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On total asymptotically nonexpansive mappings in CAT(κ) spaces

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Dedicated to Professor Shih-Sen Chang on the occasion of his 80th birthday

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Abstract
In this article, we obtain the demiclosed principle, fixed point theorems and convergence theorems for the class of total asymptotically nonexpansive mappings on CAT(κ) spaces with $\kappa > 0$. Our results generalize the results of Chang *et al.* (Appl. Math. Comput. 219:2611-2617, 2012), Tang *et al.* (Abstr. Appl. Anal. 2012:965751, 2012), Karapınar *et al.* (J. Appl. Math. 2014:738150, 2014) and many others.

Keywords: fixed point; total asymptotically nonexpansive mapping; demiclosed principle; Δ-convergence; CAT(κ) space

1 Introduction

For a real number κ, a CAT(κ) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature κ. The precise definition is given below. The letters C, A, and T stand for Cartan, Alexandrov, and Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function.

Fixed point theory in CAT(κ) spaces was first studied by Kirk [1, 2]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see, *e.g.*, [3–11]). Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$, all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \leq 0$. However, there are only a few articles that contain fixed point results in the setting of CAT(κ) spaces with $\kappa > 0$.

The concept of total asymptotically nonexpansive mappings was first introduced in Banach spaces by Alber *et al.* [12]. It generalizes the concept of asymptotically nonexpansive mappings introduced by Goebel and Kirk [13] as well as the concept of nearly asymptotically nonexpansive mappings introduced by Sahu [14]. In 2012, Chang *et al.* [15] studied the demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Since then the convergence of several iteration procedures for this type of mappings has been rapidly developed and many of articles have appeared (see, *e.g.*, [16–24]). Among other things, under some suitable assumptions, Karapınar *et al.* [24] obtained the demiclosed principle, fixed point theorems, and convergence theorems for the following iteration.

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Let K be a nonempty closed convex subset of a CAT(0) space X and $T : K \to K$ be a total asymptotically nonexpansive mapping. Given $x_1 \in K$, and let $\{x_n\} \subseteq K$ be defined by

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n\big((1-\beta_n)x_n \oplus \beta_n T^n(x_n)\big), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

In this article, we extend Karapınar *et al.*'s results to the general setting of CAT(κ) space with $\kappa > 0$.

2 Preliminaries

Let (X,ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $\rho(c(t),c(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $\rho(x,y) = l$. The image $c([0,l])$ of c is called a *geodesic segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$\rho(x,z) = (1-\alpha)\rho(x,y) \quad \text{and} \quad \rho(y,z) = \alpha\rho(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,ρ) is said to be a *geodesic space* (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $\rho(x,y) < D$). A subset K of X is said to be *convex* if K includes every geodesic segment joining any two of its points. The set K is said to be *bounded* if

$$\operatorname{diam}(K) := \sup\big\{\rho(x,y) : x,y \in K\big\} < \infty.$$

Now we introduce the model spaces M_κ^n, for more details on these spaces the reader is referred to [25]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n \quad \text{where } x = (x_1,\dots,x_n), y = (y_1,\dots,y_n).$$

Let $\mathbb{S}^n$ denote the *n-dimensional sphere* defined by

$$\mathbb{S}^n = \big\{x = (x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\big\},$$

with metric $d_{\mathbb{S}^n}(x,y) = \arccos(x|y)$, $x,y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1,\dots,u_{n+1})$ and $v = (v_1,\dots,v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic n-space* defined by

$$\mathbb{H}^n = \{u = (u_1, \dots, u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x,y) = -\langle x|y\rangle, \quad x, y \in \mathbb{H}^n.$$

Definition 2.1 Given $\kappa \in \mathbb{R}$, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$, then M_0^n is the Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$, then M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(iii) if $\kappa < 0$, then M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x,y,z)$ in (X,ρ) is a triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$\rho(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad \rho(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}), \quad \text{and} \quad \rho(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa \le 0$, then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$, then such a triangle exists whenever $\rho(x,y) + \rho(y,z) + \rho(z,x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $\rho(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

