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The well-posedness for a system of generalized quasi-variational inclusion problems

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Dedicated to Professor SS Chang on his 80th birthday.

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Abstract
We introduce the concept of Levitin-Polyak well-posedness for a system of generalized quasi-variational inclusion problems and show some characterizations of Levitin-Polyak well-posedness for the system of generalized quasi-variational inclusion problems under some suitable conditions. We also give some results concerned with the Hadamard well-posedness for the system of generalized quasi-variational inclusion problems.
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1 Introduction

Well-posedness plays a crucial role in the theory and methodology of scalar optimization problems. In 1966, Tykhonov [1] first introduced the concept of well-posedness for a global minimizing problem, which has become known as Tykhonov well-posedness. Soon after, Levitin-Polyak [2] strengthened the concept of Tykhonov well-posedness, already known as the Levitin-Polyak (for short, LP) well-posedness. Subsequently, some authors studied the LP well-posedness for convex scalar optimization problems with functional constraints [3], vector optimization problems [4], variational inequality problems [5], generalized mixed variational inequality problems [6], generalized quasi-variational inequality problems [7], generalized vector variational inequality problems [8], equilibrium problems [9], vector equilibrium problems [10], generalized vector quasi-equilibrium problems [11] and generalized quasi-variational inclusion and disclusion problems [12]. Another important notion of the well-posedness for a minimizing problem is the well-posedness by perturbations or the extended well-posedness due to Zolezzi [13]. The notion of the well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. Recently, Lemaire *et al.* [14] introduced the well-posedness by perturbations for variational inequalities and Fang *et al.* [15] considered the well-posedness by perturbations for mixed variational inequalities in Banach spaces. For more details about the well-posedness by perturbations, we refer readers to [16, 17] and the references therein.

On the other hand, for optimization problems, there is another concept of well-posedness, which has become known as the Hadamard well-posedness. The concept of Hadamard well-posedness was inspired by the classical idea of Hadamard, which goes back to the beginning of the last century. It requires the existence and uniqueness of the optimal solution together with continuous dependence on the problem data. Some results about the Hadamard well-posedness can be found in [18–20]. Recently, the concept of Hadamard well-posedness has been extended to vector optimization problems and vector equilibrium problems. Li and Zhang [21] investigated the Hadamard well-posedness for vector optimization problems. Zeng *et al.* [22] obtained a sufficient condition for the Hadamard well-posedness of a set-valued optimization problem. Salamon [23] investigated the generalized Hadamard well-posedness for parametric vector equilibrium problems with trifunctions.

Very recently, Lin and Chuang [24] studied the well-posedness in the generalized sense for variational inclusion problems and variational disclusion problems, the well-posedness for optimization problems with variational inclusion problems, variational disclusion problems as constraints. Motivated by Lin, Wang *et al.* [25] investigated the well-posedness for generalized quasi-variational inclusion problems and for optimization problems with generalized quasi-variational inclusion problems as constraints. A system of generalized quasi-variational inclusion problems, which consists of a family of generalized quasi-variational inclusion problems defined on a product set, was first introduced by Lin [26]. It is well known that the system of generalized quasi-variational inclusion problems contains the system of variational inequalities, the system of equilibrium problems, the system of vector equilibrium problems, the system of vector quasi-equilibrium problems, the system of generalized vector quasi-equilibrium problems, the system of variational inclusions problems and variational disclusions problems as special cases. For more details, one can refer to [27–33] and the references therein. Nonetheless, to the best of our knowledge, there is no paper dealing with the Levitin-Polyak and Hadamard well-posedness for the system of generalized quasi-variational inclusion problems. Therefore, it is very interesting to generalize the concept of Levitin-Polyak and Hadamard well-posedness to the system of generalized quasi-variational inclusion problems.

Motivated and inspired by research work mentioned above, in this paper, we study the LP and Hadamard well-posedness for the system of generalized quasi-variational inclusion problems. This paper is organized as follows. In Section 2, we introduce the concept of LP well-posedness for the system of generalized quasi-variational inclusion problems. Some characterizations of the LP well-posedness for the system of generalized quasi-variational inclusion problems are shown in Section 3. Some results concerned with Hadamard well-posedness for the system of generalized quasi-variational inclusion problems are given in Section 4.

2 Preliminaries

Let I be an index set and (P, d_0) be a metric space. For each $i \in I$, let X_i be a metric space, Y_i and Z_i be Hausdorff topological vector spaces, $K_i \subset X_i$ be a nonempty closed and convex subset. Set $X = \Pi_{i\in I} X_i$, $K = \Pi_{i\in I} K_i$ and $Y = \Pi_{i\in I} Y_i$. For each $i \in I$, let $A_i : K \to 2^{X_i}$, $T_i : K \to 2^{Y_i}$ and $G_i : K \times Y \times K_i \to 2^{Z_i}$ be set-valued mappings. Let $e_i : K \to Z_i$ be a continuous mapping. Throughout this paper, unless otherwise specified, we use these notations and assumptions.

Now, we consider the following system of *generalized quasi-variational inclusion problems* (for short, SQVIP).

Find $\overline{x} = (\overline{x}_i)_{i\in I} \in K$ such that, for each $i \in I$, $\overline{x}_i \in A_i(\overline{x})$ and there exists $\overline{y} = (\overline{y}_i)_{i\in I} \in T_i(\overline{x})$ satisfying

$$0 \in G_i(\overline{x}, \overline{y}, z_i)$$

for all $z_i \in A_i(\overline{x})$. We denote by S the solution set of (SQVIP).

