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On a more accurate multidimensional Mulholland-type inequality

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Abstract

In this paper, by using the way of weight coefficients and technique of real analysis, a more accurate multidimensional discrete Mulholland-type inequality with the best possible constant factor is given, which is an extension of the Mulholland inequality. The equivalent form, the operator expression with the norm as well as a few particular cases are also considered.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; weight coefficient; equivalent form; operator; norm

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = \{\int_0^\infty f^p(x)\,dx\}^{\frac{1}{p}} > 0$, $\|g\|_q > 0$. We have the following Hardy-Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, we have the following Hardy-Hilbert inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (*cf.* [1–6]). Also we have the following Mulholland inequality (*cf.* [1]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=2}^\infty \frac{a_m^p}{m^{1-p}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^\infty \frac{b_n^q}{n^{1-q}} \right\}^{\frac{1}{q}}. \tag{3}$$

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) for $p = q = 2$. Yang [5] gave some extensions of (1) and (2) as follows: If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1}\,dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) \left| f(x) \right|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y)\, dx\, dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left\{ \sum_{n=1}^\infty \phi(n) |a_n|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, it follows that

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \tag{5}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (3) reduces to (1), while (5) reduces to (2). Some other results including the multidimensional Hilbert-type integral inequalities are provided by [8–24].

About half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [25] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ $(0 < \lambda \leq 2)$ by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [26] gave a half-discrete Hardy-Hilbert inequality with the best possible constant factor. Zhong *et al.* [27–33] investigated several half-discrete Hilbert-type inequalities with particular kernels. Using the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x,n) a_n\, dx < k(\lambda_1) \|f\|_{p,\phi} \|a\|_{q,\psi} \tag{6}$$

(see Yang and Chen [34]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and the best constant factor is given by Yang [35].

In this paper, by using the way of weight coefficients and technique of real analysis, a more accurate multidimensional discrete Mulholland-type inequality with the best possible constant factor is given, which is an extension of (3). The equivalent form, the operator expression with the norm as well as a few particular cases are also considered.

2 Some lemmas

Lemma 1 *If* $(-1)^i h^{(i)}(t) > 0$ $(t > 0; i = 1, 2)$, *then for* $b > 0$, $0 < \alpha \le 1$,

$$(-1)^i \frac{d^i}{dx^i} h\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr) > 0 \quad (x > 1; i = 1, 2). \tag{7}$$

Proof We find

$$\begin{aligned}
\frac{d}{dx} h\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr) &= \frac{1}{x} h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} x < 0, \\
\frac{d^2}{dx^2} h\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr) &= \frac{d}{dx}\left[\frac{1}{x} h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} x\right] \\
&= -\frac{1}{x^2} h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} x \\
&\quad + \frac{1}{x^2} h''\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{2}{\alpha}-2} \ln^{2\alpha-2} x \\
&\quad + \alpha\left(\frac{1}{\alpha} - 1\right)\frac{1}{x^2} h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}-2} \ln^{2\alpha-2} x \\
&\quad + (\alpha - 1)\frac{1}{x^2} h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}-1} \ln^{\alpha-2} x \\
&= \bigl[-h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x) \ln x \\
&\quad + h''\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)(b + \ln^\alpha x)^{\frac{1}{\alpha}} \ln^\alpha x \\
&\quad + b(\alpha - 1) h'\bigl((b + \ln^\alpha x)^{\frac{1}{\alpha}}\bigr)\bigr]\frac{1}{x^2}(b + \ln^\alpha x)^{\frac{1}{\alpha}-2} \ln^{\alpha-2} x > 0.
\end{aligned}$$

Then we have (7). □

If $i_0, j_0 \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we set

$$\|x\|_\alpha := \left(\sum_{k=1}^{i_0} |x_k|^\alpha\right)^{\frac{1}{\alpha}} \quad \bigl(x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}^{i_0}\bigr), \tag{8}$$

$$\|y\|_\beta := \left(\sum_{k=1}^{j_0} |y_k|^\beta\right)^{\frac{1}{\beta}} \quad \bigl(y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}^{j_0}\bigr). \tag{9}$$

Lemma 2 *If* $s \in \mathbf{N}$, $\gamma, M > 0$, $\Psi(u)$ *is a non-negative measurable function in* $(0, 1]$, *and*

$$D_M := \left\{x \in \mathbf{R}_+^s; \sum_{i=1}^s x_i^\gamma \le M^\gamma\right\} = \left\{x; \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \le 1\right\},$$

then we have (*cf.* [36])

$$\begin{aligned}
&\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s \\
&\quad = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1}\, du.
\end{aligned} \tag{10}$$