A geodesic triangle $\triangle(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p, q \in \triangle(x,y,z)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, one has

$$\rho(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Definition 2.2 If $\kappa \le 0$, then X is called a CAT(κ) *space* if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then X is called a CAT(κ) *space* if and only if X is D_κ-geodesic and any geodesic triangle $\triangle(x,y,z)$ in X with $\rho(x,y) + \rho(y,z) + \rho(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space (X,ρ) if $x, y, z \in X$, then the CAT(0) inequality implies

$$\text{(CN)} \quad \rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho^2(x,y) + \frac{1}{2}\rho^2(x,z) - \frac{1}{4}\rho^2(y,z).$$

This is the *(CN) inequality* of Bruhat and Tits [26]. This inequality is extended by Dhompongsa and Panyanak [27] as

$$\text{(CN}^*) \quad \rho^2\big(x, (1-\alpha)y \oplus \alpha z\big) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \alpha(1-\alpha)\rho^2(y,z)$$

for all $\alpha \in [0,1]$ and $x, y, z \in X$. In fact, if X is a geodesic space, then the following statements are equivalent:

(i) X is a CAT(0) space;
(ii) X satisfies (CN);
(iii) X satisfies (CN*).

Let $R \in (0,2]$. Recall that a geodesic space (X,ρ) is said to be *R-convex* for R (see [28]) if for any three points $x,y,z \in X$, we have

$$\rho^2\big(x,(1-\alpha)y \oplus \alpha z\big) \leq (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y,z). \tag{1}$$

It follows from (CN*) that a geodesic space (X,ρ) is a CAT(0) space if and only if (X,ρ) is R-convex for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [28].

Lemma 2.3 *Let $\kappa > 0$ and (X,ρ) be a* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then (X,ρ) is R-convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

The following lemma is also needed.

Lemma 2.4 ([25, p.176]) *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then*

$$\rho\big((1-\alpha)x \oplus \alpha y, z\big) \leq (1-\alpha)\rho(x,z) + \alpha\rho(y,z)$$

for all $x,y,z \in X$ and $\alpha \in [0,1]$.

We now collect some elementary facts about CAT(κ) spaces. Most of them are proved in the setting of CAT(1) spaces. For completeness, we state the results in CAT(κ) with $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X,ρ). For $x \in X$, we set

$$r\big(x,\{x_n\}\big) = \limsup_{n\to\infty} \rho(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x,\{x_n\}\big) : x \in X\big\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x,\{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

It is known from Proposition 4.1 of [8] that in a CAT(κ) space X with $\operatorname{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.5 ([6, 29]) A sequence $\{x_n\}$ in X is said to *Δ-converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.6 *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then the following statements hold*:

(i) [8, Corollary 4.4] *Every sequence in* X *has a* Δ*-convergent subsequence*;
(ii) [8, Proposition 4.5] *If* $\{x_n\} \subseteq X$ *and* $\Delta\text{-}\lim_n x_n = x$, *then* $x \in \bigcap_{k=1}^{\infty} \overline{\operatorname{conv}}\{x_k, x_{k+1}, \ldots\}$, *where* $\overline{\operatorname{conv}}(A) = \bigcap\{B : B \supseteq A$ *and* B *is closed and convex*$\}$.

By the uniqueness of asymptotic centers, we can obtain the following lemma (*cf.* [27, Lemma 2.8]).

Lemma 2.7 *Let* $\kappa > 0$ *and* (X, ρ) *be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \pi/2)$. *If* $\{x_n\}$ *is a sequence in* X *with* $A(\{x_n\}) = \{x\}$ *and* $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{\rho(x_n, u)\}$ *converges, then* $x = u$.

Definition 2.8 Let K be a nonempty subset of a $\mathrm{CAT}(\kappa)$ space (X, ρ). A mapping $T : K \to K$ is called *total asymptotically nonexpansive* if there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$\rho\bigl(T^n(x), T^n(y)\bigr) \leq \rho(x, y) + \nu_n \psi\bigl(\rho(x, y)\bigr) + \mu_n \quad \text{for all } n \in \mathbb{N}, x, y \in K.$$

A point $x \in K$ is called a *fixed point* of T if $x = T(x)$. We denote with $F(T)$ the set of fixed points of T. A sequence $\{x_n\}$ in K is called *approximate fixed point sequence* for T (AFPS in short) if

$$\lim_{n\to\infty} \rho\bigl(x_n, T(x_n)\bigr) = 0.$$

Algorithm 1 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n(y_n),$$
$$y_n = (1 - \beta_n)x_n \oplus \beta_n T^n(x_n), \quad n \in \mathbb{N},$$

is called an *Ishikawa iterative sequence* (see [30]).