If the mapping $G_i : K \times Y \times K_i \to 2^{Z_i}$ is perturbed by a parameter $p \in P$, that is, $G_i : P \times K \times Y \times K_i \to 2^{Z_i}$ such that, for some $p^* \in P$, $G_i(p^*, x, y, z_i) = G_i(x, y, z_i)$ for all $(x, y, z_i) \in K \times Y \times K_i$, then, for any $p \in P$, we define a parametric system of generalized quasi-variational inclusion problem (for short, PSQVIP): Find $\overline{x} = (\overline{x}_i)_{i\in I} \in K$ such that, for each $i \in I$, $\overline{x}_i \in A_i(\overline{x})$ and there exists $\overline{y} = (\overline{y}_i)_{i\in I} \in T_i(\overline{x})$ satisfying

$$0 \in G_i(p, \overline{x}, \overline{y}, z_i)$$

for all $z_i \in A_i(\overline{x})$.

Some special cases of (SQVIP) are as follows:

(I) If, for each $i \in I$, $F_i : K \times Y \times K_i \to Z_i$ is a mapping, $C_i : K \to 2^{Z_i}$ is a pointed, closed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for every $x \in K$, $G_i(x, y, z_i)$ reduces to a single-valued mapping and $G_i(x, y, z_i) = F_i(\overline{x}, \overline{y}, z_i) + Z_i \setminus \operatorname{int} C_i(x)$ for all $(x, y, z_i) \in K \times Y \times K_i$, then (SQVIP) reduces to the system of vector equilibrium problems: Find $\overline{x} \in (\overline{x}_i)_{i\in I} \in K$ such that, for each $i \in I$, $\overline{x}_i \in A_i(\overline{x})$ and there exists $\overline{y}_i \in T_i(\overline{x})$ satisfying

$$F_i(\overline{x}, \overline{y}, z_i) \notin -\operatorname{int} C_i(\overline{x})$$

for all $z_i \in A_i(\overline{x})$, which has been studied by Peng and Wu [34] and the references therein.

(II) If, for each $i \in I$, $F_i : K \times Y \times K_i \to 2^{Z_i}$ and $\Psi_i : K \times K_i \to 2^{Z_i}$ are set-valued mappings, $C_i : K \to 2^{Z_i}$ is a pointed, closed and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in K$, $G_i(x, y, z_i) = F_i(x, y, z_i) + \Psi_i(x, z_i) + Z_i \setminus \operatorname{int} C_i(x)$ for all $(x, y, z_i) \in K \times Y \times K_i$, then (SQVIP) reduces to the system of set-valued vector quasi-equilibrium problems of Chen *et al.* [35]: Find $\overline{x} = (\overline{x}_i)_{i\in I} \in K$ such that, for each $i \in I$, $\overline{x}_i \in A_i(\overline{x})$ and there exists $\overline{y}_i \in T_i(\overline{x})$ satisfying

$$F_i(\overline{x}, \overline{y}, z_i) + \Psi_i(x, z_i) \nsubseteq -\operatorname{int} C_i(\overline{x})$$

for all $z_i \in A_i(\overline{x})$.

(III) If the index set I is a single set, then (SQVIP) reduces to the generalized quasi-variational inclusion problem studied in Wang *et al.* [12, 25] and the references therein.

Definition 2.1 Let $p^* \in P$ and $\{p^n\} \subset P$ be a sequence such that $p^n \to p^*$. A sequence $\{x^n\} \subset K$ is called a *LP approximating solution sequence* for (SQVIP) corresponding to

$\{p^n\}$ if, for each $i \in I$ and $n \in \mathbf{N}$, there exists a sequence of nonnegative real numbers $\{\epsilon^n\}$ with $\epsilon^n \to 0$ and $y_i^n \in T_i(x^n)$ such that

$$d_i(x_i^n, A_i(x^n)) \leq \epsilon^n$$

and

$$0 \in G_i(p^n, x^n, y^n, z_i) + B^+(0, \epsilon^n) e_i(x^n)$$

for all $z_i \in A_i(x^n)$, where $B^+(0, \epsilon^n)$ denote the closed interval $[0, \epsilon^n]$.

Definition 2.2 (1) (SQVIP) is said to be *LP well-posed by perturbations* if it has a unique solution and, for all $\{p^n\} \subset P$ with $p^n \to p^*$, every LP approximating solution sequence for (SQVIP) corresponding to $\{p^n\}$ converges strongly to the unique solution.

(2) (SQVIP) is said to be *generalized LP well-posed by perturbations* if the solution set S for (SQVIP) is nonempty and, for all sequences $\{p^n\} \subset P$ with $p^n \to p^*$, every LP approximating solution sequence for (SQVIP) corresponding to $\{p^n\}$ has some subsequence which converges strongly to some point of S.

Definition 2.3 [36] Let E_1, E_2 be two topological spaces. A set-valued mapping $F : E_1 \to 2^{E_2}$ is said to be:

(1) *upper semicontinuous* (for short, u.s.c.) at $x \in E_1$ if, for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(\overline{x}) \subset V$ for all $\overline{x} \in U$;
(2) *lower semicontinuous* (for short, l.s.c.) at $x \in E_1$ if, for each open set V in E_2 with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $F(x') \cap V \neq \emptyset$ for all $x' \in U(x)$;
(3) *u.s.c.* (resp., *l.s.c.*) on E_1 if it is u.s.c. (resp., l.s.c.) on every point $x \in E_1$;
(4) *continuous* on E_1 if it is both u.s.c. and l.s.c. on E_1;
(5) *closed* if the graph of F is closed, *i.e.*, the set $\mathrm{gph}(F) = \{(x, y) \in E_1 \times E_2 : y \in F(x)\}$ is closed in $E_1 \times E_2$.

Definition 2.4 [37] Let Z_1 and Z_2 be two metric spaces. A set-valued mapping $F : Z_1 \to 2^{Z_2}$ is said to be (s,s)-*subcontinuous* if, for any sequence $\{x_n\}$ converging strongly in Z_1, the sequence $\{y_n\}$ with $y_n \in F(x_n)$ has a strongly convergent subsequence.

