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Barnes-type Peters polynomial with umbral calculus viewpoint

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Abstract
In this paper, we consider the Barnes-type Peters polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.
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1 Introduction

The aim of this paper is to use umbral calculus to obtain several new and interesting identities of Barnes-type Peters polynomials. Umbral calculus has been used in numerous problems of mathematics (for example, see [1–10]). Umbral techniques have been used in different areas of physics; for example, it was used in group theory and quantum mechanics by Biedenharn *et al.* [11, 12] (for other examples, see [3, 10, 13–18]).

Let $r \in \mathbb{Z}_{>0}$. Here we will consider the polynomials $S_n(x) = S_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$ and $\hat{S}_n(x) = \hat{S}_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$, which are called Barnes-type *Peters polynomials of the first kind and of the second kind*, respectively, and are given by

$$\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^x = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!}, \tag{1.1}$$

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^x = \sum_{n\geq 0} \hat{S}_n(x)\frac{t^n}{n!}, \tag{1.2}$$

where $\lambda_1,\ldots,\lambda_r,\mu_1,\ldots,\mu_r \in \mathbb{C}$ with $\lambda_1,\ldots,\lambda_r \neq 0$. If $r=1$, then these polynomials are generalizations of Boole polynomials, see [19]. If $\mu_1 = \cdots = \mu_r = 1$, then $S_n(x|\boldsymbol{\lambda}) = S_n(x|\lambda_1,\ldots,\lambda_r) = S_n(x|\lambda_1,\ldots,\lambda_r;1,\ldots,1)$ and $\hat{S}_n(x|\boldsymbol{\lambda}) = \hat{S}_n(x|\lambda_1,\ldots,\lambda_r) = \hat{S}_n(x|\lambda_1,\ldots,\lambda_r;1,\ldots,1)$ are called Barnes-type *Boole polynomials of the first kind and of the second kind*. So,

$$\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-1}(1+t)^x = \sum_{n\geq 0} S_n(x|\boldsymbol{\lambda})\frac{t^n}{n!},$$

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)(1+t)^x = \sum_{n\geq 0} \hat{S}_n(x|\boldsymbol{\lambda})\frac{t^n}{n!}.$$

We introduce the polynomials $E_n(x|\boldsymbol{\lambda};\boldsymbol{\mu}) = E_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)$ with the generating function

$$\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} e^{xt} = \sum_{n\geq 0} E_n(x|\boldsymbol{\lambda};\boldsymbol{\mu})\frac{t^n}{n!}.$$

These polynomials may be called *generalized Barnes-type Euler polynomials*. When $\mu_1 = \cdots = \mu_r = 1$, $E_n(x|\boldsymbol{\lambda}) = E_n(x|\lambda_1,\ldots,\lambda_r) = E_n(x|\lambda_1,\ldots,\lambda_r;1,\ldots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1 = \cdots = \lambda_r = 1$, $E_n^{(r)}(x) = E_n(x|1,\ldots,1;1,\ldots,1)$ are called the *Euler polynomials of order r*. When $x = 0$, $S_n = S_n(\boldsymbol{\lambda};\boldsymbol{\mu}) = S_n(0|\boldsymbol{\lambda};\boldsymbol{\mu})$ and $\hat{S}_n = \hat{S}_n(\boldsymbol{\lambda};\boldsymbol{\mu}) = \hat{S}_n(0|\boldsymbol{\lambda};\boldsymbol{\mu})$ are called *Barnes-type Peters numbers of the first kind and of the second kind*, respectively.

Let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$, and let Π^* be the vector space of all linear functionals on Π. We denote the action of a linear functional L on a polynomial $p(x)$ by $\langle L|p(x)\rangle$, and we define the vector space structure on Π^* by

$$\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle,$$

where $c, c' \in \mathbb{C}$ (see [19–22]). We define the algebra of formal power series in a single variable t to be

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1.3}$$

The formal power series in the variable t defines a linear functional on Π by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \geq 0 \text{ (see [19–22])}. \tag{1.4}$$

By (1.3) and (1.4), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad \text{for all } n,k \geq 0 \text{ (see [19–22])}, \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. From (1.5), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$. Therefore, $\mathcal{H}$ is thought of as a set of both formal power series and linear functionals. We call $\mathcal{H}$ *umbral algebra*. *Umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [19–22]). If $O(f(t)) = 1$ (respectively, $O(f(t)) = 0$), then $f(t)$ is called a *delta* (respectively, an *invertible*) series. Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n,k \geq 0$ [19, Theorem 2.3.1]. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t),f(t))$ which is denoted by $s_n(x) \sim (g(t),f(t))$ (see [19–22]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$ and

$$f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle \frac{t^n}{n!}, \qquad p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle \frac{x^n}{n!} \tag{1.6}$$

(see [19–22]). From (1.6), we obtain

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \qquad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{1.7}$$

where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. So, by (1.7), we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$ for all $k \geq 0$ (see [19–22]).

Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!}, \tag{1.8}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [19–22]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let

$$s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x), \tag{1.9}$$

then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}\big(\ell(\bar{f}(t))\big)^k \middle| x^n \right\rangle \tag{1.10}$$

(see [19–22]).

