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New iteration scheme for numerical reckoning fixed points of nonexpansive mappings

Dipti Thakur[1], Balwant Singh Thakur[1] and Mihai Postolache[2*]

[*]Correspondence:
emscolar@yahoo.com
[2]Department of Mathematics & Informatics, University Politehnica of Bucharest, Bucharest, 060042, Romania
Full list of author information is available at the end of the article

Abstract

The purpose of this paper is to introduce a new three step iteration scheme for approximation of fixed points of the nonexpansive mappings. We show that our iteration process is faster than all of the Picard, the Mann, the Agarwal *et al.*, and the Abbas *et al.* iteration processes. We support our analytic proof by a numerical example in which we approximate the fixed point by a computer using Matlab program. We also prove some weak convergence and strong convergence theorems for the nonexpansive mappings.

MSC: 47H09; 47H10

Keywords: fixed point; nonexpansive mapping; strong and weak convergence theorems

1 Introduction

Many nonlinear equations are naturally formulated as fixed point problems,

$$x = Tx, \tag{1.1}$$

where T, the fixed point mapping, may be nonlinear. A solution x^* of the problem (1.1) is called a *fixed point* of the mapping T. Consider a *fixed point iteration*, which is given by

$$x_{n+1} = Tx_n. \tag{1.2}$$

The iterative method (1.2) is also called a *Richardson iteration*, a *Picard iteration*, or *the method of successive substitution*. The standard result for a fixed point iteration is the *contraction mapping theorem*. Indeed, the contraction mapping theorem holds on an arbitrary complete metric space; that is, if E is a complete metric space with metric d and $T: E \to E$ such that $d(Tx, Ty) \leq kd(x, y)$ for some $0 \leq k < 1$ and all $x, y \in E$, then T has a unique fixed point x^* and the iterates (1.2) converge to the fixed point x^*. The Picard iteration has been successfully employed in approximating the fixed point of contraction mappings and its variants. This success, however, has not extended to nonexpansive mappings T even when the existence of a fixed point of T is known. Consider the simple example of a self mapping in $[0, 1]$ defined by $Tx = 1 - x$ for $0 \leq x \leq 1$. Then T is a nonexpansive mapping with a unique fixed point at $x = \frac{1}{2}$. If one chooses as a starting value $x = a$, $a \neq \frac{1}{2}$, then the successive iterations of T yield the sequence $\{1 - a, a, 1 - a, a, \ldots\}$. Thus when a fixed point of nonexpansive mappings exists, other approximation techniques are needed to approximate it.

Consider an average mapping of the form $T_{\frac{1}{2}} = \frac{1}{2}I + \frac{1}{2}T$, where I is the identity operator. This average mapping is nonexpansive because T is nonexpansive, and both have the same fixed point set. Krasnosel'skii [1] was first to notice the regularization effect of this average mapping. Schaefer [2] proved a convergence result for a general $T_\lambda = \lambda I + (1-\lambda)T$ $(0 < \lambda < 1)$. An approximation of fixed points of a nonexpansive mapping using Mann's algorithm [3] has extensively been studied in the literature (see, *e.g.*, [4, 5] and references therein). Mann's algorithm generates, for an arbitrary $x_0 \in C$, a sequence $\{x_n\}$ according to the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is a real control sequence in the interval $(0, 1)$.

In 1974, Ishikawa [6] introduced an iteration process where $\{x_n\}$ is defined iteratively for each positive integer $n \geq 0$ by

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n. \end{aligned}\right\} \tag{1.4}$$

In 2000, Noor [7] introduced the following iterative scheme: for any fixed $x_0 \in C$, construct $\{x_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \end{aligned}\right\} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$.

In 2007, Agarwal *et al.* [8] introduced the following iteration process: for an arbitrary $x_0 \in C$ construct a sequence $\{x_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \in \mathbb{N}, \end{aligned}\right\} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. They showed that this process converges at a rate that is the same as that of the Picard iteration and faster than the Mann iteration for contractions.

Recently, Abbas and Nazir [9] introduced the following iteration: for an arbitrary $x_0 \in C$ construct $\{x_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1 - \alpha_n)Ty_n + \alpha_n Tz_n, \\ y_n &= (1 - \beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \end{aligned}\right\} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $(0, 1)$. They showed that this process converges faster than the Agarwal *et al.* [8] iteration process.