Lemma 3 *If* $s \in \mathbf{N}$, $\gamma > 0$, $\varepsilon > 0$, $d = (d_1, \ldots, d_s) \in [\frac{1}{2}, 1]^s$, *then*

$$\begin{aligned} A_s(\varepsilon) &:= \sum_m \bigl\| \ln(m+d) \bigr\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s (m_i + d_i)} \\ &= \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1) \quad (\varepsilon \to 0^+). \end{aligned} \tag{11}$$

Proof For $M > s^{1/\gamma}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{s}{M^\gamma}, \\ (M u^{1/\gamma})^{-s-\varepsilon}, & \frac{s}{M^\gamma} \le u \le 1. \end{cases}$$

Then by the decreasing property and (10), it follows that

$$\begin{aligned} A_s(\varepsilon) \quad &\ge \quad \int_{\{x \in \mathbf{R}_+^s; x_i \ge e - d_i\}} \bigl\| \ln(x+d) \bigr\|_\gamma^{-s-\varepsilon} \frac{dx}{\prod_{i=1}^s (x_i + d_i)} \\ &\overset{u_i = \ln(x_i + d_i)}{=} \int_{\{u \in \mathbf{R}_+^s; u_i \ge 1\}} \|u\|_\gamma^{-s-\varepsilon} \, du \\ &= \lim_{M \to \infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{s/M^\gamma}^1 \bigl(M u^{1/\gamma} \bigr)^{-s-\varepsilon} u^{\frac{s}{\gamma} - 1} \, du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \end{aligned}$$

In the following, by mathematical induction we prove that, for any $s \in \mathbf{N}$,

$$A_s(\varepsilon) \le O_s(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} \quad (\varepsilon \to 0^+). \tag{12}$$

For $s = 1$, by the Hermite-Hadamard inequality (*cf.* [37]), it follows that

$$\begin{aligned} A_1(\varepsilon) \quad &= \quad \sum_{m_1=1}^2 \frac{\ln^{-1-\varepsilon}(m_1 + d_1)}{m_1 + d_1} + \sum_{m_1=3}^\infty \frac{\ln^{-1-\varepsilon}(m_1 + d_1)}{m_1 + d_1} \\ &\le \quad O_1(1) + \int_{\frac{5}{2}}^\infty \frac{\ln^{-1-\varepsilon}(x + d_1)\, dx}{x + d_1} \le O_1(1) + \int_{e-d_1}^\infty \frac{\ln^{-1-\varepsilon}(x + d_1)\, dx}{x + d_1} \\ &\overset{u = \ln(x + d_1)}{=} O_1(1) + \int_1^\infty u^{-1-\varepsilon} \, du = O_1(1) + \frac{1}{\varepsilon}, \end{aligned}$$

and then (12) is valid. Assuming that (12) is valid for $s - 1 \in \mathbf{N}$, then for s, we set

$$\begin{aligned} A_s(\varepsilon) = &\sum_{\{m \in \mathbf{N}^s; \exists i_0, m_{i_0} = 1,2\}} \bigl\| \ln(m+d) \bigr\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s (m_i + d_i)} \\ &+ \sum_{\{m \in \mathbf{N}^s; m_i \ge 3\}} \bigl\| \ln(m+d) \bigr\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s (m_i + d_i)}. \end{aligned}$$

There exist constants $a, b \in \mathbf{R}_+$, such that

$$\begin{aligned}
&\sum_{\{m\in\mathbf{N}^s;\exists i_0, m_{i_0}=1,2\}} \|\ln(m+d)\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s(m_i+d_i)} \\
&\quad\le a + b \sum_{\{m\in\mathbf{N}^{s-1}; m_i\ge 1\}} \|\ln(m+d)\|_\gamma^{-(s-1)-(1+\varepsilon)} \frac{1}{\prod_{i=1}^{s-1}(m_i+d_i)}.
\end{aligned}$$

By the assumption of mathematical induction for $s-1$, we find

$$\begin{aligned}
&\sum_{\{m\in\mathbf{N}^{s-1}; m_i\ge 1\}} \|\ln(m+d)\|_\gamma^{-(s-1)-(1+\varepsilon)} \frac{1}{\prod_{i=1}^{s-1}(m_i+d_i)} \\
&\quad\le O_{s-1}(1) + \frac{\Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma}\gamma^{s-2}\Gamma(\frac{s-1}{\gamma})},
\end{aligned}$$

and then

$$\sum_{\{m\in\mathbf{N}^s;\exists i_0, m_{i_0}=1,2\}} \|\ln(m+d)\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s(m_i+d_i)} \le O_s(1).$$

