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On solving Lipschitz pseudocontractive operator equations

Ahmed A Abdelhakim[1*] and Feng Gu[2]

*Correspondence:
ahmedmaths2004@gmail.com
[1]Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt
Full list of author information is available at the end of the article

Abstract

We analyze the convergence of the Mann-type double sequence iteration process to the solution of a Lipschitz pseudocontractive operator equation on a bounded closed convex subset of arbitrary real Banach space into itself. Our results extend the result in (Moore in Comp. Math. Appl. 43: 1585-1589, 2002).

MSC: 47H10; 54H25

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1 Introduction

Let E be a real Banach space and E^* be the dual space of E. Let J be the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}$$

for all $x \in E$ where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. A single-valued duality map will be denoted by j.

An operator $T : E \to E$ is said to be

- pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$$

  for any $x, y \in E$;
- accretive if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq 0;$$

- strongly pseudocontractive if there exist $j(x-y) \in J(x-y)$ and a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \lambda\|x-y\|^2$$

  for any $x, y \in E$;

- strongly accretive if for any $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a constant $t \in (0,1)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq t\|x-y\|^2$$

for all $x, y \in E$.

As a consequence of a result of Kato [1], the concept of pseudocontractive operators can equivalently be defined as follows:

T is strongly pseudocontractive if there exists $\lambda \in (0,1)$ such that the inequality

$$\|x-y\| \leq \big\|x - y + r\big[(I - T - \lambda I)x - (I - T - \lambda I)y\big]\big\| \tag{1.1}$$

holds for all $x, y \in E$ and $r > 0$. If $\lambda = 0$ in the inequality (1.1), then T is pseudocontractive.

It is easy to see that T is pseudocontractive if and only if $I - T$ is accretive where I denotes the identity mapping on E.

Let C be a compact convex subset of a real Hilbert space and let $T : C \to C$ be a Lipschitz pseudocontraction. It remains as an open question whether the Mann iteration process always converges to a fixed point of T. In [2] it was proved that the Ishikawa iteration process converges strongly to a fixed point of T. In 2001, Mutangadura and Chidume [3] constructed the following example to demonstrate that the Mann iteration process is not guaranteed to converge to a fixed point of a Lipschitz pseudocontraction mapping a compact convex subset of a real Hilbert space H into itself.

Example [3] Let $H = \Re^2$ with the usual Euclidean inner product, and for $x = (a, b) \in H$ define $x^\perp = (b, -a)$. Now, let $C = B_1(o)$; the closed unit ball in H and let $C_1 = \{x \in H : \|x\| \leq \frac{1}{2}\}$, $C_2 = \{x \in H : \frac{1}{2} \leq \|x\| \leq 1\}$. Define the map $T : C \to C$ by

$$Tx = \begin{cases} x + x^\perp, & \text{if } x \in C_1; \\ \frac{x}{\|x\|} - x + x^\perp, & \text{if } x \in C_2. \end{cases}$$

Observe that T is pseudocontractive, Lipschitz continuous (with Lipschitz constant 5) and has the origin $(0,0)$ as its unique fixed point; C is compact and convex. However, for any $x \in C_1$, we have

$$\big\|(1-\lambda)x + \lambda Tx\big\|^2 = \big(1 + \lambda^2\big)\|x\|^2 > \|x\|^2, \quad \forall \lambda \in (0,1),$$

while for any $x \in C_2$, we have

$$\big\|(1-\lambda)x + \lambda Tx\big\|^2 \geq \frac{1}{2}\|x\|^2, \quad \forall \lambda \in (0,1),$$

and therefore no Mann sequence can converge to $(0,0)$, the unique fixed point of T, unless the initial guess is the fixed point itself.

Moore [4] introduced the concept of a Mann-type double sequence iteration process and proved that it converges strongly to a fixed point of a continuous pseudocontraction which maps a bounded closed convex nonempty subset of a real Hilbert space into itself.

Definition 1.1 [4] Let $\mathcal{N}$ denote the set of all nonnegative integers (the natural numbers) and let E be a normed linear space. By a double sequence in E is meant a function $f : \mathcal{N} \times \mathcal{N} \to E$ defined by $f(n,m) = x_{n,m} \in E$. A double sequence $\{x_{n,m}\}$ is said to converge strongly to x^* if given any $\epsilon > 0$, there exist $N, M > 0$ such that $\|x_{n,m} - x^*\| < \epsilon$ for all $n \geq N$, $m \geq M$. If $\forall n, r \geq N$, $\forall m, t \geq M$, we have $\|x_{n,r} - x_{m,t}\| < \epsilon$, then the double sequence is said to be Cauchy. Furthermore, if for each fixed n, $x_{n,m} \to x_n^*$ as $m \to \infty$ and then $x_n^* \to x^*$ as $n \to \infty$, then $x_{n,m} \to x^*$ as $n, m \to \infty$.

Theorem 1.1 [4] *Let C be a bounded closed convex nonempty subset of a (real) Hilbert space H, and let $T : C \to C$ be a continuous pseudocontractive map. Let $\{\alpha_n\}_{n\geq 0}, \{a_k\}_{k\geq 0} \subset (0,1)$ be real sequences satisfying the following conditions*:

(i) $\lim_{k\to\infty} a_k = 1$,
(ii) $\lim_{k,r\to\infty} (a_k - a_r)/(1 - a_k) = 0$, $\forall 0 < r \leq k$,
(iii) $\lim_{n\to\infty} \alpha_n = 0$,
(iv) $\sum_{n\geq 0} \alpha_n = \infty$.