Motivated and inspired by the above work, in this paper we introduce a new iterative scheme, where the sequence $\{x_n\}$ is generated from arbitrary $x_0 \in C$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n, \end{aligned}\right\} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $(0,1)$.

The purpose of this paper is to prove that our process (1.8) converges faster than all of the Picard, the Mann, the Ishikawa, the Noor, the Agarwal *et al.*, and the Abbas *et al.* iteration processes for contractions in the sense of Berinde [10]. We also prove weak and strong convergence theorems for nonexpansive mapping using iteration (1.8). In the last section, using a numerical example, we compare the behavior of iteration (1.8) with respect to the above mentioned iteration processes.

2 Rate of convergence

Berinde [10] proposed a method to compare the fastness of two sequences.

Definition 2.1 Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.1}$$

(i) If $l = 0$, then it can be said that $\{a_n\}$ converges faster to a than $\{b_n\}$ to b.
(ii) If $0 < l < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose that, for two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \le a_n, \quad n = 0,1,2,\ldots, \tag{2.2}$$

$$\|v_n - p\| \le b_n, \quad n- = 0,1,2,\ldots, \tag{2.3}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers (converging to zero).

Then, in view of Definition 2.1, Berinde [10] adopted the following concept.

Definition 2.2 Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (2.2) and (2.3), respectively. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{u_n\}$ converges faster than $\{v_n\}$ to p.

In recent years, Definition 2.2 has been used as a standard tool to compare the fastness of two fixed point iterations. Using this technique Sahu [11] established that the Agarwal *et al.* iteration (1.6) converges faster than the Mann (1.3) and the Picard (1.2) iterations and supported the claim by the following example.

Example 1 Let $X = \mathbb{R}$ and $K = [0,\infty)$. Let $T: K \to K$ be a mapping defined by $Tx = (3x + 18)^{\frac{1}{3}}$ for all $x \in K$. For $x_0 = 1{,}000$ and $\alpha_n = \beta_n = \frac{1}{2}$, $n = 0,1,2,\ldots$, Agarwal *et al.* iteration is faster than both the Mann and the Picard iteration.

Using a similar technique Abbas and Nazir [9] established that the Abbas *et al.* iteration (1.7) converges faster than the Agarwal *et al.* iteration (1.6) and hence it converges faster than the Mann (1.3) and the Picard (1.2) iterations also. An example is also given in support of the claim.

Example 2 Let $X = \mathbb{R}$ and $K = [1, 50]$. Let $T : K \to K$ be a mapping defined by $Tx = \sqrt{x^2 - 8x + 40}$ for all $x \in K$. For $x_0 = 30$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, $n = 0, 1, 2, \ldots$, the Abass *et al.* iteration (1.7) is faster than the Agarwal *et al.* iteration (1.6). Since Sahu [11] already has shown that the iteration (1.6) is faster than the Mann iteration (1.3), the iteration (1.7) is faster than the iterations (1.2), (1.3), and (1.6).

We now show that our process (1.8) converges faster than (1.7) in the sense of Berinde [10].

Theorem 2.3 *Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction factor $k \in (0, 1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process* (1.7) *and $\{x_n\}$ by* (1.8), *where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $[\varepsilon, 1 - \varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0, 1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process* (1.8) *converges faster than* (1.7).

Proof As proved in Theorem 3 of Abbas and Nazir [9],

$$\|u_{n+1} - p\| \le k^n \bigl[1 - (1 - k)\alpha\beta\gamma\bigr]^n \|u_1 - p\|,$$

for all $n \in \mathbb{N}$. Let

$$a_n = k^n \bigl[1 - (1 - k)\alpha\beta\gamma\bigr]^n \|u_1 - p\|.$$

Now

$$\begin{aligned}
\|z_n - p\| &= \bigl\|(1 - \gamma_n)x_n + \gamma_n Tx_n - p\bigr\| \\
&\le (1 - \gamma_n)\|x_n - p\| + k\gamma_n\|x_n - p\| \\
&= \bigl(1 - (1 - k)\gamma_n\bigr)\|x_n - p\|,
\end{aligned}$$

