Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Hyponormal Toeplitz operators with polynomial symbols on weighted Bergman spaces

In Sung Hwang[1], Jongrak Lee[1*] and Se Won Park[2]

*Correspondence: jjonglak@skku.edu
[1]Department of Mathematics, Sungkyunkwan University, Suwon, 440-746, Korea
Full list of author information is available at the end of the article

Abstract
In this note we consider the hyponormality of Toeplitz operators T_φ on weighted Bergman space $A^2_\alpha(\mathbb{D})$ with symbol in the class of functions $f+\overline{g}$ with polynomials f and g.

MSC: Primary 47B20; 47B35

Keywords: Toeplitz operators; hyponormal; weighted Bergman space; polynomial symbols

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. For $-1 < \alpha < \infty$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ of the unit disk $\mathbb{D}$ is the space of analytic functions in $L^2(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha dA(z).$$

The space $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the inner product

$$\langle f,g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dA_\alpha(z) \quad \left(f,g \in L^2(\mathbb{D}, dA_\alpha)\right).$$

If $\alpha = 0$ then $A^2_0(\mathbb{D})$ is the Bergman space $A^2(\mathbb{D})$. For any nonnegative integer n, let

$$e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}}\,z^n \quad (z \in \mathbb{D}),$$

where $\Gamma(s)$ stands for the usual Gamma function. It is easy to check that $\{e_n\}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$ [1]. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ, and the Hankel operator H_φ on $A^2_\alpha(\mathbb{D})$ are defined by

$$T_\varphi f := P_\alpha(\varphi \cdot f) \quad \text{and} \quad H_\varphi f := (I-P_\alpha)(\varphi\cdot f) \quad \left(f \in A^2_\alpha(\mathbb{D})\right),$$

where P_α denotes the orthogonal projection that maps from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$. The reproducing kernel in $A^2_\alpha(\mathbb{D})$ is given by

$$K^{(\alpha)}_z(\omega) = \frac{1}{(1-z\overline{\omega})^{2+\alpha}},$$

for $z, \omega \in \mathbb{D}$. We thus have

$$(T_\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(\omega) f(\omega)}{(1 - z\overline{\omega})^{2+\alpha}}\, dA_\alpha(\omega),$$

for $f \in A^2_\alpha(\mathbb{D})$ and $\omega \in \mathbb{D}$.

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive (semidefinite). The hyponormality of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ has been studied by Cowen [2], Curto, Hwang and Lee [3–5] and others [6]. Recently, in [7] and [8], the hyponormality of T_φ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$ was studied. In [2], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. Here we shall employ an equivalent variant of Cowen's theorem that was first proposed by Nakazi and Takahashi [9].

Cowen's theorem ([2, 9]) *For $\varphi \in L^\infty(\mathbb{T})$, write*

$$\mathcal{E}(\varphi) := \left\{ k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T}) \right\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The solution is based on a dilation theorem of Sarason [10]. For the weighted Bergman space, no dilation theorem (similar to Sarason's theorem) is available. In [11], the first named author characterized the hyponormality of T_φ on $A^2_\alpha(\mathbb{D})$ in terms of the coefficients of the trigonometric polynomial φ under certain assumptions as regards the coefficients of φ on the *weighted Bergman space* when $\alpha \geq 0$ and in [12], extended for all $-1 < \alpha < \infty$.

Theorem A ([12]) *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1 z + a_2 z^2$ $g(z) = a_{-1} z + a_{-2} z^2$. If $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$ and $-1 < \alpha < \infty$, then*

$$T_\varphi \text{ on } A^2_\alpha(\mathbb{D}) \text{ is hyponormal}$$

$$\iff \quad \begin{cases} \frac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \geq \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & \text{if } |a_{-2}| \leq |a_2|, \\ 4(|a_{-2}|^2 - |a_2|^2) \leq |a_1|^2 - |a_{-1}|^2 & \text{if } |a_2| \leq |a_{-2}|. \end{cases}$$

In this note we consider the hyponormality of Toeplitz operators T_φ on $A^2_\alpha(\mathbb{D})$ with symbol in the class of functions $f + \overline{g}$ with polynomials f and g. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. The following relations can easily be proved:

$$T_{\varphi+\psi} = T_\varphi + T_\psi \quad (\varphi, \psi \in L^\infty); \tag{1.1}$$

$$T^*_\varphi = T_{\overline{\varphi}} \quad (\varphi \in L^\infty); \tag{1.2}$$

$$T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi} \quad \text{if } \varphi \text{ or } \psi \text{ is analytic.} \tag{1.3}$$

The purpose of this paper is to prove Theorem A for the Toeplitz operators on $A^2_\alpha(\mathbb{D})$ when f and g of degree N.