If $\beta_n = 0$ for all $n \in \mathbb{N}$, then Algorithm 1 reduces to the following.

Algorithm 2 The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n(x_n), \quad n \in \mathbb{N},$$

is called a *Mann iterative sequence* (see [31]).

The following lemma is also needed.

Lemma 2.9 ([32, Lemma 1]) *Let* $\{s_n\}$ *and* $\{t_n\}$ *be sequences of nonnegative real numbers satisfying*

$$s_{n+1} \leq s_n + t_n \quad \textit{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\lim_{n\to\infty} s_n$ *exists*.

3 Main results

3.1 Existence theorems

Theorem 3.1 *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a continuous total asymptotically nonexpansive mapping. Then T has a fixed point in K.*

Proof Fix $x \in K$. We can consider the sequence $\{T^n(x)\}_{n=1}^{\infty}$ as a bounded sequence in K. Let $\phi : K \to [0,\infty)$ be a function defined by

$$\phi(u) := \limsup_{n\to\infty} \rho\bigl(T^n(x), u\bigr) \quad \text{for all } u \in K.$$

Then there exists $w \in K$ such that $\phi(w) = \inf\{\phi(u) : u \in K\}$. Since T is total asymptotically nonexpansive, for each $n, m \in \mathbb{N}$, we have

$$\rho\bigl(T^{n+m}(x), T^m(w)\bigr) \leq \rho\bigl(T^n(x), w\bigr) + \nu_m \psi\bigl(\rho\bigl(T^n(x), w\bigr)\bigr) + \mu_m. \tag{2}$$

Let $M = \operatorname{diam}(K)$. Taking $n \to \infty$ in (2), we get that

$$\phi\bigl(T^m(w)\bigr) \leq \phi(w) + \nu_m \psi(M) + \mu_m.$$

This implies that

$$\lim_{m\to\infty} \phi\bigl(T^m(w)\bigr) \leq \phi(w). \tag{3}$$

In view of (1), we have

$$\rho\left(T^n(x), \frac{1}{2}T^m(w) \oplus \frac{1}{2}T^h(w)\right)^2 \leq \frac{1}{2}\rho\bigl(T^n(x), T^m(w)\bigr)^2 + \frac{1}{2}\rho\bigl(T^n(x), T^h(w)\bigr)^2 - \frac{R}{8}\rho\bigl(T^m(w), T^h(w)\bigr)^2.$$

Taking $n \to \infty$, we get that

$$\phi(w)^2 \leq \phi\left(\frac{1}{2}T^m(w) \oplus \frac{1}{2}T^h(w)\right)^2 \leq \frac{1}{2}\phi\bigl(T^m(w)\bigr)^2 + \frac{1}{2}\phi\bigl(T^h(w)\bigr)^2 - \frac{R}{8}\rho\bigl(T^m(w), T^h(w)\bigr)^2,$$

yielding

$$\frac{R}{8}\rho\bigl(T^m(w), T^h(w)\bigr)^2 \leq \frac{1}{2}\phi\bigl(T^m(w)\bigr)^2 + \frac{1}{2}\phi\bigl(T^h(w)\bigr)^2 - \phi(w)^2. \tag{4}$$

By (3) and (4), we have $\lim_{m,h\to\infty} \rho(T^m(w), T^h(w))^2 \leq 0$. Therefore, $\{T^n(w)\}_{n=1}^{\infty}$ is a Cauchy sequence in K and hence converges to some point $v \in K$. Since T is continuous,

$$T(v) = T\Bigl(\lim_{n\to\infty} T^n(w)\Bigr) = \lim_{n\to\infty} T^{n+1}(w) = v. \qquad \square$$

From Theorem 3.1 we shall now derive a result for CAT(0) spaces which can also be found in [24].