so that

$$\begin{aligned}
\|y_n - p\| &= \bigl\|(1 - \beta_n)z_n + \beta_n Tz_n - p\bigr\| \\
&\le (1 - \beta_n)\|z_n - p\| + k\beta_n\|z_n - p\| \\
&\le (1 - \beta_n)\bigl(1 - (1 - k)\gamma_n\bigr)\|x_n - p\| + k\beta_n\bigl(1 - (1 - k)\gamma_n\bigr)\|x_n - p\| \\
&= \bigl(1 - (1 - k)\beta_n\bigr)\bigl(1 - (1 - k)\gamma_n\bigr)\|x_n - p\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - p\| &= \bigl\|(1 - \alpha_n)Tx_n + \alpha_n Ty_n - p\bigr\| \\
&\le (1 - \alpha_n)k\|x_n - p\| + k\alpha_n\|y_n - p\|
\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)k\|x_n-p\| + k\alpha_n\bigl(1-(1-k)\beta_n\bigr)\bigl(1-(1-k)\gamma_n\bigr)\|x_n-p\| \\
&= k\bigl[1-\alpha_n+\alpha_n\bigl(1-(1-k)\beta_n\bigr)\bigl(1-(1-k)\gamma_n\bigr)\bigr]\|x_n-p\| \\
&= k\bigl[1-\alpha_n+\bigl(\alpha_n-(1-k)\alpha_n\beta_n\bigr)\bigl(1-(1-k)\gamma_n\bigr)\bigr]\|x_n-p\| \\
&= k\bigl[1-\alpha_n+\alpha_n-(1-k)\alpha_n\gamma_n-(1-k)\alpha_n\beta_n+(1-k)^2\alpha_n\beta_n\gamma_n\bigr]\|x_n-p\| \\
&\le k\bigl[1-(1-k)\alpha_n\beta_n\gamma_n-(1-k)\alpha_n\beta_n\gamma_n+(1-k)^2\alpha_n\beta_n\gamma_n\bigr]\|x_n-p\| \\
&= k\bigl(1-(1-k)(1+k)\alpha_n\beta_n\gamma_n\bigr)\|x_n-p\| \\
&= k\bigl(1-\bigl(1-k^2\bigr)\alpha_n\beta_n\gamma_n\bigr)\|x_n-p\|.
\end{aligned}$$

Let

$$b_n = k^n\bigl(1-\bigl(1-k^2\bigr)\alpha\beta\gamma\bigr)^n\|x_1-p\|.$$

Then

$$\begin{aligned}
\frac{b_n}{a_n} &= \frac{k^n(1-(1-k^2)\alpha\beta\gamma)^n\|x_1-p\|}{k^n[1-(1-k)\alpha\beta\gamma]^n\|u_1-p\|} \\
&= \frac{(1-(1-k^2)\alpha\beta\gamma)^n}{(1-(1-k)\alpha\beta\gamma)^n}\frac{\|x_1-p\|}{\|u_1-p\|} \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

Now, we present an example which shows that the new iteration process (1.8) converges at a rate faster than the existing iteration schemes mentioned above.

Example 3 Let $E=\mathbb{R}$ and $C=[1,50]$. Let $T\colon C\to C$ be a mapping defined by $T(x)=\sqrt{x^2-8x+40}$ for all $x\in C$. Choose $\alpha_n=0.85$, $\beta_n=0.65$, $\gamma_n=0.45$, with the initial value $x_1=40$. Our corresponding iteration process, the Abbas and Nazir iteration process (1.7), the Agarwal *et al.* iteration process (1.6), the Noor iteration process (1.7), the Ishikawa iteration process (1.4), the Mann iteration process (1.3), and the Picard iteration processes (1.2) are, respectively, given in Table 1.

All sequences converge to $x^*=5$. Comparison shows that our iteration process (1.8) converges fastest among all the iterations considered in the example.

3 Convergence theorems

In this section, we give some convergence theorems using our iteration process (1.8); please, see Table 1 and Figure 1.