2 Main result

In this section we establish a necessary and sufficient condition for the hyponormality of the Toeplitz operator T_φ on the weighted Bergman space under a certain additional assumption concerning the symbol φ. The assumption is related on the symmetry, so it is reasonable in view point of the Hardy space [13]. We expect that this approach would provide some clue for the future study of the symmetry case.

Lemma 1 ([11]) *For any s, t nonnegative integers,*

$$P_\alpha\left(\overline{z}^t z^s\right) = \begin{cases} \frac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t, \\ 0 & \text{if } s < t. \end{cases}$$

For $0 \leq i \leq N-1$, write

$$k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}.$$

The following two lemmas will be used for proving the main result of this section.

Lemma 2 *For $0 \leq m \leq N$, we have*

(i) $$\left\|\overline{z}^m k_i(z)\right\|_\alpha^2 = \sum_{n=0}^{\infty} \frac{\Gamma(Nn+i+m+1)\Gamma(\alpha+2)}{\Gamma(Nn+i+m+\alpha+2)} |c_{Nn+i}|^2,$$

(ii) $$\left\|P_\alpha\left(\overline{z}^m k_i(z)\right)\right\|_\alpha^2 = \begin{cases} \sum_{n=0}^{\infty} \frac{\Gamma(Nn+i+1)^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & \text{if } m \leq i, \\ \sum_{n=1}^{\infty} \frac{\Gamma(Nn+i+1)^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & \text{if } m > i. \end{cases}$$

Proof Let $0 \leq m \leq N$. Then we have

$$\begin{aligned} \left\|\overline{z}^m k_i(z)\right\|_\alpha^2 &= \left\| \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m} \right\|_\alpha^2 \\ &= \sum_{n=0}^{\infty} |c_{Nn+i}|^2 \left\| z^{Nn+i+m} \right\|_\alpha^2 \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(Nn+i+m+1)\Gamma(\alpha+2)}{\Gamma(Nn+i+m+\alpha+2)} |c_{Nn+i}|^2. \end{aligned}$$

This proves (i). For (ii), if $m \leq i$ then by Lemma 1 we have

$$\begin{aligned} \left\|P_\alpha\left(\overline{z}^m k_i(z)\right)\right\|_\alpha^2 &= \left\| \sum_{n=0}^{\infty} \frac{\Gamma(Nn+i+1)\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)\Gamma(Nn+i-m+1)} c_{Nn+i} z^{Nn+i-m} \right\|_\alpha^2 \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(Nn+i+1)^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2. \end{aligned}$$

If instead $m > i$, a similar argument gives the result. □

Lemma 3 ([14]) *Let* $f(z) = a_{N-1}z^{N-1} + a_N z^N$ *and* $g(z) = a_{-(N-1)}z^{N-1} + a_{-N}z^N$. *If* $a_{N-1}\overline{a_N} = a_{-(N-1)}\overline{a_{-N}}$, *then for* $i \neq j$, *we have*

$$\langle H_{\overline{f}}k_i(z), H_{\overline{f}}k_j(z)\rangle_\alpha = \langle H_{\overline{g}}k_i(z), H_{\overline{g}}k_j(z)\rangle_\alpha.$$

Our main result now follows.

Theorem 4 *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_{N-1}z^{N-1} + a_N z^N \quad \textit{and} \quad g(z) = a_{-(N-1)}z^{N-1} + a_{-N}z^N.$$

If $a_{N-1}\overline{a_N} = a_{-(N-1)}\overline{a_{-N}}$ *and* $|a_{-N}| \leq |a_N|$, *then* T_φ *on* $A^2_\alpha(\mathbb{D})$ *is hyponormal if and only if*

$$\frac{1}{N+\alpha+1}\big(|a_N|^2 - |a_{-N}|^2\big) \geq \frac{1}{N}\big(|a_{-(N-1)}|^2 - |a_{N-1}|^2\big).$$

Proof For $0 \leq i < N$, put

$$K_i := \left\{ k_i(z) \in A^2_\alpha(\mathbb{D}) : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i}z^{Nn+i} \right\}.$$

Then a straightforward calculation shows that T_φ is hyponormal if and only if

$$\left\langle \big(H^*_{\overline{f}}H_{\overline{f}} - H^*_{\overline{g}}H_{\overline{g}}\big)\sum_{i=0}^{N-1}k_i(z), \sum_{i=0}^{N-1}k_i(z)\right\rangle_\alpha \geq 0 \quad \text{for all } k_i \in K_i \ (i = 0,1,\ldots,N-1). \tag{2.1}$$