Corollary 3.2 *Let (X,ρ) be a complete* CAT(0) *space and K be a nonempty bounded closed convex subset of X. If $T : K \to K$ is a continuous total asymptotically nonexpansive mapping, then T has a fixed point.*

Proof It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (*cf.* [25]). Then (K,ρ) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that K is R-convex for $R = 2$. Since K is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $\kappa > 0$ so that $\operatorname{diam}(K) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 3.1. □

3.2 Demiclosed principle

Theorem 3.3 *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta\text{-}\lim_n x_n = w$, then $w \in K$ and $w = T(w)$.*

Proof By Lemma 2.6, $w \in K$. As in Theorem 3.1, we define $\phi(u) := \limsup_n \rho(x_n, u)$ for each $u \in K$. Since $\lim_n \rho(x_n, T(x_n)) = 0$, by induction we can show that $\lim_n \rho(x_n, T^m(x_n)) = 0$ for all $m \in \mathbb{N}$ (*cf.* [16]). This implies that

$$\phi(u) = \limsup_{n\to\infty} \rho\bigl(T^m(x_n), u\bigr) \quad \text{for each } u \in K \text{ and } m \in \mathbb{N}. \tag{5}$$

In (5), taking $u = T^m(w)$, we have

$$\begin{aligned}\phi\bigl(T^m(w)\bigr) &= \limsup_{n\to\infty} \rho\bigl(T^m(x_n), T^m(w)\bigr)\\ &\leq \limsup_{n\to\infty}\bigl(\rho(x_n,w) + \nu_m \psi\bigl(\rho(x_n,w)\bigr) + \mu_m\bigr).\end{aligned}$$

Hence

$$\limsup_{m\to\infty} \phi\bigl(T^m(w)\bigr) \leq \phi(w). \tag{6}$$

In view of (1), we have

$$\rho\left(x_n, \frac{1}{2}w \oplus \frac{1}{2}T^m(w)\right)^2 \leq \frac{1}{2}\rho(x_n,w)^2 + \frac{1}{2}\rho\bigl(x_n, T^m(w)\bigr)^2 - \frac{R}{8}\rho\bigl(w, T^m(w)\bigr)^2,$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Since $\Delta\text{-}\lim_n x_n = w$, letting $n \to \infty$, we get that

$$\phi(w)^2 \leq \phi\left(\frac{1}{2}w \oplus \frac{1}{2}T^m(w)\right)^2 \leq \frac{1}{2}\phi(w)^2 + \frac{1}{2}\phi\bigl(T^m(w)\bigr)^2 - \frac{R}{8}\rho\bigl(w, T^m(w)\bigr)^2,$$

yielding

$$\rho\bigl(w, T^m(w)\bigr)^2 \leq \frac{4}{R}\bigl[\phi\bigl(T^m(w)\bigr)^2 - \phi(w)^2\bigr]. \tag{7}$$

By (6) and (7), we have $\lim_{m\to\infty} \rho(w, T^m(w)) = 0$. Since T is continuous,

$$T(w) = T\Big(\lim_{m\to\infty} T^m(w)\Big) = \lim_{m\to\infty} T^{m+1}(w) = w. \qquad \square$$

As we have observed in Corollary 3.2, we can derive the following result from Theorem 3.3.

Corollary 3.4 ([24, Theorem 12]) *Let (X, ρ) be a complete* CAT(0) *space, K be a nonempty bounded closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta\text{-}\lim_n x_n = w$, then $w \in K$ and $w = T(w)$.*

3.3 Convergence theorems

We begin this section by proving a crucial lemma.

Lemma 3.5 *Let $\kappa > 0$ and (X, ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $x_1 \in K$ and $\{x_n\}$ be a sequence in K defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n(y_n), \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n(x_n), \quad n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. Then $\{x_n\}$ is an AFPS for T and $\lim_n \rho(x_n, p)$ exists for all $p \in F(T)$.

Proof It follows from Theorem 3.1 that $F(T) \neq \emptyset$. Let $p \in F(T)$ and $M = \operatorname{diam}(K)$. Since T is total asymptotically nonexpansive, by Lemma 2.4 we have

$$\begin{aligned} \rho(y_n, p) &= \rho\big((1-\beta_n)x_n \oplus \beta_n T^n(x_n), p\big) \\ &\le (1-\beta_n)\rho(x_n, p) + \beta_n \rho\big(T^n(x_n), T^n(p)\big) \\ &\le \rho(x_n, p) + \beta_n \nu_n \psi(M) + \beta_n \mu_n. \end{aligned}$$