It is immediate from (1.1)-(1.2), we see that $S_n(x)$ and $\hat{S}_n(x)$ are the Sheffer sequences for the pairs

$$S_n(x) \sim \left(\prod_{j=1}^{r}\big(1 + e^{\lambda_j t}\big)^{\mu_j}, e^t - 1\right), \tag{1.11}$$

$$\hat{S}_n(x) \sim \left(\prod_{j=1}^{r}\left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j}, e^t - 1\right). \tag{1.12}$$

The aim of the present paper is to present several new identities for the Peters polynomials by the use of umbral calculus.

2 Explicit expressions

It is well known that

$$(x)_n = \sum_{m=0}^{n} S_1(n, m)x^m \sim \big(1, e^t - 1\big), \tag{2.1}$$

where $S_1(n, m)$ is the Stirling number of the first kind. By (1.11) and (1.12) we have

$$\prod_{j=1}^{r}\big(1 + e^{\lambda_j t}\big)^{\mu_j} S_n(x) \sim \big(1, e^t - 1\big) \quad \text{and} \quad \prod_{j=1}^{r}\left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j} \hat{S}_n(x) \sim \big(1, e^t - 1\big). \tag{2.2}$$

So

$$\begin{aligned} S_n(x) &= \prod_{j=1}^{r}\left(1+e^{\lambda_j t}\right)^{-\mu_j}(x)_n = \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(1+e^{\lambda_j t}\right)^{-\mu_j} x^m \\ &= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^m \\ &= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) E_m(x|\boldsymbol{\lambda};\boldsymbol{\mu}), \end{aligned} \tag{2.3}$$

which implies

$$\begin{aligned} \hat{S}_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}(x)_n = e^{\sum_{j=1}^{r}\lambda_j\mu_j t}\prod_{j=1}^{r}\left(1+e^{\lambda_j t}\right)^{-\mu_j}(x)_n \\ &= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) e^{\sum_{j=1}^{r}\lambda_j\mu_j t}\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^m \\ &= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) e^{\sum_{j=1}^{r}\lambda_j\mu_j t} E_m(x|\boldsymbol{\lambda};\boldsymbol{\mu}) \\ &= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) E_m\left(x+\sum_{j=1}^{r}\lambda_j\mu_j \,\middle|\, \boldsymbol{\lambda};\boldsymbol{\mu}\right). \end{aligned} \tag{2.4}$$

Thus, we have the following result.

Theorem 1 *For all* $n \geq 0$,

$$S_n(x) = 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) E_m(x|\boldsymbol{\lambda};\boldsymbol{\mu}),$$

$$\hat{S}_n(x) = 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n} S_1(n,m) E_m\left(x+\sum_{j=1}^{r}\lambda_j\mu_j \,\middle|\, \boldsymbol{\lambda};\boldsymbol{\mu}\right).$$

By (1.6), (1.8), (1.11) and (1.12), we have

$$S_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\log(1+t)\right)^j \,\middle|\, x^n\right\rangle x^j,$$

$$\hat{S}_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\log(1+t)\right)^j \,\middle|\, x^n\right\rangle x^j,$$

where

$$\begin{aligned} &\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\log(1+t)\right)^j \,\middle|\, x^n\right\rangle \\ &\quad = \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \,\middle|\, \left(\log(1+t)\right)^j x^n\right\rangle \end{aligned}$$

$$
\begin{aligned}
&= j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) \left\langle \prod_{j=1}^{r} \bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j} \middle| x^{n-\ell} \right\rangle \\
&= j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) S_{n-\ell}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \bigl(\log(1+t)\bigr)^j \middle| x^n \right\rangle \\
&\quad = \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| \bigl(\log(1+t)\bigr)^j x^n \right\rangle \\
&\quad = j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-\ell} \right\rangle \\
&\quad = j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) \hat{S}_{n-\ell}.
\end{aligned}
$$

Hence, we can state the following formulas.

Theorem 2 *For all* $n \geq 0$,

$$
S_n(x) = \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) S_{n-\ell} \right) x^j \quad \textit{and} \quad \hat{S}_n(x) = \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j) \hat{S}_{n-\ell} \right) x^j.
$$

Also, by the definitions, (2.1), (1.11) and (1.12), we have

$$
\begin{aligned}
S_n(y) &= \left\langle \sum_{i\geq 0} S_i(y) \frac{t^i}{i!} \middle| x^n \right\rangle = \left\langle \prod_{j=1}^{r} \bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j} (1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j} \middle| (1+t)^y x^n \right\rangle \\
&= \sum_{m=0}^{n} (y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r} \bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j} \middle| x^{n-m} \right\rangle \\
&= \sum_{m=0}^{n} (y)_m \binom{n}{m} S_{n-m}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{S}_n(y) &= \left\langle \sum_{i\geq 0} \hat{S}_i(y) \frac{t^i}{i!} \middle| x^n \right\rangle = \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| (1+t)^y x^n \right\rangle
\end{aligned}
$$

$$= \sum_{m=0}^{n} (y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-m} \right\rangle$$

$$= \sum_{m=0}^{n} (y)_m \binom{n}{m} \hat{S}_{n-m},$$

which implies the following formulas.