Also we have

$$\begin{aligned}
&\left\langle H^*_{\overline{f}}H_{\overline{f}}\sum_{i=0}^{N-1}k_i(z), \sum_{i=0}^{N-1}k_i(z)\right\rangle_\alpha \\
&\quad = \sum_{i=0}^{N-1}\langle H_{\overline{f}}k_i(z), H_{\overline{f}}k_i(z)\rangle_\alpha + \sum_{i\neq j,i,j\geq 0}^{N-1}\langle H_{\overline{f}}k_i(z), H_{\overline{f}}k_k(z)\rangle_\alpha
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\left\langle H^*_{\overline{g}}H_{\overline{g}}\sum_{i=0}^{N-1}k_i(z), \sum_{i=0}^{N-1}k_i(z)\right\rangle_\alpha \\
&\quad = \sum_{i=0}^{N-1}\langle H_{\overline{g}}k_i(z), H_{\overline{g}}k_i(z)\rangle_\alpha + \sum_{i\neq j,i,j\geq 0}^{N-1}\langle H_{\overline{g}}k_i(z), H_{\overline{g}}k_k(z)\rangle_\alpha.
\end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), it follows from Lemma 3 that

$$\begin{aligned}
T_\varphi\text{: hyponormal} \quad &\iff \quad \sum_{i=0}^{N-1}\big\langle\big(H^*_{\overline{f}}H_{\overline{f}} - H^*_{\overline{g}}H_{\overline{g}}\big)k_i(z), k_i(z)\big\rangle_\alpha \geq 0 \\
&\iff \quad \sum_{i=0}^{N-1}\big(\|\overline{f}k_i\|^2_\alpha - \|\overline{g}k_i\|^2_\alpha + \big\|P_\alpha(\overline{g}k_i)\big\|^2_\alpha - \big\|P_\alpha(\overline{f}k_i)\big\|^2_\alpha\big) \geq 0.
\end{aligned}$$

Therefore it follows from Lemma 2 that T_φ is hyponormal if and only if

$$\begin{aligned}
&\left(|a_{N-1}|^2 - |a_{-(N-1)}|^2\right)\left[\sum_{i=0}^{N-2}\left\{\frac{\Gamma(i+N)\Gamma(\alpha+2)}{\Gamma(i+N+\alpha+1)}|c_i|^2 + \sum_{n=1}^{\infty}\left(\frac{\Gamma(Nn+i+N)\Gamma(\alpha+2)}{\Gamma(Nn+i+N+\alpha+1)}\right.\right.\right.\\
&\quad\left.\left.-\frac{\Gamma(Nn+i+1)^2\Gamma(Nn+i-N+\alpha+3)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-N+2)}\right)|c_{Nn+i}|^2\right\}\\
&\quad\left.+\sum_{n=0}^{\infty}\left(\frac{\Gamma(Nn+2N-1)\Gamma(\alpha+2)}{\Gamma(Nn+2N+\alpha)} - \frac{\Gamma(Nn+N)^2\Gamma(Nn+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+N+\alpha+1)^2\Gamma(Nn+1)}\right)|c_{Nn+N-1}|^2\right]\\
&\quad+\left(|a_N|^2 - |a_{-N}|^2\right)\left[\sum_{i=0}^{N-1}\left\{\frac{\Gamma(N+i+1)\Gamma(\alpha+2)}{\Gamma(i+n+\alpha+2)}|c_i|^2\right.\right.\\
&\quad+\sum_{n=1}^{\infty}\left(\frac{\Gamma(Nn+i+N+1)\Gamma(\alpha+2)}{\Gamma(Nn+i+N+\alpha+2)}\right.\\
&\quad\left.\left.\left.-\frac{\Gamma(Nn+i+1)^2\Gamma(Nn+i-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-N+1)}\right)|c_{Nn+i}|^2\right\}\right] \geq 0,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\left(|a_{N-1}|^2 - |a_{-(N-1)}|^2\right)\left\{\sum_{n=0}^{N-2}\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)}|c_n|^2 + \sum_{n=N-1}^{\infty}\left(\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)}\right.\right.\\
&\quad\left.\left.-\frac{\Gamma(n+1)^2\Gamma(n-N+\alpha+3)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)^2\Gamma(n-N+2)}\right)|c_n|^2\right\}\\
&\quad+\left(|a_N|^2 - |a_{-N}|^2\right)\left\{\sum_{n=0}^{N-1}\frac{\Gamma(n+N+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)}|c_n|^2 + \sum_{n=N}^{\infty}\left(\frac{\Gamma(N+n+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)}\right.\right.\\
&\quad\left.\left.-\frac{\Gamma(n+1)^2\Gamma(n-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)^2\Gamma(n-N+1)}\right)|c_n|^2\right\} \geq 0. \qquad (2.4)
\end{aligned}$$

