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Bounds on normalized Laplacian eigenvalues of graphs

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Abstract

Let G be a simple connected graph of order n, where $n \geq 2$. Its normalized Laplacian eigenvalues are $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$. In this paper, some new upper and lower bounds on λ_n are obtained, respectively. Moreover, connected graphs with $\lambda_2 = 1$ (or $\lambda_{n-1} = 1$) are also characterized.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. In this paper, all graphs are simple connected of order $n \geq 2$. For $v \in V(G)$, let $d(v)$ and $N(v)$ be the degree and the set of neighbors of v, respectively. The maximum and minimum degrees of G are denoted by Δ and δ, respectively.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian* and *normalized Laplacian* matrices of G are defined as $L(G) = D(G) - A(G)$ and $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$, respectively. When only one graph G is under consideration, we sometimes use A, D, L and $\mathcal{L}$ instead of $A(G)$, $D(G)$, $L(G)$ and $\mathcal{L}(G)$, respectively. It is easy to see that $\mathcal{L}(G)$ is a symmetric positive semidefinite matrix and $D(G)^{1/2}\mathbf{1}$ is an eigenvector of $\mathcal{L}(G)$ with eigenvalue 0, where $\mathbf{1}$ is the vector with all ones. Thus, the eigenvalues $\lambda_i(G)$ $(1 \leq i \leq n)$ of $\mathcal{L}(G)$ (or the normalized Laplacian eigenvalues of G) satisfy

$$\lambda_n(G) \geq \cdots \geq \lambda_2(G) \geq \lambda_1(G) = 0.$$

Some of them may be repeated according to their multiplicities. We call $\lambda_k(G)$ the kth smallest normalized Laplacian eigenvalue of G. When only one graph G is under consideration, we sometimes write λ_k instead of $\lambda_k(G)$, for $1 \leq k \leq n$.

The normalized Laplacian is mentioned briefly in the recent monograph by Cvetković *et al.* [1]; however, the standard reference for it is the monograph by Chung [2], which deals almost entirely with this matrix. The normalized Laplacian eigenvalues can be used to give useful information about a graph [2]. For example, one can obtain the number of connected components from the multiplicity of the eigenvalue 0, the bipartiteness from its λ_n (which is at most 2), as well as the connectivity from its λ_2. Moreover, λ_2 is also closely related to the discrete Cheeger's constant, isoperimetric problems, *etc.* (see [2]). Chen and Jost [3] established the relationship between minimum vertex covers and the eigenvalues of the normalized Laplacian on trees. Some upper bounds for λ_n have been introduced by Rojo and Soto [4] and Banerjee [5], respectively. For more results on the normalized Laplacian eigenvalues of graphs can be found in [2, 6, 7].

In this paper, some new upper and lower bounds on λ_n of a graph in terms of its maximum degree, covering number *etc.*, are deduced, respectively. Moreover, connected graphs with $\lambda_2 = 1$ (or $\lambda_{n-1} = 1$) are also characterized.

2 Preliminaries

Here we recall some basic properties of the eigenvalues and eigenfunctions of the normalized Laplacian matrix of a graph G.

Let $\mathbf{g} : V(G) \to \mathbb{R}^n$ which assigns to each vertex v of G a real value $g(v)$, the coordinate of $\mathbf{g}$ according to v. Let $\mathbf{f} = D^{-1/2}\mathbf{g}$. Then we have

$$\frac{\mathbf{g}^T\mathcal{L}\mathbf{g}}{\mathbf{g}^T\mathbf{g}} = \frac{\mathbf{f}^T D^{1/2}\mathcal{L}D^{1/2}\mathbf{f}}{(D^{1/2}\mathbf{f})^T D^{1/2}\mathbf{f}} = \frac{\mathbf{f}^T L\mathbf{f}}{\mathbf{f}^T D\mathbf{f}} = \frac{\sum_{uv\in E(G)}(f(u)-f(v))^2}{\sum_{v\in V(G)} d(v)f(v)^2}.$$

Thus, the following formula for λ_n is clear:

$$\lambda_n = \sup_{\mathbf{f}\perp D\mathbf{1}} \frac{\sum_{uv\in E(G)}(f(u)-f(v))^2}{\sum_{v\in V(G)} d(v)f(v)^2}. \tag{2.1}$$

A vector $\mathbf{f}$ that satisfies equality in Eq. (2.1) is called a *harmonic eigenfunction* of $\mathcal{L}$ associated with $\lambda_n(G)$.