Definition 2.5 [38] Let A be a nonempty subset of X, the measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\left\{\epsilon > 0, A \subset \bigcup_{i=1}^{n} A_i, \operatorname{diam} A_i < \epsilon, i = 1, 2, \ldots, n\right\}.$$

Definition 2.6 [38] Let A and B be two nonempty subsets of a Banach space X. The *Hausdorff metric* $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

3 The Levitin-Polyak well-posedness for (SQVIP)

In this section, we discuss some metric characterizations of the LP well-posedness for (SQVIP). First, we introduce the following LP approximating solution set for (SQVIP):

$$\begin{aligned}\Omega(\delta,\epsilon) = \bigcup_{p\in B(p^*,\delta)} \bigl\{ & x\in K : \forall i\in I, d_i\bigl(x_i, A_i(x)\bigr) \le \epsilon \text{ and } \exists y_i \in T_i(x) \text{ such that} \\ & 0 \in G_i(p,x,y,z_i) + B^+(0,\epsilon)e_i(x), \forall z_i\in A_i(x)\bigr\}\end{aligned}$$

for all $\delta,\epsilon > 0$, where $B(p^*,\delta)$ denotes the closed ball centered at p^* with radius δ.

Clearly, we have the following:

(1) $S\subseteq \Omega(\delta,\epsilon)$ for all $\delta,\epsilon > 0$;

(2) if $0 < \delta_1 < \delta_2$ and $0 < \epsilon_1 < \epsilon_2$, then $\Omega(\delta_1,\epsilon_1)\subseteq \Omega(\delta_2,\epsilon_2)$.

Next, we present some properties of $\Omega(\delta,\epsilon)$.

Proposition 3.1 *For each $i\in I$, let $T_i : K \to 2^{Y_i}$ be compact-valued, $A_i : K\to 2^{X_i}$ be closed-valued and $(p,y)\to G_i(p,x,y,z_i)$ be closed for all $(x,z_i)\in K\times K_i$. Then $S = \bigcap_{\delta>0,\epsilon>0}\Omega(\delta,\epsilon)$.*

Proof Clearly, $S\subseteq \bigcap_{\delta>0,\epsilon>0}\Omega(\delta,\epsilon)$. Hence we only need to show that $\bigcap_{\delta>0,\epsilon>0}\Omega(\delta,\epsilon)\subseteq S$. If not, then there exists $\overline{x}\in \bigcap_{\delta>0,\epsilon>0}\Omega(\delta,\epsilon)$ such that $\overline{x}\notin S$. Thus, for any $\delta > 0$ and $\epsilon > 0$, we have $\overline{x}\in\Omega(\delta,\epsilon)\backslash S$. For each $i\in I$ and $n\in \mathbf{N}$, it follows that $\overline{x}\in\Omega(\frac{1}{n},\frac{1}{n})\backslash S$ and there exist $p^n\in B(p^*,\frac{1}{n})$ and $y_i^n\in T_i(\overline{x})$ such that

$$d_i\bigl(\overline{x}_i, A_i(\overline{x})\bigr) \le \frac{1}{n} \tag{1}$$

and

$$0\in G_i\bigl(p^n,\overline{x},y_i^n,z_i\bigr) + B^+\left(0,\frac{1}{n}\right)e_i(\overline{x}) \tag{2}$$

for all $z_i\in A_i(\overline{x})$. Clearly, $p^n\to p^*$. Since $\{y_i^n\}\subseteq T_i(\overline{x})$ and $T_i(\overline{x})$ is a compact set, there exist a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ and $\overline{y}_i\in T_i(\overline{x})$ such that $y_i^{n_k}\to\overline{y}_i$ and, for each $k\in N$,

$$0\in G_i\bigl(p^{n_k},\overline{x},y_i^{n_k},z_i\bigr) + B^+\left(0,\frac{1}{n_k}\right)e_i(\overline{x})$$

for all $z_i\in A_i(\overline{x})$. For all $z_i\in A_i(\overline{x})$, there exists $\lambda_k\in B^+(0,\frac{1}{n_k})$ such that

$$0\in G_i\bigl(p^{n_k},\overline{x},y_i^{n_k},z_i\bigr) + \lambda_k e_i(\overline{x})$$

for all $z_i\in A_i(\overline{x})$. Clearly, $\lambda_k\to 0$. Since $(p,y)\mapsto G_i(p,\overline{x},y,z_i)$ is closed for all $(\overline{x},z_i)\in K\times K_i$, this together with (2) implies that

$$0\in G_i\bigl(p^*,\overline{x},\overline{y}_i,z_i\bigr)$$

for all $z_i\in A_i(\overline{x})$. Since A_i is closed-valued, it follows from (1) that $\overline{x}_i\in A_i(\overline{x})$ and so $\overline{x}\in S$, which is a contradiction. This completes the proof. □

Example 3.1 Let I be a single set, $P = [-1,1]$, $X = Y = Z = \mathbb{R} = (-\infty,+\infty)$ and $K = [0,+\infty)$. For any $(p,x,y,z) \in P \times K \times Y \times K$, let

$$e(x) = 1, \qquad A(x) = [x,+\infty), \qquad T(x) = \{1\}, \qquad G(p,x,y,z) = (-\infty, y - x].$$

Then it is easy to see that all the conditions of Proposition 3.1 are satisfied. By Proposition 3.1, $S = \bigcap_{\delta>0,\epsilon>0} \Omega(\delta,\epsilon)$. Indeed, for all $\delta,\epsilon > 0$,

$$\begin{aligned} S &= \bigl\{x \in K : x \in A(x) \text{ and } \exists y \in T(x) \text{ s.t. } 0 \in G(p,x,y,z), \forall z \in A(x)\bigr\} \\ &= \bigl\{x \in [0,+\infty) : 1 - x \geq 0\bigr\} = [0,1] \end{aligned}$$

and

$$\begin{aligned} \Omega(\delta,\epsilon) &= \bigcup_{p \in B(p^*,\delta)} \bigl\{x \in K : d\bigl(x, A(x)\bigr) \leq \epsilon \text{ and } \exists y \in T(x) \\ &\qquad \text{s.t. } 0 \in G(p,x,y,z) + B^+(0,\epsilon)e(x), \forall z \in A(x)\bigr\} \\ &= [0, 1 + \epsilon]. \end{aligned}$$

Therefore, $\bigcap_{\delta>0,\epsilon>0} \Omega(\delta,\epsilon) = [0,1] = S$.