Theorem 3 *For all $n \geq 0$,*

$$S_n(x) = \sum_{j=0}^{n} S_{n-j} \binom{n}{j} (x)_j \quad \textit{and} \quad \hat{S}_n(x) = \sum_{j=0}^{n} \hat{S}_{n-j} \binom{n}{j} (x)_j.$$

More generally, by (2.1) and (2.2) with $p_n(x) = \prod_{j=1}^{r}(1+e^{\lambda_j t})^{\mu_j} S_n(x) = (x)_n \sim (1, e^t - 1)$, we obtain that $S_n(x+y) = \sum_{j=0}^{b} S_j(x)(y)_{n-j}\binom{n}{j}$, and with $p_n(x) = \prod_{j=1}^{r}(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}})^{\mu_j} \hat{S}_n(x) = (x)_n \sim (1, e^t - 1)$, we obtain that $\hat{S}_n(x+y) = \sum_{j=0}^{b} \hat{S}_j(x)(y)_{n-j}\binom{n}{j}$, which gives the following corollary.

Corollary 1 *For all $n \geq 0$,*

$$S_n(x+y) = \sum_{j=0}^{b} S_j(x)(y)_{n-j} \binom{n}{j} \quad \textit{and} \quad \hat{S}_n(x+y) = \sum_{j=0}^{b} \hat{S}_j(x)(y)_{n-j} \binom{n}{j}.$$

3 Recurrence relations

Note that if $a_n(x) \sim (g(t), f(t))$, then $f(t)a_n(x) = na_{n-1}(x)$, Thus, by (1.11) and (1.12), we have that $S_n(x+1) - S_n(x) = (e^t - 1)S_n(x) = nS_{n-1}(x)$ and $\hat{S}_n(x+1) - \hat{S}_n(x) = (e^t - 1)\hat{S}_n(x) = n\hat{S}_{n-1}(x)$, which give the following recurrences.

Proposition 1 *For all $n \geq 1$,*

$$S_n(x+1) - S_n(x) = nS_{n-1}(x) \quad \textit{and} \quad \hat{S}_n(x+1) - \hat{S}_n(x) = n\hat{S}_{n-1}(x).$$

Note that for $a_n(x) \sim (g(t), f(t))$, we have that $a_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)}a_n(x)$. In the case (1.11), we obtain $S_{n+1}(x) = xS_n(x-1) - e^{-t}\frac{g'(t)}{g(t)}S_n(x)$ with $g(t) = \prod_{i=1}^{r}(1+e^{\lambda_i t})^{\mu_i}$. Since $\frac{g'(t)}{g(t)} = \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}}$ and by (2.3), we get

$$\begin{aligned}
\frac{g'(t)}{g(t)} S_n(x) &= \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}} S_n(x) = \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1+e^{\lambda_i t}} S_n(x) \\
&= \sum_{i=1}^{r} \left(\frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1+e^{\lambda_i t}} 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}} \right)^{\mu_j} x^m \right) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{i=1}^{r} \left(\frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1+e^{\lambda_i t}} \prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}} \right)^{\mu_j} x^m \right) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{i=1}^{r} \frac{\lambda_i \mu_i}{2} E_m(x + \lambda_i | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i),
\end{aligned}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is a vector with 1 in the ith coordinate. Thus,

$$S_{n+1}(x) = xS_{n-1}(x) - 2^{-1-\sum_{i=1}^r \mu_j} \sum_{m=0}^{n} \sum_{i=1}^{r} S_1(n,m) \lambda_i \mu_i E_m(x + \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i). \tag{3.1}$$

On the other hand, by Theorem 2, we have

$$\begin{aligned}
\frac{g'(t)}{g(t)} S_n(x) &= \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{1 + e^{\lambda_i t}} S_n(x) = \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1 + e^{\lambda_i t}} S_n(x) \\
&= \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1 + e^{\lambda_i t}} \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} \right) x^j \\
&= \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \frac{2}{1 + e^{\lambda_i t}} x^j \right) \\
&= \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} \sum_{i=1}^{r} \frac{\lambda_i \mu_i e^{\lambda_i t}}{2} \lambda_i^j E_j(x/\lambda_i) \right) \\
&= \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} \sum_{i=1}^{r} \frac{\lambda_i^{j+1} \mu_i}{2} E_j(1 + x/\lambda_i) \right)
\end{aligned}$$

(note that $E_n(x) = \frac{2}{1+e^t} x^n = (E + x)^n = \sum_{j=0}^{n} \binom{n}{j} E_j x^{n-j}$ and $\frac{2}{1+e^{\lambda_j t}} x^j = \lambda_i^j E_i(x/\lambda_i)$), which implies

$$S_{n+1}(x) = xS_n(x-1) - \sum_{j=0}^{n} \sum_{\ell=j}^{n} \sum_{i=1}^{r} \frac{\lambda_i^{j+1} \mu_i}{2} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} E_j\bigl(1 + (x-1)/\lambda_i\bigr).$$

Thus, by (3.1), we can state the following result.