Proposition 2.1 ([2]) *Let G be a graph and $\mathbf{f}$ be a harmonic eigenfunction of $\mathcal{L}$ associated with $\lambda_n(G)$. Then for any $v \in V(G)$, we have*

$$f(v) - \frac{1}{d(v)} \sum_{uv\in E(G)} f(u) = \lambda_n(G)f(v).$$

3 Main result

We call G a *triangulation*, if every pair of adjacent vertices of G have at least one common adjacent vertex. A planar graph is called a *maximal planar graph* if for every pair of nonadjacent vertices u and v of G, the graph $G + uv$ is nonplanar. Lu *et al.* [8] and Guo *et al.* [9] gave the upper bounds for the Laplacian spectral radius of a triangulation and a maximal planar graph, respectively. For the normalized Laplacian spectral radius, we have the following somewhat similar result.

Theorem 3.1 *Let $G = (V, E)$ be a triangulation of order n. Then*

$$\lambda_n \le \max\left\{ \frac{2d(v_i) - 1 + \sqrt{4d(v_i)m(v_i) - 4d(v_i) + 1}}{2d(v_i)} : v_i \in V \right\},$$

where $m(v_i) = \sum_{v_j\in N(v_i)} d(v_j)/d(v_i)$ is the average 2-degree of the vertex v_i. Moreover, the equality holds if $G \cong K_3$.

Proof Let $V = \{v_1, \dots, v_n\}$, and let $\mathbf{f} = (f(v_1), \dots, f(v_n))^T$ be the harmonic eigenfunction of $\mathcal{L}(G)$ corresponding to λ_n. Then by Proposition 2.1, we have for each $v_i \in V$,

$$d(v_i)(1-\lambda_n)f(v_i) = \sum_{v_iv_j \in E} f(v_j).$$

Hence by the Lagrange identity, we have for each v_i,

$$d(v_i)^2(1-\lambda_n)^2 f(v_i)^2 = d(v_i)\sum_{v_iv_j \in E} f(v_j)^2 - \sum_{\substack{1\le j<k\le n \\ v_j,v_k \in N(v_i)}} \bigl(f(v_j)-f(v_k)\bigr)^2.$$

Sum over v_i to obtain

$$\begin{aligned}
&\sum_{i=1}^n d(v_i)^2(1-\lambda_n)^2 f(v_i)^2 \\
&\quad = \sum_{i=1}^n d(v_i)\sum_{v_iv_j \in E} f(v_j)^2 - \sum_{i=1}^n \sum_{\substack{1\le j<k\le n \\ v_j,v_k \in N(v_i)}} \bigl(f(v_j)-f(v_k)\bigr)^2 \\
&\quad = \sum_{i=1}^n d(v_i)m(v_i)f(v_i)^2 - \sum_{i=1}^n \sum_{\substack{1\le j<k\le n \\ v_j,v_k \in N(v_i)}} \bigl(f(v_j)-f(v_k)\bigr)^2,
\end{aligned} \tag{3.1}$$

where $m(v_i) = \sum_{v_j \in N(v_i)} d(v_j)/d(v_i)$.

Note that G is a triangulation. Then by Eq. (2.1), we have

$$\sum_{i=1}^n \sum_{\substack{1\le j<k\le n \\ v_j,v_k \in N(v_i)}} \bigl(f(v_j)-f(v_k)\bigr)^2 \ge \sum_{\substack{1\le j<k\le n \\ v_jv_k \in E(G)}} \bigl(f(v_j)-f(v_k)\bigr)^2 = \lambda_n \sum_{i=1}^n d(v_i)f(v_i)^2. \tag{3.2}$$

Thus, combining Eqs. (3.1) and (3.2), we have

$$\sum_{i=1}^n \bigl[d(v_i)^2(1-\lambda_n)^2 - d(v_i)m(v_i) + \lambda_n d(v_i)\bigr] f(v_i)^2 \le 0.$$

This implies that there exists at least one vertex v_i such that

$$d(v_i)^2(1-\lambda_n)^2 - d(v_i)m(v_i) + \lambda_n d(v_i) \le 0.$$

That is,

$$\lambda_n \le \max\left\{\frac{2d(v_i)-1+\sqrt{4d(v_i)m(v_i)-4d(v_i)+1}}{2d(v_i)} : v_i \in V\right\}.$$

For $G = K_3$, it is easy to check that the equality holds. □

Furthermore, we have the following more general result.