Proposition 3.2 *For each $i \in I$, assume that*
- (i) *P is a finite-dimensional space*;
- (ii) *$T_i : K \to 2^{Y_i}$ is u.s.c. and compact-valued*;
- (iii) *$A_i : K \to 2^{X_i}$ is (s,s)-subcontinuous, l.s.c. and closed*;
- (iv) *$G_i : P \times K \times Y \times K_i \to 2^{Z_i}$ is closed.*

Then $\Omega(\delta,\epsilon)$ is closed for any $\delta,\epsilon > 0$.

Proof For any $\delta,\epsilon \geq 0$, let $\{x^n\} \subset \Omega(\delta,\epsilon)$ and $x^n \to \overline{x}$. Then there exists $p^n \in B(p^*,\delta)$ such that, for each $i \in I$,

$$d_i\bigl(x_i^n, A_i\bigl(x^n\bigr)\bigr) \leq \epsilon \tag{3}$$

and there exists $y_i^n \in T_i(x^n)$ such that

$$0 \in G_i\bigl(p^n, x^n, y^n, z_i\bigr) + B^+(0,\epsilon)e_i\bigl(x^n\bigr)$$

for all $z_i \in A_i(x^n)$. Since P is a finite-dimensional space, we can suppose that $p^n \to \overline{p} \in B(p^*,\delta)$. In order to prove that $\overline{x} \in \Omega(\delta,\epsilon)$, we first prove that, for each $i \in I$,

$$d_i\bigl(\overline{x}_i, A_i(\overline{x})\bigr) \leq \liminf_{n\to\infty} d_i\bigl(x_i^n, A_i\bigl(x^n\bigr)\bigr) \leq \epsilon.$$

Assume that the left inequality does not hold. Then there exists $\gamma > 0$ such that

$$\liminf_{n\to\infty} d_i\bigl(x_i^n, A_i\bigl(x^n\bigr)\bigr) \leq \gamma < d_i\bigl(\overline{x}_i, A_i(\overline{x})\bigr).$$

Thus there exist an increasing sequence $\{n_k\}$ and a sequence $\{u_i^k\}$ with $u_i^k \in A_i(x^{n_k})$ such that

$$d_i(x_i^{n_k}, u_i^k) < \gamma.$$

Since, for each $i \in I$, A_i is closed and (s,s)-subcontinuous, the sequence $\{u_i^k\}$ has a subsequence, which is still denoted by $\{u_i^k\}$, converging strongly to a point $\overline{u}_i \in A_i(\overline{x})$. It follows that, for each $i \in I$,

$$\gamma < d_i(\overline{x}_i, A_i(\overline{x})) \le d_i(\overline{x}_i, \overline{u}_i) \le \liminf_{n\to\infty} d_i(x_i^{n_k}, u_i^k) \le \gamma,$$

which is a contradiction. Thus, for each $i \in I$, $d_i(\overline{x}, A_i(\overline{x})) \le \epsilon$. Since, for each $i \in I$, $T_i : K \to 2^{Y_i}$ is u.s.c. and compact-valued, there exist a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ and $\overline{y}_i \in T_i(\overline{x})$ such that $y_i^{n_k} \to \overline{y}_i$. For any $\overline{z}_i \in A_i(\overline{x})$, since A_i is l.s.c., there exists a sequence $\{z_i^k\}$ with $z_i^k \in A_i(x^{n_k})$ such that $z_i^k \to \overline{z}_i$ and, for each $k \in \mathbf{N}$,

$$0 \in G_i(p^{n_k}, x^{n_k}, y^{n_k}, z_i^k) + B^+(0,\epsilon)e_i(x^{n_k}).$$

Since G_i is closed and e_i is continuous, we obtain

$$0 \in G_i(\overline{p}, \overline{x}, \overline{y}, \overline{z}_i) + B^+(0,\epsilon)e_i(\overline{x})$$

for all $\overline{z}_i \in A_i(\overline{x})$. Thus $\overline{x} \in \Omega(\delta,\epsilon)$ and so $\Omega(\delta,\epsilon)$ is closed. This completes the proof. □

Remark 3.1 If I is a single set and $x \in A(x)$ for all $x \in K$, then Propositions 3.1 and 3.2 can be considered as a generalization of Properties 3.1 and 3.2 of [25], respectively.

In this paper, let $d(x,y) = \sup_{i\in I} d_i(x_i, y_i)$ for all $x, y \in X$. It is clear that (X,d) is a metric space.

Theorem 3.1 *For each $i \in I$, let X_i be complete. We assume that*
(i) $T_i : K \to 2^{Y_i}$ *is u.s.c. and compact-valued;*
(ii) $A_i : K \to 2^{X_i}$ *is (s,s)-subcontinuous, l.s.c. and closed;*
(iii) $G_i : P \times K \times Y \times K_i \to 2^{Z_i}$ *is closed.*
Then (SQVIP) *is LP well-posed by perturbations if and only if, for any $\delta, \epsilon > 0$,*

$$\Omega(\delta,\epsilon) \ne \emptyset, \qquad \operatorname{diam}\Omega(\delta,\epsilon) \to 0 \tag{4}$$

as $(\delta,\epsilon) \to (0,0)$.

Proof Suppose that (SQVIP) is LP well-posed by perturbations. Then (SQVIP) has a unique solution $x^* \in \Omega(\delta,\epsilon)$ for any $\delta,\epsilon > 0$. This implies that $\Omega(\delta,\epsilon) \ne \emptyset$ for any $\delta,\epsilon > 0$.