This implies that

$$\begin{aligned} \rho(x_{n+1}, p) &= \rho\big((1-\alpha_n)x_n \oplus \alpha_n T^n(y_n), p\big) \\ &\le (1-\alpha_n)\rho(x_n, p) + \alpha_n \rho\big(T^n(y_n), T^n(p)\big) \\ &\le (1-\alpha_n)\rho(x_n, p) + \alpha_n\big[\rho(y_n, p) + \nu_n \psi(M) + \mu_n\big] \\ &\le \rho(x_n, p) + \alpha_n(1+\beta_n)\big(\nu_n \psi(M) + \mu_n\big). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$, by Lemma 2.9 $\lim_n \rho(x_n, p)$ exists. Next, we show that $\{x_n\}$ is an AFPS for T. In view of (1), we have

$$\begin{aligned}
\rho(x_{n+1}, p)^2 &= \rho\big((1-\alpha_n)x_n \oplus \alpha_n T^n(y_n), p\big)^2 \\
&\leq (1-\alpha_n)\rho(x_n, p)^2 + \alpha_n \rho\big(T^n(y_n), p\big)^2 \\
&\leq (1-\alpha_n)\rho(x_n, p)^2 + \alpha_n \big[\rho(y_n, p) + \nu_n \psi(M) + \mu_n\big]^2 \\
&\leq (1-\alpha_n)\rho(x_n, p)^2 + \alpha_n \rho(y_n, p)^2 \\
&\quad + \alpha_n \big[2\rho(y_n, p)\big(\nu_n \psi(M) + \mu_n\big) + \big(\nu_n \psi(M) + \mu_n\big)^2\big].
\end{aligned}$$

This implies that

$$\rho(x_{n+1}, p)^2 \leq (1-\alpha_n)\rho(x_n, p)^2 + \alpha_n \rho(y_n, p)^2 + A\nu_n + B\mu_n \quad \text{for some } A, B \geq 0. \tag{8}$$

Again by (1), we have

$$\begin{aligned}
\rho(y_n, p)^2 &= \rho\big((1-\beta_n)x_n \oplus \beta_n T^n(x_n), p\big)^2 \\
&\leq (1-\beta_n)\rho(x_n, p)^2 + \beta_n \rho\big(T^n(x_n), T^n(p)\big)^2 - \frac{R}{2}\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2 \\
&\leq (1-\beta_n)\rho(x_n, p)^2 + \beta_n \big[\rho(x_n, p) + \nu_n \psi(M) + \mu_n\big]^2 \\
&\quad - \frac{R}{2}\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2 \\
&\leq \rho(x_n, p)^2 + \beta_n \big[2\rho(x_n, p)\big(\nu_n \psi(M) + \mu_n\big) + \big(\nu_n \psi(M) + \mu_n\big)^2\big] \\
&\quad - \frac{R}{2}\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2.
\end{aligned}$$

Substituting this into (8), we get that

$$\begin{aligned}
\rho(x_{n+1}, p)^2 &\leq \rho(x_n, p)^2 + \alpha_n\beta_n \big[2\rho(x_n, p)\big(\nu_n \psi(M) + \mu_n\big) + \big(\nu_n \psi(M) + \mu_n\big)^2\big] \\
&\quad - \frac{R}{2}\alpha_n\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2 + A\nu_n + B\mu_n,
\end{aligned}$$

yielding

$$\frac{R}{2}\alpha_n\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2 \leq \rho(x_n, p)^2 - \rho(x_{n+1}, p)^2 + C\nu_n + D\mu_n \quad \text{for some } C, D \geq 0.$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$, we have

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)\rho\big(x_n, T^n(x_n)\big)^2 < \infty.$$

This implies by $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$ that

$$\lim_{n\to\infty} \rho\big(x_n, T^n(x_n)\big) = 0. \tag{9}$$

By the uniform continuity of T, we have

$$\lim_{n\to\infty} \rho\big(T(x_n), T^{n+1}(x_n)\big) = 0. \tag{10}$$

It follows from (9) and the definitions of x_{n+1} and y_n that

$$\begin{aligned}\rho(x_n, x_{n+1}) &\le \rho\big(x_n, T^n(y_n)\big)\\ &\le \rho\big(x_n, T^n(x_n)\big) + \rho\big(T^n(x_n), T^n(y_n)\big)\\ &\le \rho\big(x_n, T^n(x_n)\big) + \rho(x_n, y_n) + \nu_n\psi(M) + \mu_n\\ &\le (1+\beta_n)\rho\big(x_n, T^n(x_n)\big) + \nu_n\psi(M) + \mu_n \longrightarrow 0 \quad \text{as } n\to\infty. \end{aligned} \tag{11}$$