Theorem 3.2 *Let $G = (V, E)$ be a simple connected graph of order n with m edges. If each edge of G belongs to at least t triangles ($t \geq 1$), then*

$$\lambda_n \leq \max\left\{\frac{2d(v_i) - t + \sqrt{4d(v_i)m(v_i) - 4td(v_i) + t^2}}{2d(v_i)} : v_i \in V\right\}, \tag{3.3}$$

the equality occurs if G is the complete graph K_{t+2}.

Proof For $G = K_{t+2}$, it is easy to check that the equality in Eq. (3.3) holds. If we replace Eq. (3.2) in the proof of Theorem 3.1 by

$$\sum_{i=1}^{n} \sum_{\substack{1\leq j<k\leq n \\ v_j, v_k \in N(v_i)}} \bigl(f(v_j) - f(v_k)\bigr)^2 \geq t \sum_{\substack{1\leq j<k\leq n \\ v_j v_k \in E}} \bigl(f(v_j) - f(v_k)\bigr)^2 = t\lambda_n \sum_{i=1}^{n} d(v_i) f(v_i)^2,$$

then the result follows. □

For the maximal planar graphs, we have the following upper bound.

Theorem 3.3 *Let G be a maximal planar graph of order $n \geq 4$ with m edges. Then*

$$\lambda_n \leq \max\left\{\frac{d(v_i) - 1 + \sqrt{d(v_i)m(v_i) - 2d(v_i) + 1}}{d(v_i)} : v_i \in V(G)\right\}. \tag{3.4}$$

Proof Note that for any maximal planar graph G, each edge of G belongs to at least 2 triangles. Then the result follows from Theorem 3.2. □

In what follows, we turn to some lower bounds on λ_n. The following result due to Chung [2] concerns the lower bound on $\lambda_n(G)$.

Lemma 3.4 ([2]) *Let G be a connected graph of order n. Then $\lambda_n(G) \geq \frac{n}{n-1}$, the equality holds if and only if $G \cong K_n$, where K_n is the complete graph of order n.*

Let $G = (V, E)$ be a graph and $X \subseteq V$ be a subset of the vertices. Let $\overline{X} = V \setminus X$ be the complement of the set X. The volume of X is defined to be the sum of the degrees of the vertices in G, that is,

$$\operatorname{vol}(X) = \sum_{v\in X} d(v).$$

Note that $\operatorname{vol}(V)$ is equal to twice the number of edges in the graph.

Theorem 3.5 *Let G be a connected graph of order n with m edges. For any nonempty subset $X \subseteq V$, we have*

$$\lambda_n \geq \frac{2m|E_X|}{\operatorname{vol}(X)(2m - \operatorname{vol}(X))},$$

where E_X is the set of all edges with one end in X and the other end in $\overline{X}$. Moreover, if the equality holds, then $\frac{\sum_{u\in N(v)} f(u)}{d(v)} = x$ for each $v \in X$ and $\frac{\sum_{u\in N(v)} f(u)}{d(v)} = y$ for each $v \in \overline{X}$, where x and y are constant such that $\frac{x}{y} = -\frac{\operatorname{vol}(\overline{X})}{\operatorname{vol}(X)}$.