By Lemma 1 and the Hermite-Hadamard inequality (*cf.* [37]), we obtain

$$\begin{aligned}
&\sum_{\{m\in\mathbf{N}^s; m_i\ge 3\}} \|\ln(m+d)\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s(m_i+d_i)} \\
&\quad\le \int_{\{x\in\mathbf{R}_+^s; x_i\ge\frac{5}{2}\}} \|\ln(x+d)\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s(x_i+d_i)}\,dx \\
&\quad\le \int_{\{x\in\mathbf{R}_+^s; x_i\ge e-d_i\}} \|\ln(x+d)\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s(x_i+d_i)}\,dx \\
&\quad\overset{u_i=\ln(x_i+d_i)}{=} \int_{\{u\in\mathbf{R}_+^s; u_i\ge 1\}} \|u\|_\gamma^{-s-\varepsilon}\,du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.
\end{aligned}$$

Hence we prove that (12) is valid for $s\in\mathbf{N}$. Therefore, we have (11). □

Lemma 4 *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C}\cup\{\infty\}$, $z_k \in \mathbf{C}\backslash\{z|\operatorname{Re} z\ge 0, \operatorname{Im} z = 0\}$ $(k=1,2,\ldots,n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* z_i $(i=1,2,\ldots,n)$, *and* $z=\infty$ *is a zero point of* $f(z)$ *whose order is not less than* 1, *then for* $\alpha\in\mathbf{R}$, *we have*

$$\int_0^\infty f(x)x^{\alpha-1}\,dx = \frac{2\pi i}{1-e^{2\pi\alpha i}} \sum_{k=1}^n \operatorname{Re} s\big[f(z)z^{\alpha-1}, z_k\big], \tag{13}$$

where $0<\operatorname{Im}\ln z = \arg z < 2\pi$. *In particular, if* z_k $(k=1,\ldots,n)$ *are all poles of order* 1, *setting* $\varphi_k(z) = (z-z_k)f(z)$ $(\varphi_k(z_k)\ne 0)$, *then*

$$\int_0^\infty f(x)x^{\alpha-1}\,dx = \frac{\pi}{\sin\pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha-1}\varphi_k(z_k). \tag{14}$$

Proof By [38, p.118], we have (13). We find

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i\sin 2\pi\alpha \\ &= -2i\sin\pi\alpha(\cos\pi\alpha + i\sin\pi\alpha) = -2ie^{i\pi\alpha}\sin\pi\alpha. \end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$\operatorname{Re} s\big[f(z)z^{\alpha-1}, -a_k\big] = {z_k}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (13), we obtain (14). □

Example 1 For $s \in \mathbf{N}$, we set

$$k_\lambda(x,y) = \prod_{k=1}^{s} \frac{1}{(x^{\lambda/s} + c_k y^{\lambda/s})} \quad (0 < c_1 < \cdots < c_s, 0 < \lambda \le s).$$

For $0 < \lambda_1 \le i_0$, $0 < \lambda_2 \le j_0$, $\lambda_1 + \lambda_2 = \lambda$, by (14), we find

$$\begin{aligned} k_s(\lambda_1) &:= \int_0^\infty \prod_{k=1}^{s} \frac{1}{t^{\lambda/s} + c_k} t^{\lambda_1 - 1}\, dt \\ &\overset{u=t^{\lambda/s}}{=} \frac{s}{\lambda} \int_0^\infty \prod_{k=1}^{s} \frac{1}{u + c_k} u^{\frac{s\lambda_1}{\lambda} - 1}\, du \\ &= \frac{\pi s}{\lambda \sin(\frac{\pi s \lambda_1}{\lambda})} \sum_{k=1}^{s} c_k^{\frac{s\lambda_1}{\lambda} - 1} \prod_{j=1\,(j\ne k)}^{s} \frac{1}{c_j - c_k} \in \mathbf{R}_+. \end{aligned} \tag{15}$$

In particular, for $s = 1$, we obtain

$$k_1(\lambda_1) = \frac{1}{\lambda} \int_0^\infty \frac{u^{(\lambda_1/\lambda)-1}}{u + c_1}\, du = \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} c_1^{\frac{\lambda_1}{\lambda} - 1}.$$