Define ζ_α by

$$\zeta_\alpha(n) := \frac{\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)} - \frac{\Gamma(n+1)^2\Gamma(n-N+\alpha+3)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)^2\Gamma(n-N+2)}}{\frac{\Gamma(N+n+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)} - \frac{\Gamma(n+1)^2\Gamma(n-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)^2\Gamma(n-N+1)}} \quad (n \geq 1).$$

Then a direct calculation gives

$$\zeta_\alpha(n) < \frac{\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)}}{\frac{\Gamma(N+n+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)}}.$$

Observe that

$$\begin{aligned}
\frac{N+\alpha+1}{N} &\geq \frac{N+n+\alpha+1}{N+n} \geq \frac{N+N_i+\alpha+1}{N+N_i}\\
&\geq \zeta_\alpha(N_i) \quad \text{for all } N_i \geq N \text{ and } n = 1, 2, \ldots, N-1; \qquad (2.5)
\end{aligned}$$

and

$$\frac{N+\alpha+1}{N} \geq \frac{\frac{\Gamma(2N-1)\Gamma(\alpha+2)}{\Gamma(2N+\alpha)} - \frac{\Gamma(N)^2\Gamma(\alpha+2)^2}{\Gamma(N+\alpha+1)^2}}{\frac{\Gamma(2N)\Gamma(\alpha+2)}{\Gamma(2N+\alpha+1)}}.$$

Therefore (2.4) and (2.5) show that T_φ is hyponormal if and only if

$$\frac{1}{N+\alpha+1}\big(|a_N|^2 - |a_{-N}|^2\big) \geq \frac{1}{N}\big(|a_{-(N-1)}|^2 - |a_{N-1}|^2\big).$$

This completes the proof. □

Remark 5 Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = a_{N-1}z^{N-1} + a_N z^N \quad \text{and} \quad g(z) = a_{-(N-1)}z^{N-1} + a_{-N}z^N.$$

If $a_{N-1}\overline{a_N} = a_{-(N-1)}\overline{a_{-N}}$, $|a_N| \leq |a_{-N}|$, and T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal. Then

$$|a_{-N}|^2 - |a_N|^2 \leq \left\{\frac{2N+\alpha}{2N-1} - \frac{\Gamma(N)^2\Gamma(2N+\alpha+1)\Gamma(\alpha+2)}{\Gamma(2N)\Gamma(N+\alpha+1)^2}\right\}\big(|a_{N-1}|^2 - |a_{-(N-1)}|^2\big).$$

Proof If we let $c_j = 1$ for $0 \leq j \leq N-1$ and the other c_j's be 0 into (2.4), then we have

$$\begin{aligned}&\big(|a_{N-1}|^2 - |a_{-(N-1)}|^2\big)\left\{\sum_{n=0}^{N-2}\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)}\right.\\ &\quad\left.+\left(\frac{\Gamma(2N-1)\Gamma(\alpha+2)}{\Gamma(2N+\alpha)} - \frac{\Gamma(N)^2\Gamma(\alpha+2)^2}{\Gamma(N+\alpha+1)^2\Gamma(1)}\right)\right\}\\ &\quad+\big(|a_N|^2 - |a_{-N}|^2\big)\sum_{n=0}^{N-1}\frac{\Gamma(n+N+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)} \geq 0. \end{aligned} \tag{2.6}$$

Define ξ_α by

$$\xi_\alpha(n) := \frac{\frac{\Gamma(N+n)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+1)}}{\frac{\Gamma(N+n+1)\Gamma(\alpha+2)}{\Gamma(N+n+\alpha+2)}} \quad (0 \leq n \leq N-1).$$

Then $\xi_\alpha(n)$ is a strictly decreasing function and

$$\begin{aligned}&\frac{N+n+\alpha+1}{N+n} \geq \frac{2N+\alpha}{2N-1} \geq \frac{2N+\alpha}{2N-1} - \frac{\Gamma(N)^2\Gamma(2N+\alpha+1)\Gamma(\alpha+2)}{\Gamma(2N)\Gamma(N+\alpha+1)^2}\\ &\quad\text{for all } n = 0,1,\ldots,N-1.\end{aligned} \tag{2.7}$$

Therefore (2.6) and (2.7) give that if T_φ is hyponormal then

$$\left\{\frac{2N+\alpha}{2N-1} - \frac{\Gamma(N)^2\Gamma(2N+\alpha+1)\Gamma(\alpha+2)}{\Gamma(2N)\Gamma(N+\alpha+1)^2}\right\}\big(|a_{N-1}|^2 - |a_{-(N-1)}|^2\big) \geq |a_{-N}|^2 - |a_N|^2.$$

This completes the proof. □

Example 6 Let $\varphi(z) = 2\overline{z}^2 + \frac{3}{2}\overline{z} + \frac{7}{2}z + \frac{6}{7}z^2$ and $\alpha = 0$. Then by Theorem A, T_φ is not hyponormal. But φ satisfies the inequality in Remark 5, hence the inverse of Remark 5 is not satisfied.