Lemma 3.1 *Let C be a nonempty closed convex subset of a norm space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by* (1.8) *and $F(T)\neq\emptyset$. Then $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$.*

Table 1 Comparative results

Step	Picard	Mann	Ishikawa	Noor	Agarwal	Abbas	New iter.
1	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000
2	36.3318042492	36.8820336118	34.8751575132	33.9816211055	34.3249281505	34.2399531822	32.9458774280
3	32.7008496221	33.7905308732	29.8335259837	28.0882816012	28.7529148550	28.5873914017	26.0696692526
4	29.1159538575	30.7306375124	24.9067432334	22.3811620460	23.3289757744	23.0905148078	19.4826041425
5	25.5892777970	27.7090706072	20.1467307646	16.9736024952	18.1321892967	17.8350979079	13.4242477938
6	22.1381326176	24.7347891266	15.6449263114	12.0962209155	13.3147454600	12.9887334680	8.4745882697
7	18.7880774656	21.8200359935	11.5741197024	8.2289280979	9.1939307941	8.9032413368	5.7279660470
8	15.5784221001	18.9820007784	8.2638548016	6.0182077910	6.3717274607	6.2123720180	5.0765141830
9	12.5721859009	16.2455313784	6.1736938982	5.2517005165	5.2434387591	5.2064678069	5.0064676549
10	9.8733161157	13.6475866165	5.3185408455	5.0576355955	5.0298139084	5.0252795464	5.0005330507
11	7.6482574613	11.2442765494	5.0768890301	5.0129587850	5.0033662656	5.0028941692	5.0000438381
12	6.1081734180	9.1201110370	5.0179832209	5.0029016212	5.0003761718	5.0003285479	5.0000036046
13	5.3333287129	7.3913650188	5.0041744485	5.0006491038	5.0000419870	5.0000372607	5.0000002964
14	5.0771808572	6.1732610225	5.0009673150	5.0001451769	5.0000046858	5.0000042253	5.0000000244
15	5.0160062399	5.4814708358	5.0002240577	5.0000324684	5.0000005229	5.0000004791	5.0000000020
16	5.0032258274	5.1725897008	5.0000518932	5.0000072614	5.0000000584	5.0000000543	5.0000000002
17	5.0006461643	5.0576419946	5.0000120186	5.0000016240	5.0000000065	5.0000000062	5.0000000000
18	5.0001292729	5.0187159301	5.0000027835	5.0000003632	5.0000000007	5.0000000007	5.0000000000
19	5.0000258562	5.0060176595	5.0000006447	5.0000000812	5.0000000001	5.0000000001	5.0000000000
20	5.0000051713	5.0019286052	5.0000001493	5.0000000182	5.0000000000	5.0000000000	5.0000000000
21	5.0000010343	5.0006174572	5.0000000346	5.0000000041	5.0000000000	5.0000000000	5.0000000000
22	5.0000002069	5.0001976174	5.0000000080	5.0000000009	5.0000000000	5.0000000000	5.0000000000
23	5.0000000414	5.0000632408	5.0000000019	5.0000000002	5.0000000000	5.0000000000	5.0000000000
24	5.0000000083	5.0000202374	5.0000000004	5.0000000000	5.0000000000	5.0000000000	5.0000000000
25	5.0000000017	5.0000064760	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000
26	5.0000000003	5.0000020723	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
27	5.0000000001	5.0000006631	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
28	5.0000000000	5.0000002122	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
29	5.0000000000	5.0000000679	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
30	5.0000000000	5.0000000217	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
31	5.0000000000	5.0000000070	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
32	5.0000000000	5.0000000022	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
33	5.0000000000	5.0000000007	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
34	5.0000000000	5.0000000002	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
35	5.0000000000	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000

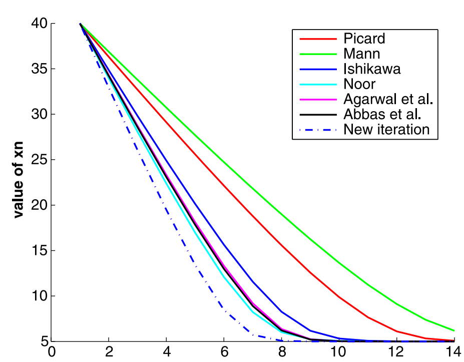

Figure 1 Convergence behavior of the Picard, the Mann, the Ishikawa, the Noor, the Agarwal *et al.*, the Abbas *et al.* iterations, and new iteration for the function given in Example 3.