By (9), (10), and (11), we have

$$\begin{aligned}\rho\big(x_n, T(x_n)\big) &\le \rho(x_n, x_{n+1}) + \rho\big(x_{n+1}, T^{n+1}(x_{n+1})\big)\\ &\quad + \rho\big(T^{n+1}(x_{n+1}), T^{n+1}(x_n)\big) + \rho\big(T^{n+1}(x_n), T(x_n)\big)\\ &\le \rho(x_n, x_{n+1}) + \rho\big(x_{n+1}, T^{n+1}(x_{n+1})\big) + \rho(x_{n+1}, x_n)\\ &\quad + \nu_{n+1}\psi(M) + \mu_{n+1} + \rho\big(T^{n+1}(x_n), T(x_n)\big) \longrightarrow 0 \quad \text{as } n\to\infty. \end{aligned}$$

□

Now, we are ready to prove our Δ-convergence theorem.

Theorem 3.6 *Let $\kappa > 0$ and (X,ρ) be a complete* CAT(κ) *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T: K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $x_1 \in K$ and $\{x_n\}$ be a sequence in K defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n(y_n),\\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n(x_n), \quad n\in\mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ *such that* $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. *Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n)$ is contained in $F(T)$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in K$. Hence $v \in F(T)$ by Lemma 3.5 and Theorem 3.3. Since $\lim_n \rho(x_n, v)$ exists, $u = v$ by Lemma 2.7. This shows that $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, by Lemma 3.5 $\lim_n \rho(x_n, u)$ exists. Again, by Lemma 2.7, $x = u$. This completes the proof. □

As a consequence of Theorem 3.6, we obtain the following.

Corollary 3.7 ([24, Theorem 17]) *Let (X,ρ) be a complete $\mathrm{CAT}(0)$ space, K be a nonempty bounded closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $x_1 \in K$ and $\{x_n\}$ be a sequence in K defined by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n(y_n),$$
$$y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Recall that a mapping $T : K \to K$ is said to be *semi-compact* if K is closed and each bounded AFPS for T in K has a convergent subsequence. Now, we prove a strong convergence theorem for uniformly continuous total asymptotically nonexpansive semi-compact mappings.

Theorem 3.8 *Let $\kappa > 0$ and (X,ρ) be a complete $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $x_1 \in K$ and $\{x_n\}$ be a sequence in K defined by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n(y_n),$$
$$y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. Suppose that T^m is semi-compact for some $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof By Lemma 3.5, $\lim_n \rho(x_n, T(x_n)) = 0$. Since T is uniformly continuous, we have

$$\rho\bigl(x_n, T^m(x_n)\bigr) \le \rho\bigl(x_n, T(x_n)\bigr) + \rho\bigl(T(x_n), T^2(x_n)\bigr) + \cdots + \rho\bigl(T^{m-1}(x_n), T^m(x_n)\bigr) \longrightarrow 0$$

as $n \to \infty$. That is, $\{x_n\}$ is an AFPS for T^m. By the semi-compactness of T^m, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$\rho\bigl(T(p),p\bigr) \le \rho\bigl(T(p), T(x_{n_j})\bigr) + \rho\bigl(T(x_{n_j}), x_{n_j}\bigr) + \rho(x_{n_j}, p) \longrightarrow 0 \quad \text{as } j \to \infty.$$

That is, $p \in F(T)$. By Lemma 3.5, $\lim_n \rho(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. □

Corollary 3.9 ([24, Theorem 22]) *Let (X,ρ) be a complete $\mathrm{CAT}(0)$ space, K be a nonempty bounded closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $x_1 \in K$ and $\{x_n\}$ be a sequence in K defined by*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T^n(y_n),$$
$$y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *such that* $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. *Suppose that* T^m *is semi-compact for some* $m \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 3.10 The results in this article also hold for the class of weakly total asymptotically nonexpansive mappings in the following sense. A mapping $T : K \to K$ is called *weakly total asymptotically nonexpansive* if there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\rho\big(T^n(x), T^n(y)\big) \le \rho(x,y) + \nu_n\psi\big(\rho(x,y)\big) + \mu_n \quad \text{for all } n \in \mathbb{N}, x, y \in K.$$

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author completed the paper himself. The author read and approved the final manuscript.

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