Theorem 4 *For all* $n \geq 0$,

$$\begin{aligned}
S_{n+1}(x) &= xS_n(x-1) - 2^{-1-\sum_{i=1}^r \mu_j} \sum_{m=0}^{n} \sum_{i=1}^{r} S_1(n,m) \lambda_i \mu_i E_m(x + \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i), \\
S_{n+1}(x) &= xS_n(x-1) - \sum_{j=0}^{n} \sum_{\ell=j}^{n} \sum_{i=1}^{r} \frac{\lambda_i^{j+1} \mu_i}{2} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} E_j\bigl(1 + (x-1)/\lambda_i\bigr).
\end{aligned}$$

As a corollary, we get the following identity.

Corollary 2 *For all* $n \geq 0$,

$$\begin{aligned}
&2^{-1-\sum_{i=1}^r \mu_j} \sum_{m=0}^{n} \sum_{i=1}^{r} S_1(n,m) \lambda_i \mu_i E_m(x + \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i), \\
&\quad = \sum_{j=0}^{n} \sum_{\ell=j}^{n} \sum_{i=1}^{r} \frac{\lambda_i^{j+1} \mu_i}{2} \binom{n}{\ell} S_1(\ell, j) S_{n-\ell} E_j\bigl(1 + (x-1)/\lambda_i\bigr).
\end{aligned}$$

In the case (1.12), we obtain $\hat{S}_{n+1}(x) = x\hat{S}_n(x-1) - e^{-t}\frac{g'(t)}{g(t)}\hat{S}_n(x)$ with $g(t) = \prod_{i=1}^r (\frac{1+e^{\lambda_j t}}{e^{\lambda_i t}})^{\mu_j}$. Since $\frac{g'(t)}{g(t)} = \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \sum_{i=1}^r \lambda_i\mu_i$ and by (2.4), we get

$$\frac{g'(t)}{g(t)}S_n(x) = \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}}\hat{S}_n(x) - \boldsymbol{\lambda\mu}\hat{S}_n(x),$$

where $\boldsymbol{\lambda\mu} = \sum_{j=1}^r \lambda_j\mu_j$ and

$$\begin{aligned}
&\sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}}\hat{S}_n(x) \\
&\quad = \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{2}\frac{2}{1+e^{\lambda_i t}}\hat{S}_n(x) \\
&\quad = \sum_{i=1}^r \left(\frac{\lambda_i\mu_i e^{\lambda_i t}}{2}\frac{2}{1+e^{\lambda_i t}}2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^n S_1(n,m)e^{\sum_{j=1}^r \lambda_j\mu_j t}\prod_{j=1}^r \left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^m\right) \\
&\quad = 2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^n S_1(n,m)\sum_{i=1}^r \left(\frac{\lambda_i\mu_i}{2}e^{\lambda_i t+\sum_{j=1}^r \lambda_j\mu_j t}\frac{2}{1+e^{\lambda_i t}}\prod_{j=1}^r \left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^m\right) \\
&\quad = 2^{-1-\sum_{j=1}^r \mu_j}\sum_{m=0}^n S_1(n,m)\sum_{i=1}^r \lambda_i\mu_i E_n\bigl(x+\boldsymbol{\lambda}(\boldsymbol{\mu}+e_i)\,|\,\boldsymbol{\lambda};\boldsymbol{\mu}+e_i\bigr).
\end{aligned}$$

So

$$\begin{aligned}
\hat{S}_{n+1}(x) &= (x+\boldsymbol{\lambda\mu})\hat{S}_n(x-1) \\
&\quad - 2^{-1-\sum_{j=1}^r \mu_j}\sum_{m=0}^n S_1(n,m)\sum_{i=1}^r \lambda_i\mu_i E_n\bigl(x+\boldsymbol{\lambda}(\boldsymbol{\mu}+e_i)-1\,|\,\boldsymbol{\lambda};\boldsymbol{\mu}+e_i\bigr).
\end{aligned} \tag{3.2}$$

On the other hand, by Theorem 2, we have

$$\begin{aligned}
\frac{g'(t)}{g(t)}\hat{S}_n(x) &= \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{1+e^{\lambda_i t}}\hat{S}_n(x) - \boldsymbol{\lambda\mu}\hat{S}_n(x) = \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{2}\frac{2}{1+e^{\lambda_i t}}\hat{S}_n(x) - \boldsymbol{\lambda\mu}\hat{S}_n(x) \\
&= \sum_{i=1}^r \frac{\lambda_i\mu_i e^{\lambda_i t}}{2}\frac{2}{1+e^{\lambda_i t}}\sum_{j=0}^n\left(\sum_{\ell=j}^n \binom{n}{\ell}S_1(\ell,j)\hat{S}_{n-\ell}\right)x^j - \boldsymbol{\lambda\mu}\hat{S}_n(x) \\
&= \sum_{j=0}^n\left(\sum_{\ell=j}^n \binom{n}{\ell}S_1(\ell,j)\hat{S}_{n-\ell}\sum_{i=1}^r \frac{\lambda_i^{j+1}\mu_i}{2}E_j(1+x/\lambda_i)\right) - \boldsymbol{\lambda\mu}\hat{S}_n(x).
\end{aligned}$$

Therefore, by (3.2), we have the following result.