Definition 1 For $s \in \mathbf{N}$, $0 < \alpha, \beta \le 1$, $0 < c_1 < \cdots < c_s$, $0 < \lambda \le s$, $0 < \lambda_1 \le i_0$, $0 < \lambda_2 \le j_0$, $\lambda_1 + \lambda_2 = \lambda$, $\tau = (\tau_1, \ldots, \tau_{i_0}) \in [\frac{1}{2}, 1]^{i_0}$, $\sigma = (\sigma_1, \ldots, \sigma_{j_0}) \in [\frac{1}{2}, 1]^{j_0}$, $\ln(m + \tau) = (\ln(m_1 + \tau_1), \ldots, \ln(m_{i_0} + \tau_{i_0})) \in \mathbf{R}_+^{i_0}$, $\ln(n + \sigma) = (\ln(n_1 + \sigma_1), \ldots, \ln(n_{j_0} + \sigma_{j_0})) \in \mathbf{R}_+^{j_0}$, we define two weight coefficients $w_\lambda(\lambda_2, n)$ and $W_\lambda(\lambda_1, m)$ as follows:

$$\begin{aligned} w_\lambda(\lambda_2, n) &:= \sum_m \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} \|\ln(m+\tau)\|_\alpha^{\lambda_1 - i_0}}{\prod_{k=1}^{s}[\|\ln(m+\tau)\|_\alpha^{\lambda/s} + c_k \|\ln(n+\sigma)\|_\beta^{\lambda/s}] \prod_{i=1}^{i_0}(m_i + \tau_i)}, \\ W_\lambda(\lambda_1, m) &:= \sum_n \frac{\|\ln(m+\tau)\|_\alpha^{\lambda_1} \|\ln(n+\sigma)\|_\beta^{\lambda_2 - j_0}}{\prod_{k=1}^{s}[\|\ln(m+\tau)\|_\alpha^{\lambda/s} + c_k \|\ln(n+\sigma)\|_\beta^{\lambda/s}] \prod_{j=1}^{j_0}(n_j + \sigma_j)}, \end{aligned} \tag{16}$$

where $\sum_m = \sum_{m_{i_0}=1}^\infty \cdots \sum_{m_1=1}^\infty$ and $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$.

Lemma 5 *Let the assumptions as in Definition* 1 *be fulfilled. Then*:

(i) *we have*

$$w_\lambda(\lambda_2, n) < K_2 \quad \bigl(n \in \mathbf{N}^{j_0}\bigr), \tag{17}$$

$$W_\lambda(\lambda_1, m) < K_1 \quad \bigl(m \in \mathbf{N}^{i_0}\bigr), \tag{18}$$

where

$$K_1 := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} k_s(\lambda_1), \qquad K_2 := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k_s(\lambda_1), \tag{19}$$

and $k_s(\lambda_1)$ *is indicated by* (15);

(ii) *for* $p > 1$, $0 < \varepsilon < p\min\{\lambda_1, 1-\lambda_2\}$, *setting* $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, *we have*

$$0 < \widetilde{K}_2\bigl(1 - \widetilde{\theta}_\lambda(n)\bigr) < w_\lambda(\widetilde{\lambda}_2, n), \tag{20}$$

where

$$\begin{aligned}\widetilde{\theta}_\lambda(n) &:= \frac{1}{k_s(\widetilde{\lambda}_1)} \int_0^{i_0^{\lambda/(\alpha s)}/\|\ln(n+\sigma)\|_\beta^{\lambda/s}} \frac{v^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^s (v + c_k)}\, dv \\ &= O\biggl(\frac{1}{\|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_1}}\biggr),\end{aligned} \tag{21}$$

$$\widetilde{K}_2 = \frac{\Gamma^{i_0}(\frac{1}{\alpha}) k_s(\widetilde{\lambda}_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \in \mathbf{R}_+. \tag{22}$$

Proof By Lemma 1, the Hermite-Hadamard inequality (*cf.* [37]), (10), and (15), it follows that

$$\begin{aligned}
&w_\lambda(\lambda_2, n) \\
&\quad< \int_{(\frac{1}{2},\infty)^{i_0}} \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} \|\ln(x+\tau)\|_\alpha^{\lambda_1 - i_0}\, dx}{\prod_{k=1}^s [\|\ln(x+\tau)\|_\alpha^{\lambda/s} + c_k \|\ln(n+\sigma)\|_\beta^{\lambda/s}] \prod_{i=1}^{i_0}(x_i + \tau_i)} \\
&\quad\overset{u_i = \ln(x_i+\tau_i)}{=} \int_{\{u \in \mathbf{R}_+^{i_0}; u_i > \ln(\frac{1}{2}+\tau_i)\}} \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} \|u\|_\alpha^{\lambda_1 - i_0}}{\prod_{k=1}^s [\|u\|_\alpha^{\lambda/s} + c_k \|\ln(n+\sigma)\|_\beta^{\lambda/s}]}\, du \\
&\quad\le \int_{\mathbf{R}_+^{i_0}} \frac{\|n-\sigma\|_\beta^{\lambda_2} \|u\|_\alpha^{\lambda_1 - i_0}}{\prod_{k=1}^s [\|u\|_\alpha^{\lambda/s} + c_k \|\ln(n+\sigma)\|_\beta^{\lambda/s}]}\, du \\
&\quad= \lim_{M\to\infty} \int_{\mathbf{D}_M} \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} M^{\lambda_1 - i_0} [\sum_{i=1}^{j_0} (\frac{u_i}{M})^\alpha]^{(\lambda_1 - i_0)/\alpha}}{\prod_{k=1}^s \{M^{\frac{\lambda}{s}} [\sum_{i=1}^{i_0} (\frac{u_i}{M})^\alpha]^{\frac{\lambda}{\alpha s}} + c_k \|\ln(n+\sigma)\|_\beta^{\frac{\lambda}{s}}\}}\, du \\
&\quad= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} M^{\lambda_1 - i_0} t^{(\lambda_1 - i_0)/\alpha} t^{\frac{i_0}{\alpha}-1}}{\prod_{k=1}^s (M^{\frac{\lambda}{s}} t^{\frac{\lambda}{\alpha s}} + c_k \|\ln(n+\sigma)\|_\beta^{\frac{\lambda}{s}})}\, dt \\
&\quad= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{\|\ln(n+\sigma)\|_\beta^{\lambda_2} t^{\frac{\lambda_1}{\alpha}-1}}{\prod_{k=1}^s (M^{\frac{\lambda}{s}} t^{\frac{\lambda}{\alpha s}} + c_k \|\ln(n+\sigma)\|_\beta^{\frac{\lambda}{s}})}\, dt
\end{aligned}$$