Remark 7 Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero. Suppose T_φ on $H^2(\mathbb{T})$ is hyponormal. It is well known [15] that

$$N - m \leq \operatorname{rank}\left[T_\varphi^*, T_\varphi\right] \leq N.$$

However, the result cannot be extended to the case of $A^2(\mathbb{D})$; for example, if $\varphi(z) = a_{-1}\overline{z} + a_1 z$ then a straightforward calculation shows that the selfcommutator of Toeplitz operator T_φ on $A^2(\mathbb{D})$ is given by

$$\left[T_\varphi^*, T_\varphi\right] = \left(|a_1|^2 - |a_{-1}|^2\right) \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & 0 & \cdots \\ 0 & 0 & \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\alpha_n = \frac{1}{n(n+1)}$. Thus $\operatorname{rank}[T_\varphi^*, T_\varphi] = \infty$ and the trace of the selfcommutator $\operatorname{tr}[T_\varphi^*, T_\varphi] = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sungkyunkwan University, Suwon, 440-746, Korea. [2]Department of Mathematics, Shingyeong University, Hwaseong, 445-741, Korea.

Acknowledgements
This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (2011-0022577). The authors are grateful to the referee for several helpful suggestions.

Received: 16 May 2014 Accepted: 26 August 2014 Published: 02 Sep 2014

References
1. Zhu, K: Theory of Bergman Spaces. Springer, New York (2000)
2. Cowen, C: Hyponormality of Toeplitz operators. Proc. Am. Math. Soc. **103**, 809-812 (1988)
3. Curto, RE, Hwang, IS, Lee, WY: Hyponormality and subnormality of block Toeplitz operators. Adv. Math. **230**, 2094-2151 (2012)
4. Curto, RE, Lee, WY: Joint hyponormality of Toeplitz pairs. Mem. Am. Math. Soc. **150**, 712 (2001)
5. Hwang, IS, Lee, WY: Hyponormality of trigonometric Toeplitz operators. Trans. Am. Math. Soc. **354**, 2461-2474 (2002)
6. Hwang, IS, Kim, IH, Lee, WY: Hyponormality of Toeplitz operators with polynomial symbol. Math. Ann. **313**, 247-261 (1999)
7. Lu, Y, Shi, Y: Hyponormal Toeplitz operators on the weighted Bergman space. Integral Equ. Oper. Theory **65**, 115-129 (2009)
8. Lu, Y, Liu, C: Commutativity and hyponormality of Toeplitz operators on the weighted Bergman space. J. Korean Math. Soc. **46**, 621-642 (2009)
9. Nakazi, T, Takahashi, K: Hyponormal Toeplitz operators and extremal problems of Hardy spaces. Trans. Am. Math. Soc. **338**, 759-769 (1993)
10. Sarason, D: Generalized interpolation in H^∞. Trans. Am. Math. Soc. **127**, 179-203 (1967)
11. Hwang, IS, Lee, JR: Hyponormal Toeplitz operators on the weighted Bergman spaces. Math. Inequal. Appl. **15**, 323-330 (2012)
12. Lee, J, Lee, Y: Hyponormality of Toeplitz operators on the weighted Bergman spaces. Honam Math. J. **35**, 311-317 (2013)
13. Farenick, DR, Lee, WY: Hyponormality and spectra of Toeplitz operators. Trans. Am. Math. Soc. **348**, 4153-4174 (1996)

14. Hwang, IS: Hyponormal Toeplitz operators on the Bergman spaces. J. Korean Math. Soc. **42**, 387-403 (2005)
15. Farenick, DR, Lee, WY: On hyponormal Toeplitz operators with polynomial and circulant-type symbols. Integral Equ. Oper. Theory **29**, 202-210 (1997)

10.1186/1029-242X-2014-335
Cite this article as: Hwang et al.: **Hyponormal Toeplitz operators with polynomial symbols on weighted Bergman spaces.** *Journal of Inequalities and Applications* 2014, 2014:335