Now, we show that

$$\operatorname{diam}\Omega(\delta,\epsilon) \to 0$$

as $(\delta,\epsilon) \to (0,0)$. If not, then there exist $\gamma > 0$, sequences $\{\delta_n\}$ and $\{\epsilon_n\}$ of nonnegative real numbers with $(\delta_n,\epsilon_n) \to (0,0)$, and the sequences $\{x^n\}$ and $\{\overline{x}^n\}$ with $x^n, \overline{x}^n \in \Omega(\delta,\epsilon)$

satisfying

$$d(x^n, \overline{x}^n) > \gamma \tag{5}$$

for all $n \in \mathbf{N}$. Since $x^n, \overline{x}^n \in \Omega(\delta, \epsilon)$, there exist $p^n, \overline{p}^n \in B(p^*, \delta_n)$ and $y_i^n \in T_i(x^n)$ and $\overline{y}_i^n \in T_i(\overline{x}^n)$ such that

$$d_i(x_i^n, A_i(x^n)) \leq \epsilon_n, \qquad 0 \in G_i(p^n, x^n, y^n, z_i)$$

for all $z_i \in A_i(x^n)$ and

$$d_i(\overline{x}_i^n, A_i(\overline{x}^n)) \leq \epsilon_n, \qquad 0 \in G_i(\overline{p}^n, \overline{x}^n, \overline{y}^n, \overline{z}_i)$$

for all $\overline{z}_i \in A_i(\overline{x}^n)$. Clearly, $p^n \to p^*$ and $\overline{p}^n \to p^*$. Thus $\{x^n\}$ and $\{\overline{x}^n\}$ are both the LP approximating solution sequences for (SQVIP) corresponding to $\{p^n\}$ and $\{\overline{p}^n\}$, respectively. Since (SQVIP) is LP well-posed by perturbations, $\{x^n\}$ and $\{\overline{x}^n\}$ have to converge strongly to the unique solution x^* of (SQVIP), which is a contradiction to (5).

Conversely, suppose that (4) holds. Let $\{p^n\} \subseteq P$ be any sequence with $p^n \to p^*$ and $\{x^n\}$ be the LP approximating solution sequence for (SQVIP) corresponding to $\{p^n\}$. Then there exist a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n)$ such that

$$d_i(x_i^n, A_i(x^n)) \leq \epsilon_n \tag{6}$$

and

$$0 \in G_i(p^n, x^n, y^n, z_i) + B^+(0, \epsilon_n) e_i(x^n) \tag{7}$$

for all $n \in \mathbf{N}$. Set $\delta_n = d_0(p^n, p^*)$. Then $p^n \in B(p^*, \delta_n)$ and $x^n \in \Omega(\delta_n, \epsilon_n)$ and $\delta_n \to 0$. It follows from (4) that $\{x^n\}$ is a Cauchy sequence and so it converges strongly to a point $\overline{x} \in K$. By the similar arguments as in the proof of Proposition 3.2, we can show that $\overline{x}_i \in A_i(\overline{x})$ and there exists $\overline{y}_i \in T_i(\overline{x})$ such that

$$0 \in G_i(p^*, \overline{x}, \overline{y}, z_i) \tag{8}$$

for all $z_i \in A_i(\overline{x})$. Thus $\overline{x}$ is a solution of (SQVIP).

Finally, to complete the proof, it is sufficient to prove that (SQVIP) has a unique solution. If (SQVIP) has two distinct solutions x and $\overline{x}$, then it is easy to see that $x, \overline{x} \in \Omega(\delta, \epsilon)$ for any $\delta, \epsilon > 0$. It follows that

$$0 < d(x, \overline{x}) \leq \operatorname{diam} \Omega(\delta, \epsilon)$$

for all $\delta, \epsilon > 0$, which contradicts (4). Thus (SQVIP) has a unique solution. This completes the proof. □

Remark 3.2 If I is a single set, $x \in A(x)$ for all $x \in K$, then Theorem 3.1 can be seen as a generalization of Theorem 3.1 of [25].

Example 3.2 Let I be a single set, $P = [-1,1]$, $X = Y = Z = \mathbb{R} = (-\infty,+\infty)$ and $K = [-1,0]$. For all $(p,x,y,z) \in P \times K \times Y \times K$, let

$$e(x) = 1, \qquad A(x) = [x,0], \qquad T(x) = \{-1\},$$
$$G(p,x,y,z) = \bigl(-\infty, \bigl(p^2+1\bigr)(y-x)\bigr].$$

Then A is (s,s)-subcontinuous, l.s.c. and closed, T is u.s.c. and compact-valued and G is closed. For any $\delta, \epsilon > 0$, we have

$$\begin{aligned} S &= \bigl\{x \in K : x \in A(x) \text{ and } \exists y \in T(x) \text{ s.t. } 0 \in G(p,x,y,z), \forall z \in A(x)\bigr\} \\ &= \bigl\{x \in [-1,0] : -1 - x \geq 0\bigr\} \\ &= \{-1\} \end{aligned}$$

and

$$\begin{aligned} \Omega(\delta,\epsilon) = \bigcup_{p \in B(p^*,\delta)} &\bigl\{x \in K : d\bigl(x, A(x)\bigr) \leq \epsilon \text{ and } \exists y \in T(x) \\ &\text{s.t. } 0 \in G(p,x,y,z) + B^+(0,\epsilon)e(x), \forall z \in A(x)\bigr\} \\ = &\begin{cases} [-1, \frac{\epsilon}{(p-\delta)^2+1} - 1], & p^* > 0, \\ [-1, \epsilon - 1], & p^* = 0, \\ [-1, \frac{\epsilon}{(p+\delta)^2+1} - 1], & p^* < 0 \end{cases} \end{aligned}$$

for sufficiently small $\delta > 0$. Therefore, $\operatorname{diam} \Omega(\delta,\epsilon) \to 0$ as $(\delta,\epsilon) \to (0,0)$.