Theorem 5 *For all* $n \geq 0$,

$$\begin{aligned}
\hat{S}_{n+1}(x) &= (x+\boldsymbol{\lambda\mu})\hat{S}_n(x-1) \\
&\quad - 2^{-1-\sum_{j=1}^r \mu_j}\sum_{m=0}^n S_1(n,m)\sum_{i=1}^r \lambda_i\mu_i E_n\bigl(x+\boldsymbol{\lambda}(\boldsymbol{\mu}+e_i)-1\,|\,\boldsymbol{\lambda};\boldsymbol{\mu}+e_i\bigr),
\end{aligned}$$

$$\hat{S}_{n+1}(x) = (x + \boldsymbol{\lambda\mu})\hat{S}_n(x-1) - \sum_{j=0}^{n}\sum_{\ell=j}^{n}\sum_{i=1}^{r}\frac{\lambda_i^{j+1}\mu_i}{2}\binom{n}{\ell}S_1(\ell,j)\hat{S}_{n-\ell}E_j\big(1+(x-1)/\lambda_i\big).$$

As a corollary, we get the following identity.

Corollary 3 *For all* $n \geq 0$,

$$\begin{aligned}
&2^{-1-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{i=1}^{r}\lambda_i\mu_iE_n\big(x+\boldsymbol{\lambda}(\boldsymbol{\mu}+e_i)-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i\big),\\
&\quad=\sum_{j=0}^{n}\sum_{\ell=j}^{n}\sum_{i=1}^{r}\frac{\lambda_i^{j+1}\mu_i}{2}\binom{n}{\ell}S_1(\ell,j)\hat{S}_{n-\ell}E_j\big(1+(x-1)/\lambda_i\big).
\end{aligned}$$

Recall that for $a_n(x) \sim (g(t), f(t))$, we have $\frac{d}{dx}a_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)|x^{n-\ell}\rangle a_\ell(x)$. Hence, in the case (1.11), namely $a_n(x) = S_n(x)$, we have

$$\begin{aligned}
\big\langle\bar{f}(t)|x^{n-\ell}\big\rangle &= \big\langle\log(1+t)|x^{n-\ell}\big\rangle\\
&= \left\langle\sum_{m\geq 1}\frac{(-1)^{m-1}x^m}{m}\bigg|x^{n-\ell}\right\rangle = (-1)^{n-\ell-1}(n-\ell-1)!,
\end{aligned}$$

which implies $d/dxS_n(x) = n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-\ell-1}}{\ell!(n-\ell)}S_\ell(x)$. In the same way, we obtain the case $\hat{S}_n(x)$, which leads to the following result.

Theorem 6 *For all* $n \geq 1$,

$$\frac{d}{dx}S_n(x) = n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-\ell-1}}{\ell!(n-\ell)}S_\ell(x) \quad \textit{and} \quad \frac{d}{dx}\hat{S}_n(x) = n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-\ell-1}}{\ell!(n-\ell)}\hat{S}_\ell(x).$$

Now we find another recurrence relation by using the derivative operator. For $n \geq 1$, by (1.11) we have

$$\begin{aligned}
S_n(y) &= \left\langle\sum_{i\geq 0}S_i(y)\frac{t^i}{i!}\bigg|x^n\right\rangle = \left\langle\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y\bigg|x^n\right\rangle\\
&= \left\langle\frac{d}{dt}\left(\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y\right)\bigg|x^{n-1}\right\rangle\\
&= \left\langle\frac{d}{dt}\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y\bigg|x^{n-1}\right\rangle\\
&\quad+\left\langle\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\frac{d}{dt}(1+t)^y\bigg|x^{n-1}\right\rangle\\
&= \left\langle\frac{d}{dt}\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y\bigg|x^{n-1}\right\rangle + yS_{n-1}(y-1).
\end{aligned}$$

Observe that $\frac{d}{dt}\prod_{j=1}^{r}(1+(1+t)^{\lambda_j})^{-\mu_j} = -\prod_{j=1}^{r}(1+(1+t)^{\lambda_j})^{\mu_j}\sum_{i=1}^{r}\lambda_i\mu_i\frac{(1+t)^{\lambda_i-1}}{1+(1+t)^{\lambda_i}}$. Thus,

$$\begin{aligned}
&\left\langle \frac{d}{dt}\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{\mu_j}(1+t)^y \middle| x^{n-1}\right\rangle \\
&\quad = -\sum_{i=1}^{r}\lambda_i\mu_i\left\langle \bigl(1+(1+t)^{\lambda_i}\bigr)^{-1}\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^{y+\lambda_i-1}\middle| x^{n-1}\right\rangle \\
&\quad = -\sum_{i=1}^{r}\lambda_i\mu_i S_{n-1}(y+\lambda_i-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i).
\end{aligned}$$

Hence,

$$S_n(x) = xS_{n-1}(x-1) - \sum_{i=1}^{r}\lambda_i\mu_i S_{n-1}(x+\lambda_i-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i). \tag{3.3}$$