Proof Let $X \subseteq V$ and $\mathbf{f}$ be a vector such that

$$f(u) = \begin{cases} -\operatorname{vol}(\overline{X}) & \text{if } u \in X, \\ \operatorname{vol}(X) & \text{if } u \notin X. \end{cases}$$

Clearly, $\sum_{u\in V} d(u)f(u) = -\operatorname{vol}(\overline{X})\operatorname{vol}(X) + \operatorname{vol}(\overline{X})\operatorname{vol}(X) = 0$. Moreover, note that $\operatorname{vol}(X) + \operatorname{vol}(\overline{X}) = \operatorname{vol}(V) = 2m$. Then, by Eq. (2.1), we have

$$\begin{aligned} \lambda_n &\geq \frac{\sum_{uv\in E(G)}(f(u)-f(v))^2}{\sum_{v\in V(G)} d(v)f(v)^2} \\ &= \frac{|E_X|(\operatorname{vol}(X)+\operatorname{vol}(\overline{X}))^2}{\operatorname{vol}(X)\operatorname{vol}(\overline{X})(\operatorname{vol}(X)+\operatorname{vol}(\overline{X}))} \\ &= \frac{2m|E_X|}{\operatorname{vol}(X)(2m-\operatorname{vol}(X))}. \end{aligned}$$

Moreover, if the equality holds, then $\mathbf{f}$ is the harmonic eigenfunction of $\mathcal{L}$ associated with $\lambda_n(G)$. Hence Proposition 2.1 implies that

$$\begin{cases} (\lambda_n - 1)\operatorname{vol}(\overline{X}) = \frac{\sum_{u\in N(v)} f(u)}{d(v)} & \text{for each } v \in X, \\ (1-\lambda_n)\operatorname{vol}(X) = \frac{\sum_{u\in N(v)} f(u)}{d(v)} & \text{for each } v \in \overline{X}. \end{cases}$$

Let $x = (\lambda_n - 1)\operatorname{vol}(\overline{X})$ and $y = (1-\lambda_n)\operatorname{vol}(X)$. Then $\frac{x}{y} = -\frac{\operatorname{vol}(\overline{X})}{\operatorname{vol}(X)}$. This completes the proof. □

Let $X = \{u\}$ in Theorem 3.5. Note that $\operatorname{vol}(X) = d(u) = |E_X|$. Then we have the following.

Corollary 3.6 *Let G be a graph of order n with m edges. Then*

$$\lambda_n \geq \frac{2m}{2m-\Delta}, \tag{3.5}$$

where Δ is the maximum degree of G.

Remark 3.1 Note that $2m \leq n\Delta$ holds for any graph of order n with m edges and maximum degree Δ. Thus the lower bound in Corollary 3.6 is always better than that in Lemma 3.4. Moreover, if G is a complete graph K_n or a star S_n, then it is easy to check that the equality holds in Eq. (3.5).

Similarly, let $X = \{u, v\}$ in Theorem 3.5. Then we have:

Corollary 3.7 *Let G be a graph of order n with m edges. Let $a = \max_{uv\in E(G)}\{d(u)+d(v)\}$ and $b = \max_{uv\notin E(G)}\{d(u)+d(v)\}$. Then:*

(1) $\lambda_n \geq \frac{2m(a-2)}{a(2m-a)}$, *and the equality holds if $G \cong K_n$.*
(2) $\lambda_n \geq \frac{2m}{2m-b}$, *and the equality holds if $G \cong K_{2,n-2}$, where $G \cong K_{2,n-2}$ is the complete bipartite graph with parts of cardinalities 2 and $n-2$.*

Proof Let $X = \{u, v\}$ in Theorem 3.5. If $uv \in E(G)$, then $|E_X| = d(u) + d(v) - 2$ and $\mathrm{vol}(X) = d(u) + d(v)$. Theorem 3.5 implies that

$$\lambda_n \geq \frac{2m[(d(u) + d(v)) - 2]}{(d(u) + d(v))[2m - (d(u) + d(v))]}.$$

Let $f(x) = \frac{2m(x-2)}{x(2m-x)}$ for $x > 2$. Then it is easy to see that $f(x)$ is increasing on x. Hence, we have $\lambda_n \geq \frac{2m(a-2)}{a(2m-a)}$. Moreover, it is easy to check that the equality holds when $G \cong K_n$.

If $uv \notin E(G)$, then $|E_X| = \mathrm{vol}(X) = d(u) + d(v)$. Theorem 3.5 implies that

$$\lambda_n \geq \frac{2m}{2m - (d(u) + d(v))}.$$

Hence $\lambda_n \geq \frac{2m}{2m-b}$. Moreover, it is easy to check that the equality holds when $G \cong K_{2,n-2}$. □

A set of vertices X of G is called a *cover* of G if every edge of G is incident to some vertex in X. The least cardinality of a cover of G is called the *covering number* of G and denoted by $\tau(G)$. It is clear that if a vertex set X is a vertex cover if and only if $\overline{X}$ is an independent set. The following lower bound for λ_n in terms of $\tau(G)$ is obtained.