Theorem 3.2 *For each $i \in I$, let X_i be complete and P be a finite-dimensional space. We assume that*

(i) $T_i : K \to 2^{Y_i}$ *is u.s.c. and compact-valued*;
(ii) $A_i : K \to 2^{X_i}$ *is (s,s)-subcontinuous, l.s.c. and closed*;
(iii) $G_i : P \times K \times Y \times K_i \to 2^{Z_i}$ *is closed.*

Then (SQVIP) *is generalized LP well-posed by perturbations if and only if, for any $\delta, \epsilon > 0$,*

$$\Omega(\delta,\epsilon) \neq \emptyset, \qquad \mu\bigl(\Omega(\delta,\epsilon)\bigr) \to 0 \tag{9}$$

as $(\delta,\epsilon) \to (0,0)$.

Proof Suppose that (SQVIP) is generalized LP well-posed by perturbations. Then S is nonempty.

Now, we prove that S is compact. Indeed, let $\{x^n\}$ be a sequence in S. Then $\{x^n\}$ is the LP approximating solution sequence for (SQVIP) corresponding to $\{p^*\}$. Since (SQVIP) is generalized LP well-posed by perturbations, $\{x^n\}$ has a subsequence which converges strongly to a point of S. This implies that S is compact. For any $\delta, \epsilon \geq 0$, since $S \subset \Omega(\delta,\epsilon)$, we have $\Omega(\delta,\epsilon) \neq \emptyset$ and

$$\mathcal{H}\bigl(\Omega(\delta,\epsilon), S\bigr) = \sup_{x \in \Omega(\delta,\epsilon)} d(x,S).$$

Since S is compact,

$$\mu\big(\Omega(\delta,\epsilon)\big) \le 2\mathcal{H}\big(\Omega(\delta,\epsilon),S\big) + \mu(S) = 2\sup_{x\in\Omega(\delta,\epsilon)} d(x,S).$$

In order to prove $\mu(\delta,\Omega(\epsilon)) \to 0$, we need to prove that

$$\sup_{x\in\Omega(\delta,\epsilon)} d(x,S) \to 0$$

as $(\delta,\epsilon) \to (0,0)$. Assume that this is not true. Then there exist $\alpha > 0$, and the sequences $\{\delta_n\}$ and $\{\epsilon_n\}$ of nonnegative real numbers with $(\delta_n,\epsilon_n) \to (0,0)$ and $\{x^n\}$ with $x^n \in \Omega(\delta_n,\epsilon_n)$ such that, for n sufficiently large,

$$d\big(x^n,S\big) > \alpha. \tag{10}$$

Since $x^n \in \Omega(\delta_n,\epsilon_n)$, there exists $p^n \in B(p^*,\delta_n)$ such that, for each $i \in I$, $d_i(x_i^n, A_i(x^n)) \le \epsilon_n$ and there exists $y_i^n \in T_i(x^n)$ satisfying

$$0 \in G_i\big(p^n,x^n,y^n,z_i\big) + B^+(0,\epsilon_n)e_i\big(x^n\big)$$

for all $z_i \in A_i(x^n)$, it follows that $p^n \to p^*$ and $\{x^n\}$ is the LP approximating solution sequence for (SQVIP) corresponding to $\{p^n\}$. By the generalized LP well-posedness by perturbations of (SQVIP), there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ which converges strongly to a point of S, which contradicts (10).

Conversely, suppose that (9) holds. From Propositions 3.1 and 3.2, $\Omega(\delta,\epsilon)$ is closed for any $\delta,\epsilon > 0$ and $S = \bigcap_{\delta>0,\epsilon>0}\Omega(\delta,\epsilon)$. Since $\mu(\Omega(\delta,\epsilon)) \to 0$ as $(\delta,\epsilon) \to (0,0)$, theorem on p.412 in [38] can be applied and one concludes that the set S is nonempty compact and

$$\mathcal{H}\big(\Omega(\delta,\epsilon),S\big) \to 0 \tag{11}$$

as $(\delta,\epsilon) \to (0,0)$. Let $\{p^n\}$ be any sequence in P with $p^n \to p^*$. If $\{x^n\}$ is the LP approximating solution sequence for (SQVIP) corresponding to $\{p^n\}$, then there exists a sequence $\{\epsilon_n\}$ of nonnegative numbers with $\epsilon_n \to 0$ and $\{y_i^n\}$ with $y_i^n \in T_i(x^n)$ such that

$$d_i\big(x_i^n, A_i\big(x^n\big)\big) \le \epsilon_n$$

and

$$0 \in G_i\big(p^n,x^n,y^n,z_i\big) + B^+(0,\epsilon_n)e_i\big(x^n\big)$$

for all $z_i \in A_i(x^n)$. For each $n \in \mathbf{N}$, let $\delta_n = d_0(p^n,p^*)$. Then $p^n \in B(p^*,\delta_n)$ and $x^n \in \Omega(\delta_n,\epsilon_n)$. Thus it follows from (11) that

$$d\big(x^n,S\big) \le \mathcal{H}\big(\Omega(\delta_n,\epsilon_n),S\big) \to 0$$

as $(\delta_n,\epsilon_n) \to (0,0)$. The compactness of S implies that (SQVIP) is generalized LP well-posed by perturbations. This completes the proof. □

Remark 3.3 If I is a single set, $x \in A(x)$ for every $x \in K$, then Theorem 3.2 can be considered as a generalization of Theorem 3.2 of [25].