Also, for $n \geq 1$, by (1.12) we have

$$\begin{aligned}
\hat{S}_n(y) &= \left\langle \sum_{i\geq 0}\hat{S}_i(y)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y\middle| x^n\right\rangle \\
&= \left\langle \frac{d}{dt}\left[\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y\right]\middle| x^{n-1}\right\rangle \\
&= \left\langle \frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y\middle| x^{n-1}\right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\frac{d}{dt}(1+t)^y\middle| x^{n-1}\right\rangle \\
&= \left\langle \frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y\middle| x^{n-1}\right\rangle + y\hat{S}_{n-1}(y-1).
\end{aligned}$$

Observe that $\frac{d}{dt}\prod_{j=1}^{r}(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}})^{\mu_j} = \prod_{j=1}^{r}(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}})^{\mu_j}\sum_{i=1}^{r}\lambda_i\mu_i(1+t)^{-\lambda_i-1}\frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}$. So

$$\begin{aligned}
&\left\langle \frac{d}{dt}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^y\middle| x^{n-1}\right\rangle \\
&\quad = \sum_{i=1}^{r}\lambda_i\mu_i\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^{y-\lambda_i-1}\middle| x^{n-1}\right\rangle \\
&\quad = \sum_{i=1}^{r}\lambda_i\mu_i\hat{S}_{n-1}(y-\lambda_i-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i).
\end{aligned}$$

Thus,

$$\hat{S}_n(x) = x\hat{S}_{n-1}(x-1) + \sum_{i=1}^{r} \lambda_i \mu_i \hat{S}_{n-1}(x - \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i). \tag{3.4}$$

Hence, by (3.3) and (3.4), we obtain the following result.

Theorem 7 *For* $n \geq 1$,

$$S_n(x) = xS_{n-1}(x-1) - \sum_{i=1}^{r} \lambda_i \mu_i S_{n-1}(x + \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i),$$

$$\hat{S}_n(x) = x\hat{S}_{n-1}(x-1) + \sum_{i=1}^{r} \lambda_i \mu_i \hat{S}_{n-1}(x - \lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i).$$

Another result that can be obtained is the following.

Theorem 8 *For* $n - 1 \geq m \geq 1$,

$$\begin{aligned}\sum_{\ell=0}^{n-m} \binom{n}{\ell} S_1(n-\ell, m) S_\ell &= \sum_{\ell=0}^{n-m} \binom{n-1}{\ell} S_1(n-1-\ell, m-1) S_\ell(-1) \\ &\quad - \sum_{\ell=0}^{n-1-m} \binom{n-1}{\ell} S_1(n-1-\ell, m) \sum_{i=1}^{r} \lambda_i \mu_i S_\ell(\lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i),\end{aligned}$$

$$\begin{aligned}\sum_{\ell=0}^{n-m} \binom{n}{\ell} S_1(n-\ell, m) \hat{S}_\ell &= \sum_{\ell=0}^{n-m} \binom{n-1}{\ell} S_1(n-1-\ell, m-1) \hat{S}_\ell(-1) \\ &\quad + \sum_{\ell=0}^{n-1-m} \binom{n-1}{\ell} S_1(n-1-\ell, m) \sum_{i=1}^{r} \lambda_i \mu_i \hat{S}_\ell(-\lambda_i - 1 | \boldsymbol{\lambda}; \boldsymbol{\mu} + e_i).\end{aligned}$$

Proof Because of the similarity in the two cases $S_n(x)$ and $\hat{S}_n(x)$, we only give the proof of the first identity. In order to prove the first identity, we compute the following in two different ways:

$$A = \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\log(1+t)\right)^m \middle| x^n \right\rangle.$$

On the one hand, it is equal to

$$\begin{aligned}A &= \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \middle| \left(\log(1+t)\right)^m x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \middle| m! \sum_{\ell \geq m} S_1(\ell, m) \frac{t^\ell}{\ell!} x^n \right\rangle \\ &= m! \sum_{\ell=m}^{n} S_1(\ell, m) \binom{n}{\ell} \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \middle| x^{n-\ell} \right\rangle\end{aligned}$$

$$= m! \sum_{\ell=m}^{n} S_1(\ell, m) \binom{n}{\ell} S_{n-\ell}$$
$$= m! \sum_{\ell=0}^{n-m} S_1(n-\ell, m) \binom{n}{\ell} S_\ell. \tag{3.5}$$

On the other hand,

$$A = \left\langle \frac{d}{dt} \left[\prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\log(1+t)\right)^m \right] \middle| x^{n-1} \right\rangle$$
$$= \left\langle \frac{d}{dt} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle$$
$$+ \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \frac{d}{dt} \left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle.$$