Proof Let $p \in F(T)$ for all $n \in \mathbb{N}$. From (1.8), we have

$$\begin{aligned} \|z_n - p\| &= \big\|(1-\gamma_n)x_n + \gamma_n Tx_n - p\big\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\|y_n - p\| &= \big\|(1-\beta_n)z_n + \beta_n Tz_n - p\big\| \\ &\le (1-\beta_n)\|z_n - p\| + \beta_n\|Tz_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|,\end{aligned} \tag{3.2}$$

thus from (3.1) and (3.2)

$$\begin{aligned}\|x_{n+1} - p\| &= \big\|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\big\| \\ &\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

Thus $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. □

We need following lemma to establish our next result.

Lemma 3.2 [12] *Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty}\|x_n\| \le r$, $\limsup_{n\to\infty}\|y_n\| \le r$ and $\limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

We now establish a result which will be of key importance for the main result.

Lemma 3.3 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$ and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof By Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - p\| = c$.
From (3.1) and (3.2) we have

$$\limsup_{n\to\infty}\|y_n - p\| \le c \tag{3.3}$$

and

$$\limsup_{n\to\infty}\|z_n - p\| \le c. \tag{3.4}$$

Since T is a nonexpansive mapping, it follows that

$$\|Tx_n - p\| \le \|x_n - p\|$$

and

$$\|Ty_n - p\| \le \|y_n - p\|.$$

Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \|Tx_n - p\| \le c \tag{3.5}$$

and

$$\limsup_{n\to\infty} \|Ty_n - p\| \le c. \tag{3.6}$$

Since

$$c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \big\|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\big\|,$$

by using Lemma 3.2, we have

$$\lim_{n\to\infty} \|Tx_n - Ty_n\| = 0. \tag{3.7}$$

Now

$$\|x_{n+1} - p\| = \big\|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\big\| \le \|Tx_n - p\| + \alpha_n\|Tx_n - Ty_n\|$$

yields

$$c \le \liminf_{n\to\infty} \|Tx_n - p\|, \tag{3.8}$$

so that (3.5) and (3.8) give

$$\lim_{n\to\infty} \|Tx_n - p\| = c. \tag{3.9}$$

On the other hand, we have

$$\|Tx_n - p\| \le \|Tx_n - Ty_n\| + \|Ty_n - p\| \le \|Tx_n - Ty_n\| + \|y_n - p\|,$$

which yields

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.10}$$

From (3.3) and (3.10) we get

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Since T is a nonexpansive mapping, we have from (3.1)

$$\limsup_{n\to\infty} \|Tz_n - p\| \le c. \tag{3.11}$$

From (3.4) and (3.11), by using Lemma 3.2 we obtain

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.12}$$

Since

$$\|y_n - p\| \le \|z_n - p\| + \beta_n \|Tz_n - z_n\|,$$

we write

$$c \le \limsup_{n\to\infty} \|z_n - p\|, \tag{3.13}$$

then

$$\|z_n - p\| = c, \tag{3.14}$$

so

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \big\|(1-\alpha_n)x_n + \alpha_n Tx_n - p\big\| \\ &= \lim_{n\to\infty} \big\|(1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\big\|, \end{aligned}$$

and by Lemma 3.2, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Lemma 3.4 [13] *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space and $T\colon C \to E$ be a nonexpansive mapping. Then there is a strictly increasing and continuous convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$g\Big(\big\|T\big(tx + (1-t)y\big) - \big(tTx + (1-t)Ty\big)\big\|\Big) \le \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ and $t \in [0,1]$.

Lemma 3.5 *For any $p_1, p_2 \in F(T)$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists, for all $t \in [0,1]$ under the conditions of Lemma* 3.3.

Proof By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ and hence $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex nonempty subset of C. Set

$$a_n(t) := \big\|tx_n + (1-t)p_1 - p_2\big\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and, from Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \|x_n - p_2\|$ exist.

Now it remains to show that $\lim_{n\to\infty} a_n(t)$ exists for $t \in (0,1)$.

For each $n \in N$, define $W_n : D \to D$ by

$$\begin{cases} W_n x = (1-\alpha_n)Tx + \alpha_n TV_n x, \\ V_n x = (1-\beta_n)U_n x + \beta_n TU_n x, \\ U_n x = (1-\gamma_n)x + \gamma_n Tx \end{cases}$$

for all $x \in D$.