$$\begin{aligned}
&\overset{t=\|\ln(n+\sigma)\|_\beta^\alpha M^{-\alpha}v^{\alpha s/\lambda}}{=} \frac{s\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^\infty \frac{v^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^s(v+c_k)}\,dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k_s(\lambda_1) = K_2.
\end{aligned}$$

Hence, we have (17). In the same way, we have (18).

By the decreasing property and (10), similarly to the proof of (11), we find

$$\begin{aligned}
&w_\lambda(\widetilde{\lambda}_2, n) \\
&> \int_{\{x\in\mathbf{R}_+^{i_0};x_i\ge e-\tau_i\}} \frac{\|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_2}\|\ln(x+\tau)\|_\alpha^{\widetilde{\lambda}_1-i_0}\,dx}{\prod_{k=1}^s[\|\ln(x+\tau)\|_\alpha^{\lambda/s}+c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]\prod_{i=1}^{i_0}(x_i+\tau_i)} \\
&\overset{u_i=\ln(x_i+\tau_i)}{=} \|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_2}\int_{\{u\in\mathbf{R}_+^{i_0};u_i\ge 1\}} \frac{\|u\|_\alpha^{\widetilde{\lambda}_1-i_0}\,du}{\prod_{k=1}^s[\|u\|_\alpha^{\lambda/s}+c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]} \\
&= \|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_2}\lim_{M\to\infty}\int\cdots\int_{D_M} \frac{\{\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha\}^{\frac{\widetilde{\lambda}_1-i_0}{\alpha}}M^{\widetilde{\lambda}_1-i_0}\,du_1\cdots du_{i_0}}{\prod_{k=1}^s[\{\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha\}^{\frac{\lambda}{\alpha s}}M^{\frac{\lambda}{s}}+c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]} \\
&= \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_2}\lim_{M\to\infty}\int_{\frac{i_0}{M^\alpha}}^1 \frac{t^{\frac{\widetilde{\lambda}_1-i_0}{\alpha}}M^{\widetilde{\lambda}_1-i_0}t^{\frac{i_0}{\alpha}-1}}{\prod_{k=1}^s[t^{\frac{\lambda}{\alpha s}}M^{\frac{\lambda}{s}}+c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]}\,dt \\
&= \frac{s\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_{\frac{i_0^{\lambda/(\alpha s)}}{\|\ln(n+\sigma)\|_\beta^{\lambda/s}}}^\infty \frac{v^{\frac{s\widetilde{\lambda}_1}{\lambda}-1}\,dv}{\prod_{k=1}^s(v+c_k)} = \widetilde{K}_2\bigl(1-\widetilde{\theta}_\lambda(n)\bigr) > 0, \\
&0 < \widetilde{\theta}_\lambda(n) = \frac{s}{\lambda k_s(\widetilde{\lambda}_1)}\int_0^{i_0^{\lambda/(\alpha s)}/\|\ln(n+\sigma)\|_\beta^{\lambda/s}} \frac{v^{\frac{s\widetilde{\lambda}_1}{\lambda}-1}}{\prod_{k=1}^s(v+c_k)}\,dv \\
&\le \frac{s}{\lambda k_s(\widetilde{\lambda}_1)\prod_{k=1}^s c_k}\int_0^{i_0^{\lambda/(\alpha s)}/\|\ln(n+\sigma)\|_\beta^{\lambda/s}} v^{\frac{s\widetilde{\lambda}_1}{\lambda}-1}\,dv \\
&= \frac{1}{\widetilde{\lambda}_1 k_s(\widetilde{\lambda}_1)\prod_{k=1}^s c_k}\frac{i_0^{\widetilde{\lambda}_1/\alpha}}{\|\ln(n+\sigma)\|_\beta^{\widetilde{\lambda}_1}}.
\end{aligned}$$