Here,

$$\left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \frac{d}{dt} \left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle$$
$$= m \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{-1} \middle| \left(\log(1+t)\right)^{m-1} x^{n-1} \right\rangle$$
$$= m! \sum_{\ell=m-1}^{n-1} S_1(\ell, m-1) \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{-1} \middle| \frac{t^\ell}{\ell!} x^{n-1} \right\rangle$$
$$= m! \sum_{\ell=0}^{n-m} \binom{n-1}{\ell} S_1(n-1-\ell, m-1) \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{-1} \middle| x^\ell \right\rangle$$
$$= m! \sum_{\ell=0}^{n-m} \binom{n-1}{\ell} S_1(n-1-\ell, m-1) S_\ell(-1)$$

and

$$\left\langle \frac{d}{dt} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \left(\log(1+t)\right)^m \middle| x^{n-1} \right\rangle$$
$$= -\sum_{i=1}^{r} \lambda_i \mu_i \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{\lambda_i - 1} \middle| \left(\log(1+t)\right)^m x^{n-1} \right\rangle$$
$$= -\sum_{i=1}^{r} \lambda_i \mu_i \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{\lambda_i - 1} \middle| m! \sum_{\ell \geq m} S_1(\ell, m) \frac{t^\ell}{\ell!} x^{n-1} \right\rangle$$
$$= -m! \sum_{i=1}^{r} \sum_{\ell=m}^{n-1} \lambda_i \mu_i \binom{n-1}{\ell} S_1(\ell, m)$$
$$\times \left\langle \left(1 + (1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} (1+t)^{\lambda_i - 1} \middle| x^{n-1-\ell} \right\rangle$$

$$= -m!\sum_{i=1}^{r}\sum_{\ell=0}^{n-1-m}\lambda_i\mu_i\binom{n-1}{\ell}S_1(n-1-\ell,m)$$
$$\times\left\langle\left(1+(1+t)^{\lambda_i}\right)^{-1}\prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}(1+t)^{\lambda_i-1}\middle|x^{\ell}\right\rangle$$
$$= -m!\sum_{i=1}^{r}\sum_{\ell=0}^{n-1-m}\lambda_i\mu_i\binom{n-1}{\ell}S_1(n-1-\ell,m)S_\ell(\lambda_i-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i).$$

Altogether, we have, for $n-1\geq m\geq 1$,

$$m!\sum_{\ell=0}^{n-m}\binom{n}{\ell}S_1(n-\ell,m)S_\ell$$
$$= m!\sum_{\ell=0}^{n-m}\binom{n-1}{\ell}S_1(n-1-\ell,m-1)S_\ell(-1)$$
$$- m!\sum_{i=1}^{r}\sum_{\ell=0}^{n-1-m}\lambda_i\mu_i\binom{n-1}{\ell}S_1(n-1-\ell,m)S_\ell(\lambda_i-1|\boldsymbol{\lambda};\boldsymbol{\mu}+e_i).$$

By dividing by $m!$, we complete the proof. □

4 Identities

Let $S_n(x)=\sum_{m=0}^{n}c_{n,m}(x)_m$ and $\hat{S}_n(x)=\sum_{m=0}^{n}\hat{c}_{n,m}(x)_m$. By (1.9), (1.10) and (1.11), we obtain

$$c_{n,m}=\frac{1}{m!}\left\langle\prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle|t^m x^n\right\rangle$$
$$=\binom{n}{m}\left\langle\prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\middle|x^{n-m}\right\rangle$$
$$=\binom{n}{m}S_{n-m},$$

and by (1.9), (1.10) and (1.12), we obtain

$$\hat{c}_{n,m}=\frac{1}{m!}\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|t^m x^n\right\rangle$$
$$=\binom{n}{m}\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|x^{n-m}\right\rangle$$
$$=\binom{n}{m}\hat{S}_{n-m}.$$

Hence, we have the following identities.

Theorem 9 *For all $n\geq 0$,*

$$S_n(x)=\sum_{m=0}^{n}S_{n-m}\binom{n}{m}(x)_m\quad\textit{and}\quad\hat{S}_n(x)=\sum_{m=0}^{n}\hat{S}_{n-m}\binom{n}{m}(x)_m.$$

Now, let $S_n(x) = \sum_{m=0}^{n} c_{n,m} H_m^{(s)}(x|\alpha)$ and $\hat{S}_n(x) = \sum_{m=0}^{n} \hat{c}_{n,m} H_m^{(s)}(x|\alpha)$, where $H_n^{(s)}(x|\alpha) \sim ((\frac{e^t-\alpha}{1-\alpha})^s, t)$, with $\alpha \neq 1$. Then, by (1.9), (1.10) and (1.11), we obtain

$$\begin{aligned} c_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^{\log(1+t)}-\alpha}{1-\alpha}\right)^s \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^s}\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\bigl(\log(1+t)\bigr)^m (1-\alpha+t)^s \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^s}\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\bigl(\log(1+t)\bigr)^m \middle| \sum_{j=0}^{\min\{s,n\}}\binom{s}{j}(1-\alpha)t^j x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\alpha)^{s-j}(n)_j\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\bigl(\log(1+t)\bigr)^m \middle| x^{n-j} \right\rangle, \end{aligned}$$

and by Theorem 8, we have

$$\begin{aligned} c_{n,m} &= \frac{1}{m!(1-\alpha)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\alpha)^{s-j}(n)_j\left(m!\sum_{\ell=0}^{n-j-m}\binom{n-j}{\ell}S_1(n-j-\ell,m)S_\ell\right) \\ &= \sum_{j=0}^{n-m}\sum_{\ell=0}^{n-m-j}\binom{s}{j}\binom{n-j}{\ell}(1-\alpha)^{-j}(n)_j S_1(n-j-\ell,m)S_\ell. \end{aligned}$$