Theorem 3.8 *Let G be a graph order n with m edges. Then*

$$\lambda_n \geq \frac{2m}{2m - \delta(n - \tau(G))},$$

where δ is the minimum degree of G. Moreover, the equality holds if $G \cong C_n$ when n is even, $G \cong K_{a,b}$ or $G \cong K_n$, where C_n is the cycle of order n and $K_{a,b}$ is the complete bipartite graph with parts of cardinalities a and b.

Proof Let X be a minimal covering set of G with $|X| = \tau(G)$. Then $\overline{X}$ is an independent set. Hence $\mathrm{vol}(\overline{X}) = |E_X|$ and $\mathrm{vol}(X) = 2m - |E_X|$. Then Theorem 3.5 implies that $\lambda_n \geq \frac{2m}{2m-|E_X|}$. Moreover, by the definition of covering set, we have $|E_X| \geq \delta(n - \tau(G))$. Hence we have $\lambda_n \geq \frac{2m}{2m-\delta(n-\tau(G))}$. Moreover, if $G \cong C_n$ when n is even, then $\tau(G) = \frac{n}{2}$. Hence it is easy to check that the equality holds. Similarly, if $G \cong K_{a,b}$ or $G \cong K_n$, then the equality holds. This completes the proof. □

Chung [2] proved that for any graph G of order n, $\lambda_2 \leq \frac{n}{n-1}$ with equality holding if and only if $G \cong K_n$. Moreover, the following result is also introduced.

Lemma 3.9 ([2]) *Let G ($G \neq K_n$) be a connected graph of order n. Then $\lambda_2 \leq 1$.*

In what follows, we characterize all connected graphs with $\lambda_2 = 1$. We will make use of the following lemma.

Lemma 3.10 ([7]) *Let G be a connected graph of order n with maximum degree Δ and minimum degree δ. Let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$ are the eigenvalues of $A(G)$. Then for each $1 \leq k \leq n$,*

$$\frac{|\rho_{n-k+1}|}{\Delta} \leq |1 - \lambda_k| \leq \frac{|\rho_{n-k+1}|}{\delta}.$$

Theorem 3.11 *Let G ($G \neq K_n$) be a connected graph of order n. Then $\lambda_2 = 1$ if and only if G is a complete multipartite graph.*

Proof By Lemma 3.10, if $\lambda_2 = 1$, then $\rho_{n-1} = 0$, where ρ_{n-1} is the second largest eigenvalue of $A(G)$. Hence the result follows from the fact that for any simple connected graph G of order n, $\rho_{n-1} \leq 0$ if and only if G is a complete multipartite graph [10]. On the other hand, when G ($G \neq K_n$) is a complete multipartite graph, $\rho_{n-1}(G) = 0$ [10]. This together with Lemma 3.10 imply that $\lambda_2 = 1$. The proof is completed. □

Moreover, the following result on λ_{n-1} is also obtained.

Theorem 3.12 *Let G be a connected graph of order n. Then $\lambda_{n-1} \geq 1$, the equality holds if and only if G is a complete bipartite graph.*

Proof Note that for any connected graph of order n, $\lambda_1 = 0$ and $\lambda_n \leq 2$. Since $\sum_{i=1}^{n} \lambda_i = n$, $\sum_{i=2}^{n-1} \lambda_i \geq n-2$ and hence $\lambda_{n-1} \geq 1$. Moreover, if $\lambda_{n-1} = 1$, then $\lambda_2 = \cdots = \lambda_{n-1} = 1$ and $\lambda_n = 2$ since $\sum_{i=2}^{n} \lambda_i = n$. This implies that G is bipartite [2]. Moreover, since $\lambda_2 = 1$, combining with Theorem 3.11 we find that G is complete bipartite graph. On the other hand, it is easy to check that if G is a complete bipartite graph, then $\lambda_{n-1} = 1$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JL carried out the proofs of the main results in the manuscript. J-MG and WCS participated in the design of the study and drafted the manuscript. All authors read and approved the final manuscript.

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