We see that

$$\|U_n x - U_n y\| \le \|x - y\|, \quad \forall x, y \in D,$$

and

$$\|V_n x - V_n y\| \le \|x - y\|, \quad \forall x, y \in D,$$

hence,

$$\|W_n x - W_n y\| \le \|x - y\|, \quad \forall x, y \in D.$$

Set

$$R_{n,m} = W_{n+m-1} W_{n+m-2} \cdots W_n$$

and

$$b_{n.m} = \left\| R_{n,m}\big(tx_n + (1-t)p_1\big) - \big(tR_{n,m}x_n + (1-t)p_1\big) \right\|,$$

for all $n, m \in N$. Then $R_{n,m}x_n = x_{n+m}$ and $R_{n,m}p = p$ $\forall p \in F(T)$. Also,

$$\|R_{n,m}x - R_{n.m}y\| \le \|x - y\|, \quad \forall x, y \in D.$$

By Lemma 3.4, there exists a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned} g(b_{n,m}) &\le \|x_n - p_1\| - \|R_{n,m}x_n - R_{n,m}p_1\| \\ &= \|x_n - p_1\| - \|x_{n+m} - p_1\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$, we get $\lim_{n,m\to\infty} g(b_{n,m}) = 0$ and by the property of g, we get $\lim_{n,m\to\infty} b_{n,m} = 0$.

Now,

$$\begin{aligned} a_{n+m}(t) &= \left\| tx_{n+m} + (1-t)p_1 - p_2 \right\| \\ &= \left\| tR_{n,m}x_n + (1-t)p_1 - p_2 \right\| \end{aligned}$$

$$
\begin{aligned}
&\leq b_{n,m} + \left\| R_{n,m}\left(tx_n + (1-t)p_1\right) - p_2 \right\| \\
&= b_{n,m} + \left\| R_{n,m}\left(tx_n + (1-t)p_1\right) - R_{n,m}p_2 \right\| \\
&\leq b_{n,m} + \left\| \left(tx_n + (1-t)p_1\right) - p_2 \right\| \\
&= b_{n,m} + a_n(t).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t) \\
&\leq \limsup_{m\to\infty} (b_{n,m}) + a_n(t).
\end{aligned}
$$

Since $\lim_{n,m\to\infty} b_{n,m} = 0$, we get

$$
\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t).
$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$, *i.e.*, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. □

Let E be a Banach space and $S_E = \{x \in E : \|x\| = 1\}$ unit sphere on E. The Banach space E is said to be smooth if

$$
\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3.15}
$$

exists for each x and y in S_E. In this case, the norm of E is called Gâteaux differentiable.

The space E is called Fréchet differentiable normed (see, *e.g.*, [14]); for each x in E, the above limit exists and is attained uniformly for y in E, and in this case it is also well known that

$$
\left\langle h, J(x)\right\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+h\|^2 \leq \left\langle h, J(x)\right\rangle + \frac{1}{2}\|x\|^2 + b\left(\|h\|\right) \tag{3.16}
$$

for all $x, h \in E$, where J is the Fréchet derivative of the function $\frac{1}{2}\|\cdot\|^2$ at $x \in E$, $\langle\cdot,\cdot\rangle$ is the dual pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0}\frac{b(t)}{t} = 0$.

Lemma 3.6 *Assume that the conditions of Lemma* 3.3 *are satisfied. Then, for any* $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ *exists*; *in particular,* $\langle p - q, J(p_1 - p_2)\rangle = 0$ *for all* $p, q \in \omega_w(x_n)$, *the set of all weak limits of* $\{x_n\}$.

The proof of Lemma 3.6 is similar to the proof of Lemma 2.3 of Khan and Kim [15].

A Banach space E is said to satisfy the Opial condition [16] if for each sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$
\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|
$$

for all $y \in E$ with $y \neq x$.

A Banach space E is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

We need the following to prove our next result.

Definition 3.7 A mapping $T: C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$, and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 3.8 [17] *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and T a nonexpansive mapping on C. Then $I - T$ is demiclosed at zero.*

Lemma 3.9 [8] *Let E be a reflexive Banach space satisfying the Opial condition, C a nonempty convex subset of E, and $T: C \to E$ an operator such that $I - T$ demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Lemma 3.10 [18] *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$, here $\omega_w(x_n)$ denotes the w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

We now establish a weak convergence result.