By (1.9), (1.10) and (1.12), we obtain

$$\begin{aligned} \hat{c}_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^{\log(1+t)}-\alpha}{1-\alpha}\right)^s \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\log(1+t)\bigr)^m \middle| (1-\alpha+t)^s x^n \right\rangle \\ &= \frac{1}{m!(1-\alpha)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\alpha)^{s-j}(n)_j\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\bigl(\log(1+t)\bigr)^m \middle| x^{n-j} \right\rangle, \end{aligned}$$

and by Theorem 8, we have

$$\begin{aligned} \hat{c}_{n,m} &= \frac{1}{m!(1-\alpha)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\alpha)^{s-j}(n)_j\left(m!\sum_{\ell=0}^{n-j-m}\binom{n-j}{\ell}S_1(n-j-\ell,m)\hat{S}_\ell\right) \\ &= \sum_{j=0}^{n-m}\sum_{\ell=0}^{n-m-j}\binom{s}{j}\binom{n-j}{\ell}(1-\alpha)^{-j}(n)_j S_1(n-j-\ell,m)\hat{S}_\ell. \end{aligned}$$

Therefore, we can state the following result.

Theorem 10 *For all* $n \geq 0$,

$$S_n(x) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{\ell=0}^{n-m-j}\binom{s}{j}\binom{n-j}{\ell}(1-\alpha)^{-j}(n)_j S_1(n-j-\ell,m)S_\ell\right)H_m^{(s)}(x|\alpha),$$

$$\hat{S}_n(x) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{\ell=0}^{n-m-j}\binom{s}{j}\binom{n-j}{\ell}(1-\alpha)^{-j}(n)_j S_1(n-j-\ell,m)\hat{S}_\ell\right)H_m^{(s)}(x|\alpha).$$

Finally, we express our polynomials $S_n(x)$ and $\hat{S}_n(x)$ in terms of Bernoulli polynomials of order s. Let $S_n(x) = \sum_{m=0}^{n} c_{n,m}B_m^{(s)}(x)$ and $\hat{S}_n(x) = \sum_{m=0}^{n} \hat{c}_{n,m}B_m^{(s)}(x)$, where $B_n^{(s)}(x) \sim ((\frac{e^t-1}{t})^s, t)$. Then, by (1.9), (1.10) and (1.11), we obtain

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^{\log(1+t)}-1}{\log(1+t)}\right)^s \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\log(1+t)\right)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\log(1+t)\right)^m \middle| \left(\frac{t}{\log(1+t)}\right)^s x^n\right\rangle,
\end{aligned}$$

and by the fact that $(\frac{t}{\log(1+t)})^s = \sum_{n\geq 0} C_n^{(s)}\frac{t^n}{n!}$, where $C_n^{(s)}$ is the Cauchy number of the first kind of order s, we derive

$$c_{n,m} = \frac{1}{m!}\sum_{i=0}^{n-m}\binom{n}{i}C_i^{(s)}\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\log(1+t)\right)^m \middle| x^{n-i}\right\rangle,$$

and by Theorem 8, we obtain

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!}\sum_{i=0}^{n-m}\binom{n}{i}C_i^{(s)}\left(m!\sum_{\ell=0}^{n-i-m}\binom{n-i}{\ell}S_1(n-i-\ell,m)S_\ell\right) \\
&= \sum_{i=0}^{n-m}\sum_{\ell=0}^{n-i-m}\binom{n}{i}\binom{n-i}{\ell}C_i^{(s)}S_1(n-i-\ell,m)S_\ell.
\end{aligned}$$

Also, by (1.9), (1.10) and (1.12), we obtain

$$\begin{aligned}
\hat{c}_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^{\log(1+t)}-1}{\log(1+t)}\right)^s \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\log(1+t)\right)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\log(1+t)\right)^m \middle| \left(\frac{t}{\log(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\binom{n}{i}C_i^{(s)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\log(1+t)\right)^m \middle| x^{n-i}\right\rangle,
\end{aligned}$$

and by Theorem 8, we obtain

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!}\sum_{i=0}^{n-m}\binom{n}{i}C_i^{(s)}\left(m!\sum_{\ell=0}^{n-i-m}\binom{n-i}{\ell}S_1(n-i-\ell,m)\hat{S}_\ell\right) \\
&= \sum_{i=0}^{n-m}\sum_{\ell=0}^{n-i-m}\binom{n}{i}\binom{n-i}{\ell}C_i^{(s)}S_1(n-i-\ell,m)\hat{S}_\ell.
\end{aligned}$$

Hence, we have the following identities.

Theorem 11 *For all* $n \geq 0$,

$$S_n(x) = \sum_{m=0}^{n} \left(\sum_{j=0}^{n-m} \sum_{\ell=0}^{n-m-j} \binom{n}{j} \binom{n-j}{\ell} C_j^{(s)} S_1(n-j-\ell, m) S_\ell \right) B_m^{(s)}(x),$$

$$\hat{S}_n(x) = \sum_{m=0}^{n} \left(\sum_{j=0}^{n-m} \sum_{\ell=0}^{n-m-j} \binom{n}{j} \binom{n-j}{\ell} C_j^{(s)} S_1(n-j-\ell, m) \hat{S}_\ell \right) B_m^{(s)}(x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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