Example 3.3 Let I be a single set, $P = [-1,1]$, $X = Y = Z = \mathbb{R} = (-\infty, +\infty)$ and $K = [0, +\infty)$. For all $(p,x,y,z) \in P \times K \times Y \times K$, let

$$e(x) = 1, \qquad A(x) = [0,x], \qquad T(x) = \{1\},$$
$$G(p,x,y,z) = \big(-\infty, \big(p^2+1\big)(y-x+z)\big].$$

Then all the conditions of Theorem 3.2 are satisfied. It follows that, for any $\delta, \epsilon > 0$,

$$S = \big\{x \in K : x \in A(x) \text{ and } \exists y \in T(x) \text{ s.t. } 0 \in G(p,x,y,z), \forall z \in A(x)\big\} = [0,1]$$

and

$$\begin{aligned}\Omega(\delta,\epsilon) &= \bigcup_{p \in B(p^*,\delta)} \big\{x \in K : d\big(x, A(x)\big) \leq \epsilon \text{ and } \exists y \in T(x) \\ &\qquad \text{s.t. } 0 \in G(p,x,y,z) + B^+(0,\epsilon)e(x), \forall z \in A(x)\big\} \\ &= \begin{cases} [0, \frac{\epsilon}{(p-\delta)^2+1} + 1], & p^* > 0, \\ [0, \epsilon + 1], & p^* = 0, \\ [0, \frac{\epsilon}{(p+\delta)^2+1} + 1], & p^* < 0 \end{cases}\end{aligned}$$

for sufficiently small $\delta > 0$. Therefore, $\mu(\Omega(\delta,\epsilon)) \to 0$ as $(\delta,\epsilon) \to (0,0)$. From Theorem 3.2, (SQVIP) is generalized LP well-posedness by perturbations.

4 The Hadamard well-posedness for (SQVIP)

In this section, for each $i \in I$, we assume that X_i, Y_i and Z_i are finite-dimensional spaces, $K_i \subset X_i$ is a nonempty closed and convex subset.

For each $i \in I$, let M_i be the collection of all (A_i, T_i, G_i) such that

(i) $A_i : K \to 2^{X_i}$ is continuous and bounded compact-convex-valued;
(ii) $T_i : K \to 2^{Y_i}$ is u.s.c. and bounded compact-convex-valued;
(iii) $G_i : K \times Y \times K_i \to 2^{Z_i}$ is u.s.c. and bounded compact-convex-valued.

Definition 4.1 [39] A sequence $\{D_n\}$ of nonempty subsets of $\mathbb{R}^n$ is said to be *convergent to D in the sense of Painlevé-Kuratowski* (for short, $D_n \xrightarrow{\text{P.K.}} D$) if

$$\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n,$$

where $\liminf_{n\to\infty} D_n$, the inner limit, consists of all possible limit points of the sequences $\{x_n\}$ with $x_n \in D_n$ for all $n \in \mathbf{N}$ and $\limsup_{n\to\infty} D_n$, the outer limit, consists of all possible cluster points of such sequences.

Definition 4.2 [22] A sequence $\{F_n\}$ of nonempty set-valued mappings $F_n : \mathbb{R}^k \to 2^{\mathbb{R}^h}$ is said to be *convergent to a set-valued mapping* $F : \mathbb{R}^k \to 2^{\mathbb{R}^h}$ *in the sense of Painlevé-Kuratowski* (for short, $F_n \xrightarrow{\text{P.K.}} F$) if $\text{gph}(F_n) \xrightarrow{\text{P.K.}} \text{gph}(F)$, where $\text{gph}(F_n) = \{(x,z) \in \mathbb{R}^k \times \mathbb{R}^h : x \in \text{dom}\, F_n, z \in F_n(x)\}$ and $\text{gph}(F) = \{(x,z) \in \mathbb{R}^k \times \mathbb{R}^h : x \in \text{dom}\, F, z \in F(x)\}$.

We say that, for each $i \in I$, a sequence $\{(A_i^n, T_i^n, G_i^n)\} \subset M_i$ converges to $(A_i, T_i, G_i) \in M_i$ in the sense of Painlevé-Kuratowski (for short, $(A_i^n, T_i^n, G_i^n) \xrightarrow{\text{P.K.}} (A_i, T_i, G_i)$) if $A_i^n \xrightarrow{\text{P.K.}} A_i$, $T_i^n \xrightarrow{\text{P.K.}} T_i$ and $G_i^n \xrightarrow{\text{P.K.}} G_i$.

Next, we give the definition of the Hadamard well-posedness for (SQVIP). As mentioned above, we denote by S the solution set of (SQVIP) determined by (A_i, T_i, G_i) for each $i \in I$. Similarly, we denote by S_n the solution set of $(\text{SQVIP})_n$ determined by (A_i^n, T_i^n, G_i^n) for each $i \in I$ and $n \in \mathbf{N}$, where $(\text{SQVIP})_n$ is formulated as follows:

Find $\bar{x} \in K$ such that, for each $i \in I$, there exists $\bar{y}_i \in T_i^n(\bar{x})$ satisfying

$$\bar{x}_i \in A_i^n(\bar{x}), \qquad 0 \in G_i^n(\bar{x}, \bar{y}, z_i)$$

for all $z_i \in A_i^n(\bar{x})$.

Definition 4.3 (SQVIP) is said to be *Hadamard well-posed* if its solution set $S \neq \emptyset$ and, when, for each $i \in I$, every sequence of pairs $\{(A_i^n, T_i^n, G_i^n)\} \subset M_i$ converges to $(A_i, T_i, G_i) \in M_i$ in the sense of Painlevé-Kuratowski, any sequence $\{x^n\}$ satisfying $x^n \in S_n$ has a subsequence which converges strongly to a point in S.