Theorem 3.11 *Let E be a uniformly convex Banach space and let C, T, and $\{x_n\}$ be as in Lemma* 3.3 *and $F(T) \neq \emptyset$. Assume that any of the following conditions hold*:

(a) *E satisfies the Opial condition*,

(b) *E has a Fréchet differentiable norm*,

(c) *the dual E^* of E satisfies the Kadec-Klee property.*

Then $\{x_n\}$ converges weakly to a point of $F(T)$.

Proof Let $p \in F(T)$, by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists.

We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$.

Let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.3, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, and also $I - T$ is demiclosed with respect to zero, hence by Lemma 3.8, we obtain $Tu = u$. In a similar manner, we have $v \in F(T)$.

Next, we prove the uniqueness.

First assume that (a) holds. If $u \neq v$, then, by the Opial condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{i\to\infty} \|x_{n_i} - u\| < \lim_{i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{j\to\infty} \|x_{n_j} - v\| < \lim_{j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction, so $u = v$.

Next, assume (b) holds.

By Lemma 3.6, $\langle p - q, J(p_1 - p_2)\rangle = 0$, for all $p, q \in \omega_w(x_n)$. Therefore, $\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0$ implies $u = v$.

Finally, assume that (c) is true.

Since $\lim_{n\to\infty}\|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$ by Lemma 3.5, $u = v$ by Lemma 3.10, and $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

A mapping $T: C \to C$ is said to be semicompact if any sequence $\{x_n\}$ in C, such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, has a subsequence converging strongly to some $p \in C$.

Next we establish the following strong convergence results.

Theorem 3.12 *Let E be a uniformly convex Banach space and let C, T, and $\{x_n\}$ be as in Lemma* 3.3. *If T is semicompact and $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Lemma 3.3, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$; since T is semicompact, $\{x_n\}$ has a subsequence converging to some $p \in C$ as C is closed. Continuity of T gives $\lim_{j\to\infty}\|Tx_{n_j} - Tp\| \to 0$. Then by Lemma 3.3,

$$\|Tp - p\| = 0.$$

This yields $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$, and therefore $\{x_n\}$ must itself converge to $p \in F(T)$ and this completes the proof. □

Theorem 3.13 *Let E be a uniformly convex Banach space and let C, T, $F(T)$, and $\{x_n\}$ be as in Lemma* 3.3. *Then $\{x_n\}$ converges to a point of $F(T)$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Proof Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.3, $\lim_{n\to\infty}\|x_n - w\|$ exists for all $w \in F(T)$, therefore $\lim_{n\to\infty} d(x_n, F(T))$ exists. But by hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

We will show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\varepsilon > 0$, there exists n_0 in $\mathbb{N}$ such that, for all $n \geq n_0$,

$$d\bigl(x_n, F(T)\bigr) < \frac{\varepsilon}{2}.$$

Particularly, $\inf\{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\varepsilon}{2}$. Hence, there exists $p^* \in F(T)$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. Now, for $m, n \geq n_0$,

$$\|x_{n+m} - x_n\| \leq \bigl\|x_{n+m} - p^*\bigr\| + \bigl\|x_n - p^*\bigr\| \leq 2\bigl\|x_{n_0} - p^*\bigr\| < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of a complete space, $\lim_{n\to\infty} x_n = p \in C$. Since $F(T)$ is closed, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives $d(p, F(T)) = 0$, *i.e.*, $p \in F(T)$. □

Definition 3.14 A mapping $T: C \to C$, where C is a subset of a normed space E, is said to satisfy Condition (I) [19] if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,1)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Applying Theorem 3.13, we obtain strong convergence of the process (1.8) under Condition (I) as follows.

Theorem 3.15 *Let e be a uniformly convex Banach space and let C, T, and $\{x_n\}$ be as in Lemma* 3.3. *Let T satisfy Condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof We proved in Lemma 3.3 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.17}$$

From Condition (I) and (3.17), we get

$$\lim_{n\to\infty} f\bigl(d\bigl(x_n, F(T)\bigr)\bigr) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, we have

$$\lim_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0.$$

Now all the conditions of Theorem 3.13 are satisfied, therefore, by its conclusion, $\{x_n\}$ converges strongly to a point of $F(T)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, 492010, India. [2]Department of Mathematics & Informatics, University Politehnica of Bucharest, Bucharest, 060042, Romania.

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