Hence, we have (20) and (21). □

3 Main results and operator expressions

Setting $\Phi(m) := \prod_{i=1}^{i_0}(m_i+\tau_i)^{p-1}\|\ln(m+\tau)\|_\alpha^{p(i_0-\lambda_1)-i_0}$ ($m\in\mathbf{N}^{i_0}$) and $\Psi(n) := \prod_{j=1}^{j_0}(n_j+\sigma_j)^{q-1}\|\ln(n+\sigma)\|_\beta^{q(j_0-\lambda_2)-j_0}$ ($n\in\mathbf{N}^{j_0}$), wherefrom

$$\bigl[\Psi(n)\bigr]^{1-p} = \prod_{j=1}^{j_0}(n_j+\sigma_j)^{-1}\bigl\|\ln(n+\sigma)\bigr\|_\beta^{p\lambda_2-j_0},$$

we have the following.

Theorem 1 *If* $s\in\mathbf{N}$, $0<\alpha,\beta\le 1$, $0<c_1<\cdots<c_s$, $0<\lambda\le s$, $0<\lambda_1\le i_0$, $0<\lambda_2\le j_0$, $\lambda_1+\lambda_2=\lambda$, $\tau\in[\frac{1}{2},1]^{i_0}$, $\sigma\in[\frac{1}{2},1]^{j_0}$, *then for* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\ge 0$, $0<\|a\|_{p,\Phi},\|b\|_{q,\Psi}<\infty$, *we have*

$$\begin{aligned} I &:= \sum_n \sum_m \frac{a_m b_n}{\prod_{k=1}^s [\| \ln(m+\tau)\|_\alpha^{\lambda/s} + c_k \| \ln(n+\sigma)\|_\beta^{\lambda/s}]} \\ &< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \end{aligned} \tag{23}$$

where the constant factor

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k_s(\lambda_1) \tag{24}$$

is the best possible ($k_s(\lambda_1)$ *is indicated by* (15)).

Proof By the Hölder inequality (*cf.* [37]), we have

$$\begin{aligned} I &= \sum_n \sum_m \frac{1}{\prod_{k=1}^s [\| \ln(m+\tau)\|_\alpha^{\lambda/s} + c_k \| \ln(n+\sigma)\|_\beta^{\lambda/s}]} \\ &\quad \times \left[\frac{\| \ln(m+\tau)\|_\alpha^{(i_0-\lambda_1)/q}}{\| \ln(n+\sigma)\|_\beta^{(j_0-\lambda_2)/p}} \frac{\prod_{i=1}^{i_0}(m_i+\tau_i)^{1/q}}{\prod_{j=1}^{j_0}(n_j+\sigma_j)^{1/p}} a_m \right] \\ &\quad \times \left[\frac{\| \ln(n+\sigma)\|_\beta^{(j_0-\lambda_2)/p}}{\| \ln(m+\tau)\|_\alpha^{(i_0-\lambda_1)/q}} \frac{\prod_{j=1}^{j_0}(n_j+\sigma_j)^{1/p}}{\prod_{i=1}^{i_0}(m_i+\tau_i)^{1/q}} b_n \right] \\ &\le \left\{ \sum_m W_\lambda(\lambda_1, m) \prod_{i=1}^{i_0} (m_i+\tau_i)^{p-1} \left\| \ln(m+\tau) \right\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_n w_\lambda(\lambda_2, n) \prod_{j=1}^{j_0} (n_j+\sigma_j)^{q-1} \left\| \ln(n+\sigma) \right\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (17) and (18), we have (23).

For $0 < \varepsilon < p\min\{\lambda_1, 1-\lambda_2\}$, $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we set

$$\begin{aligned} \widetilde{a}_m &= \left\| \ln(m+\tau) \right\|_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}} \frac{1}{\prod_{i=1}^{i_0}(m_i+\tau_i)}, \\ \widetilde{b}_n &= \left\| \ln(n+\sigma) \right\|_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} \frac{1}{\prod_{j=1}^{j_0}(n_j+\sigma_j)} \quad \left(m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}\right). \end{aligned}$$