Theorem 4.1 *For each $i \in I$, let $(A_i^n, T_i^n, G_i^n) \in M_i$ for all $n \in \mathbf{N}$. Then the solution set S_n for $(\text{SQVIP})_n$ is closed.*

Proof Without loss of generality, we suppose that $n = 1$. Take any sequence $\{x^n\} \subset S_1$ satisfying $x^n \to x^*$. For each $i \in I$, since K_i is closed, it follows that K is closed and $x^* \in K$. Now, $\{x^n\} \subset S_1$ implies that, for each $i \in I$, there exists $y_i^n \in T_i^1(x^n)$ such that

$$x_i^n \in A_i^1(x^n), \qquad 0 \in G_i^1(x^n, y^n, z_i)$$

for all $z_i \in A_i^1(x^n)$. For each $i \in I$, since $y_i^n \in T_i^1(x^n)$, $T_i^1(\cdot)$ is u.s.c. and compact-valued, this implies that there exist $y_i^* \in T_i^1(x^*)$ and a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ such that $y_i^{n_k} \to y_i^*$. Since $A_i^1(\cdot)$ is continuous and compact-valued, it follows that $A_i^1(\cdot)$ is closed, this implies that $x_i^* \in A_i^1(x^*)$. For each $\bar{z}_i \in A_i^1(x^*)$, since $A_i^1(\cdot)$ is continuous, there exists a sequence $\{z_i^n\} \subseteq K_i$ with $z_i^n \in A_i^1(x^n)$ such that

$$z_i^n \to \bar{z}_i, \qquad 0 \in G_i^1(x^n, y^n, z_i^n).$$

Since G_i^1 is u.s.c. and compact-valued, we know that G_i^1 is closed, which implies that $0 \in G_i^1(x^*, y^*, \bar{z}_i)$. Therefore, S_1 is closed. This completes the proof. □

Theorem 4.2 *For each $i \in I$, let K_i be a nonempty compact subset of X_i, $(A_i^n, T_i^n, G_i^n) \in M_i$ for all $n \in \mathbf{N}$, $(A_i, T_i, G_i) \in M_i$ and $(A_i^n, T_i^n, G_i^n) \xrightarrow{\text{P.K.}} (A_i, T_i, G_i)$. Then*

$$\limsup_{n \to \infty} S_n \subset S. \tag{12}$$

Proof Suppose that (12) does not hold. Then there exists x^* satisfying

$$x^* \in \limsup_{n \to \infty} S_n, \qquad x^* \notin S. \tag{13}$$

From (13), it follows that there exists $x^n \in S_n$ such that the sequence $\{x^n\}$ has a subsequence, which is still denoted by $\{x^n\}$, converging strongly to x^*. For each $i \in I$, since K_i is compact, we know that K is compact. Again, from $S_n \subset K$, $S \subset K$ and Theorem 4.1, it follows that S_n and S are both compact. Thus, for n sufficiently large, there exists $\epsilon > 0$ satisfying

$$x^n \notin B(S, \epsilon),$$

where $B(S,\epsilon) = \bigcup_{y \in S} B(y,\epsilon)$ and $B(y,\epsilon)$ denotes the ball with the center y and the radius ϵ. It follows from $x^n \in S_n$ that, for each $i \in I$, there exists $y_i^n \in T_i^n(x^n)$ such that

$$x_i^n \in A_i^n(x^n), \qquad 0 \in G_i^n(x^n, y^n, z_i^n)$$

for all $z_i^n \in A_i^n(x^n)$. For each $i \in I$, since $T_i^n \xrightarrow{\text{P.K.}} T_i$, we have

$$\limsup_{n\to\infty} \operatorname{gph}(T_i^n) \subset \operatorname{gph}(T_i). \tag{14}$$

Again, since $\{x^n\}$ is bounded and $\{T_i^n\}$ is bounded, there exists a subsequence of $\{y_i^n\}$ with $y_i^n \in T_i^n(x^n)$ converging strongly to a point $y_i^* \in Y_i$. This together with (14) implies that $y_i^* \in T_i(x^*)$. By similar arguments, we also know that $x_i^* \in A_i(x^*)$. Since $A_i^n \xrightarrow{\text{P.K.}} A_i$, we have

$$\operatorname{gph}(A_i) \subset \liminf_{n\to\infty} \operatorname{gph}(A_i^n). \tag{15}$$

By Theorem 5.37 of [39], for all $\bar{z}_i \in A_i(x^*)$, there exist a sequence $\{z_i^n\}$ converging strongly to $\bar{z}_i$ and $\{x^n\}$ such that $z_i^n \in A_i^n(x^n)$ for all $n \in \mathbf{N}$ and $x^n \to x^*$. It follows from (14) that there exists $g_i^n \in G_i^n(x^n, y^n, z_i^n)$ such that $g_i^n = 0$. Since $G_i^n \xrightarrow{\text{P.K.}} G_i$, we have

$$\limsup_{n\to\infty} \operatorname{gph}(G_i^n) \subset \operatorname{gph}(G_i). \tag{16}$$

Again, since $\{x^n\}$, $\{y^n\}$, $\{z^n\}$, and $\{G_i^n\}$ are bounded, it follows that there exists a subsequence of $\{g_i^n\}$ converging strongly to a point $g_i \in Z_i$ and so, from (16), $g_i \in G_i(x^*, y^*, \bar{z}_i)$. Since $g_i^n = 0$ for all $n \in \mathbf{N}$, we get $g_i = 0$. This implies that $x^* \in S$, which is a contradiction. This completes the proof. □

Theorem 4.3 *For each $i \in I$, let K_i be a nonempty compact subset of X_i, $(A_i, T_i, G_i) \in M_i$ and $S \neq \emptyset$. Then* (SQVIP) *is Hadamard well-posed.*

Proof For each $i \in I$, let $\{(A_i^n, T_i^n, G_i^n)\} \subset M_i$, $(A_i^n, T_i^n, G_i^n) \xrightarrow{\text{P.K.}} (A_i, T_i, G_i)$ and $\{x^n\}$ be a sequence satisfying $x^n \in S_n$. For each $i \in I$, by the compactness of K_i, we know that K is compact. Again, from $\{x^n\} \subset K$ and the compactness of K, it follows that $x^n \to x^* \in K$ and so, from Theorem 4.2,

$$\limsup_{n\to\infty} S_n \subset S.$$

Thus $\{x^n\}$ has a subsequence which converges strongly to an element in S and so (SQVIP) is Hadamard well-posed. This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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