1}^{m} x_i\right) - 2H\left(\sum_{i=1}^{m} mx_i\right), t\right) \\ & \quad = \lim_{n\to\infty} N_B\left(\sum_{i=1}^{m} f\left(m^{n+1}x_i + \sum_{j=1, j\neq i}^{m} m^n x_j\right) + f\left(\sum_{i=1}^{m} m^n x_i\right) - 2f\left(\sum_{i=1}^{m} m^{n+1} x_i\right), m^n t\right) \\ & \quad \le \lim_{n\to\infty} \varphi\left(m^n x_1, \ldots, m^n x_m, m^n t\right) = 1 \end{aligned}$$

for all $x_1, \ldots, x_m \in A$ and $t > 0$. So

$$\sum_{i=1}^{m} H\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + H\left(\sum_{i=1}^{m} x_i\right) = 2H\left(\sum_{i=1}^{m} mx_i\right)$$

for all $x_1, \ldots, x_m \in A$.

By a similar method to above, we get $\mu H(mx) = H(m\mu x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from equations (2.3), (2.4), and (2.11) that

$$\begin{aligned} N_B\big(H(xy) - H(x)H(y), t\big) &= \lim_{n\to\infty} N_B\big(f(m^n xy) - f(m^n x) f(m^n y), m^n t\big) \\ &\le \lim_{n\to\infty} \psi\left(m^n x, m^n y, m^{2n} t\right) = 1 \end{aligned}$$

for all $x, y \in A$. So $H(xy) = H(x)H(y)$ for all $x, y \in A$. Thus $H : A \to B$ is a homomorphism satisfying equation (2.7), as desired.

Also by equations (2.5), (2.6), (2.11), and by a similar method we have $H(x^*) = H(x)^*$. □

3 Stability of homomorphisms in fuzzy Lie C^*-algebras

A fuzzy C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on $\mathcal{C}$, is called a *fuzzy Lie C^*-algebra* (see [18–20]).

Definition 3.1 Let A and B be fuzzy Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *fuzzy Lie C^*-algebra homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a fuzzy Lie C^*-algebra with norm N_A and that B is a fuzzy Lie C^*-algebra with norm N_B.

We prove the generalized Hyers-Ulam stability of homomorphisms in fuzzy Lie C^*-algebras for the functional equation $D_\mu f(x_1, \ldots, x_m) = 0$.

Theorem 3.2 *Let $f : A \to B$ be a mapping for which there are functions $\varphi : A^m \times (0,\infty) \to [0,1]$ and $\psi : A^2 \times (0,\infty) \to [0,1]$ such that*

$$\lim_{j\to\infty} \varphi\bigl(m^j x_1, \ldots, m^j x_m, m^j t\bigr) = 1, \tag{3.1}$$

$$N_B\bigl(D_\mu f(x_1, \ldots, x_m), t\bigr) \ge \varphi(x_1, \ldots, x_m, t), \tag{3.2}$$

$$N_B\bigl(f\bigl([x,y]\bigr) - \bigl[f(x), f(y)\bigr], t\bigr) \ge \psi(x, y, t), \tag{3.3}$$

$$\lim_{j\to\infty} \psi\bigl(m^j x, m^j y, m^{2j} t\bigr) = 1 \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$, all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi(mx, 0, \ldots, 0, mlt) \ge \varphi(x, 0, \ldots, 0, t)$$

for all $x \in A$ and $t > 0$, then there exists a unique homomorphism $H : A \to B$ such that

$$N_B\bigl(f(x) - H(x), t\bigr) \ge \varphi\bigl(x, 0, \ldots, 0, (m - mL)t\bigr) \tag{3.5}$$

for all $x \in A$ and $t > 0$.

Proof By the same reasoning as the proof of Theorem 2.1, we can find that the mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

for all $x \in A$.

It follows from equation (3.3) that

$$\begin{aligned} N_B\bigl(H\bigl([x,y]\bigr) - \bigl[H(x), H(y)\bigr], t\bigr) &= \lim_{n\to\infty} N_B\bigl(f\bigl(m^{2n}[x,y]\bigr) - \bigl[f\bigl(m^n x\bigr), f\bigl(m^n y\bigr)\bigr], m^{2n} t\bigr) \\ &\ge \lim_{n\to\infty} \psi\bigl(m^n x, m^n y, m^{2n} t\bigr) = 1 \end{aligned}$$

for all $x, y \in A$ and $t > 0$. So

$$H\bigl([x,y]\bigr) = \bigl[H(x), H(y)\bigr]$$

for all $x, y \in A$.

Thus $H : A \to B$ is a fuzzy Lie C^*-algebra homomorphism satisfying equation (3.5), as desired. □

Corollary 3.3 *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$N_B\bigl(D_\mu f(x_1, \ldots, x_m), t\bigr) \ge \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}, \tag{3.6}$$

$$N_B\bigl(f\bigl([x,y]\bigr) - \bigl[f(x), f(y)\bigr], t\bigr) \ge \frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r} \tag{3.7}$$

for all $\mu \in \mathbb{T}^1$, *all* $x_1, \ldots, x_m, x, y \in A$ *and* $t > 0$. *Then there exists a unique homomorphism* $H : A \to B$ *such that*

$$N_B\big(f(x) - H(x), t\big) \leq \frac{t}{t + \frac{\theta}{m-m^r} \|x\|_A^r}$$

for all $x \in A$ *and* $t > 0$.

Proof The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, \ldots, x_m, t) = \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)},$$
$$\psi(x, y, t) := \frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r}$$

for all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. Putting $L = m^{r-1}$, we get the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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