Then by (11) and (20)-(22), we obtain

$$\begin{aligned} \|\widetilde{a}\|_{p,\Phi} \|\widetilde{b}\|_{q,\Psi} &= \left\{ \sum_m \prod_{i=1}^{i_0} (m_i+\tau_i)^{p-1} \left\| \ln(m+\tau) \right\|_\alpha^{p(i_0-\lambda_1)-i_0} \widetilde{a}_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_n \prod_{j=1}^{j_0} (n_j+\sigma_j)^{q-1} \left\| \ln(n+\sigma) \right\|_\beta^{q(j_0-\lambda_2)-j_0} \widetilde{b}_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_m \left\| \ln(m+\tau) \right\|_\alpha^{-i_0-\varepsilon} \frac{1}{\prod_{i=1}^{i_0}(m_i+\tau_i)} \right\}^{\frac{1}{p}} \end{aligned}$$

$$\times \left\{ \sum_n \left\| \ln(n+\sigma) \right\|_\beta^{-j_0-\varepsilon} \frac{1}{\prod_{j=1}^{j_0}(n_j+\sigma_j)} \right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right]^{\frac{1}{p}} \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \widetilde{O}(1) \right]^{\frac{1}{q}}, \tag{25}$$

$$\widetilde{I} := \sum_n \left[\sum_m \frac{\widetilde{a}_m}{\prod_{k=1}^s (\| \ln(m+\tau) \|_\alpha^{\lambda/s} + c_k \| \ln(n+\sigma) \|_\beta^{\lambda/s})} \right] \widetilde{b}_n$$

$$= \sum_n w_\lambda(\widetilde{\lambda}_2, n) \left\| \ln(n+\sigma) \right\|_\beta^{-j_0-\varepsilon} \frac{1}{\prod_{j=1}^{j_0}(n_j+\sigma_j)}$$

$$> \widetilde{K}_2 \sum_n \left(1 - O\left(\frac{1}{\| \ln(n+\sigma) \|_\beta^{\widetilde{\lambda}_1}} \right) \right) \left\| \ln(n+\sigma) \right\|_\beta^{-j_0-\varepsilon} \frac{1}{\prod_{j=1}^{j_0}(n_j+\sigma_j)}$$

$$= \widetilde{K}_2 \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) - O(1) \right]. \tag{26}$$

If there exists a constant $K \leq K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$, such that (23) is valid when replacing $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ by K, then we have

$$\left(K_2 + o(1) \right) \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \widetilde{O}(1) - \varepsilon O(1) \right]$$

$$< \varepsilon \widetilde{I} < \varepsilon K \| \widetilde{a} \|_{p,\varphi} \| \widetilde{b} \|_{q,\psi} = K \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right]^{\frac{1}{p}} \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \widetilde{O}(1) \right]^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta}) \Gamma^{i_0}(\frac{1}{\alpha}) k_s(\lambda_1)}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta}) \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \leq K \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{q}},$$

and then $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \leq K$. Hence, $K = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is the best possible constant factor of (23). □

Theorem 2 *With the assumptions of Theorem* 1, *for* $0 < \|a\|_{p,\Phi} < \infty$, *we have the following inequality with the best constant factor* $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$:

$$J := \left\{ \sum_n \left[\Psi(n) \right]^{1-p} \left(\sum_m \frac{a_m}{\prod_{k=1}^s [\| \ln(m+\tau) \|_\alpha^{\lambda/s} + c_k \| \ln(n+\sigma) \|_\beta^{\lambda/s}]} \right)^p \right\}^{\frac{1}{p}}$$

$$< K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}, \tag{27}$$

which is equivalent to (23).

Proof We set b_n as follows:

$$b_n := \left[\Psi(n) \right]^{1-p} \left(\sum_m \frac{a_m}{\prod_{k=1}^s (\| \ln(m+\tau) \|_\alpha^{\lambda/s} + c_k \| \ln(n+\sigma) \|_\beta^{\lambda/s})} \right)^{p-1}.$$

Then it follows that $J^p = \|b\|_{q,\Psi}^q$. If $J = 0$, then (27) is trivially valid, since $0 < \|a\|_{p,\Phi} < \infty$; if $J = \infty$, then it is a contradiction since the right hand side of (27) is finite. Suppose that $0 < J < \infty$. Then by (23), we find

$$\|b\|_{q,\Psi}^q = J^p = I < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}\|b\|_{q,\Psi},$$

namely, $\|b\|_{q,\Psi}^{q-1} = J < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}$, and then (27) follows.

On the other hand, assuming that (27) is valid, by the Hölder inequality, we have

$$\begin{aligned} I &= \sum_n \big(\Psi(n)\big)^{\frac{-1}{q}} \left[\sum_m \frac{a_m}{\prod_{k=1}^s(\|\ln(m+\tau)\|_\alpha^{\lambda/s} + c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s})}\right] \\ &\quad \times \left[\big(\Psi(n)\big)^{\frac{1}{q}} b_n\right] \le J\|b\|_{q,\Psi}. \end{aligned} \tag{28}$$

Then by (27), we have (23). Hence (27) and (23) are equivalent.

By the equivalency, the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (27) is the best possible. Otherwise, we would reach a contradiction by (28) that the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (23) is not the best possible. □

For $p > 1$, we define two real weight normal discrete spaces $l_{p,\varphi}$ and $l_{q,\psi}$ as follows:

$$l_{p,\varphi} := \left\{a = \{a_m\}; \|a\|_{p,\Phi} = \left\{\sum_m \Phi(m) a_m^p\right\}^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} := \left\{b = \{b_n\}; \|b\|_{q,\Psi} = \left\{\sum_n \Psi(n) b_n^q\right\}^{\frac{1}{q}} < \infty\right\}.$$

With the assumptions of Theorem 2, in view of $J < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}$, we have the following definition.

Definition 2 Define a multidimensional Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For $a \in l_{p,\Phi}$, there exists an unique representation $Ta \in l_{p,\Psi^{1-p}}$, satisfying for $n \in \mathbf{N}^{j_0}$,

$$(Ta)(n) := \sum_m \frac{a_m}{\prod_{k=1}^s[\|\ln(m+\tau)\|_\alpha^{\lambda/s} + c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]}. \tag{29}$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \sum_m \frac{a_m b_n}{\prod_{k=1}^s[\|\ln(m+\tau)\|_\alpha^{\lambda/s} + c_k\|\ln(n+\sigma)\|_\beta^{\lambda/s}]}. \tag{30}$$

Then by Theorem 1 and Theorem 2, for $0 < \|a\|_{p,\varphi}, \|b\|_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(Ta, b) < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{31}$$

$$\|Ta\|_{p,\Psi^{1-p}} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi}. \tag{32}$$

It follows that T is bounded since

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}} \le K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}. \tag{33}$$

Since the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (32) is the best possible, we have:

Corollary 1 *With the assumptions of Theorem* 2, T *is defined by Definition* 2, *it follows that*

$$\|T\| = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k_s(\lambda_1). \tag{34}$$

Remark 1 (i) Setting $\Phi_1(m) := \prod_{i=1}^{i_0}(m_i+1)^{p-1}\|\ln(m+1)\|_\alpha^{p(i_0-\lambda_1)-i_0}$ ($m \in \mathbf{N}^{i_0}$) and $\Psi_1(n) := \prod_{j=1}^{j_0}(n_j+1)^{q-1}\|\ln(n+1)\|_\beta^{q(j_0-\lambda_2)-j_0}$ ($n \in \mathbf{N}^{j_0}$), then putting $\tau = \sigma = 1$ in (23) and (27), we have the following equivalent inequalities with the best constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$:

$$\sum_n \sum_m \frac{a_m b_n}{\prod_{k=1}^s [\|\ln(m+1)\|_\alpha^{\lambda/s} + c_k\|\ln(n+1)\|_\beta^{\lambda/s}]} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi_1}\|b\|_{q,\Psi_1}, \tag{35}$$

$$\begin{aligned} &\left\{\sum_n [\Psi_1(n)]^{1-p}\left(\sum_m \frac{a_m}{\prod_{k=1}^s [\|\ln(m+1)\|_\alpha^{\lambda/s} + c_k\|\ln(n+1)\|_\beta^{\lambda/s}]}\right)^p\right\}^{\frac{1}{p}} \\ &\quad < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \|a\|_{p,\Phi_1}. \end{aligned} \tag{36}$$

Hence, (23) and (27) are more accurate inequalities than (35) and (36).

(ii) Putting $i_0 = j_0 = 1$, $\lambda = s$, $\phi_1(m) := (m+1)^{p-1}\ln^{p(1-\lambda_1)-1}(m+1)$ ($m \in \mathbf{N}$) and $\psi_1(n) := (n+1)^{q-1}\ln^{q(1-\lambda_2)-1}(n+1)$ ($n \in \mathbf{N}$), in (32), we have the following new inequality:

$$\begin{aligned} &\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{\prod_{k=1}^s \ln(m+1)(n+1)^{c_k}} \\ &\quad < \frac{\pi}{\sin(\pi\lambda_1)} \sum_{k=1}^s \prod_{j=1 (j\neq k)}^s \frac{c_k^{\lambda_1-1}}{c_j - c_k} \|a\|_{p,\phi_1}\|b\|_{q,\psi_1}. \end{aligned} \tag{37}$$

In particular, for $s = c_k = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (37), we can deduce (4). Hence, (23) is an extension of (4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QC participated in the design of the study and performed the numerical analysis. BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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