For an arbitrary but fixed $\omega \in C$, and for each $k \geq 0$, define $T_k : C \to C$ by $T_k x = (1 - a_k)\omega + a_k Tx$, $\forall x \in C$. Then the double sequence $\{x_{k,n}\}_{k\geq 0, n\geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_n T_k x_{k,n}, \quad k, n \geq 0,$$

converges strongly to a fixed point x_∞^* of T in C.

The following lemma will be useful in the sequel.

Lemma 1.2 [5] *Let $\{\delta_n\}$ and $\{\sigma_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$\delta_{n+1} \leq \gamma\delta_n + \sigma_n, \quad n \geq 0.$$

Here $\gamma \in [0,1)$. If $\lim_{n\to\infty} \sigma_n = 0$, then $\lim_{n\to\infty} \delta_n = 0$.

It is our purpose in this paper to extend Theorem 1.1 from Hilbert space to an arbitrary real Banach space with no further assumptions on the real sequences $\{\alpha_n\}_{n\geq 0}$, $\{a_k\}_{k\geq 0}$.

2 Main results

Theorem 2.1 *Let C be a bounded closed convex subset of a Banach space E and $T : C \to C$ be a Lipschitz pseudocontraction with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\geq 0}, \{a_k\}_{k\geq 0} \subset (0,1)$ be real sequences satisfying the following conditions*:

(i) $\lim_{k\to\infty} a_k = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$.

For an arbitrary but fixed $\omega \in C$, and for each $k \geq 0$, define $T_k : C \to C$ by $T_k x = (1 - a_k)\omega + a_k Tx$, $\forall x \in C$. Then the double sequence $\{x_{k,n}\}_{k\geq 0, n\geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_n T_k x_{k,n}, \quad k, n \geq 0 \tag{2.1}$$

converges strongly to a fixed point x^* of T in C.

Proof Since T is Lipschitzian, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

Since T is pseudocontractive, for each $k \geq 0$, we have

$$\langle T_k x - T_k y, j(x - y)\rangle = a_k\langle Tx - Ty, j(x - y)\rangle \leq a_k\|x - y\|^2.$$

Hence, T_k is Lipschitz and strongly pseudocontractive. Also, C is invariant under T_k for all $k \geq 0$, by convexity. Thus, for each $k \geq 0$, T_k has a unique fixed point x_k^*, say, in C.

Now, we proceed in the following steps.

(I) for each $k \geq 0$, $x_{k,n} \to x_k^* \in C$ as $n \to \infty$.

(II) $x_k^* \to x^* \in C$ as $k \to \infty$.

(III) $x^* \in F(T)$.

Proof of (I). In fact, it follows from (2.1) that

$$\begin{aligned}
x_{k,n} &= x_{k,n+1} + \alpha_n x_{k,n} - \alpha_n T_k x_{k,n} \\
&= (1 + \alpha_n)x_{k,n+1} + \alpha_n(I - T_k - \lambda I)x_{k,n+1} - (2 - \lambda)\alpha_n x_{k,n+1} + \alpha_n x_{k,n} \\
&\quad + \alpha_n(T_k x_{k,n+1} - T_k x_{k,n}) \\
&= (1 + \alpha_n)x_{k,n+1} + \alpha_n(I - T_k - \lambda I)x_{k,n+1} - (2 - \lambda)\alpha_n\bigl[(1 - \alpha_n)x_{k,n} + \alpha_n T_k x_{k,n}\bigr] \\
&\quad + \alpha_n x_{k,n} + \alpha_n(T_k x_{k,n+1} - T_k x_{k,n}) \\
&= (1 + \alpha_n)x_{k,n+1} + \alpha_n(I - T_k - \lambda I)x_{k,n+1} - (1 - \lambda)\alpha_n x_{k,n} \\
&\quad + (2 - \lambda)\alpha_n^2(x_{k,n} - T_k x_{k,n}) + \alpha_n(T_k x_{k,n+1} - T_k x_{k,n}).
\end{aligned}$$

Thus, if x_k^* is a fixed point of T_k, $k \geq 0$, then

$$\begin{aligned}
x_{k,n+1} - x_k^* = {}& (1 + \alpha_n)\bigl(x_{k,n+1} - x_k^*\bigr) + \alpha_n(I - T_k - \lambda I)\bigl(x_{k,n+1} - x_k^*\bigr) \\
& - (1 - \lambda)\alpha_n\bigl(x_{k,n} - x_k^*\bigr) + (2 - \lambda)\alpha_n^2(x_{k,n} - T_k x_{k,n}) + \alpha_n(T_k x_{k,n+1} - T_k x_{k,n}).
\end{aligned}$$

Using inequality (1.1), it follows that

$$\begin{aligned}
\bigl\|x_{k,n+1} - x_k^*\bigr\| \geq {}& (1 + \alpha_n)\bigl\|x_{k,n+1} - x_k^*\bigr\| - (1 - \lambda)\alpha_n\bigl\|x_{k,n} - x_k^*\bigr\| \\
& - (2 - \lambda)\alpha_n^2\|x_{k,n} - T_k x_{k,n}\| - \alpha_n\|T_k x_{k,n+1} - T_k x_{k,n}\|. \qquad (2.2)
\end{aligned}$$

On the other hand, by (2.1) we obtain

$$\begin{aligned}
\|x_{k,n+1} - x_{k,n}\| &= \alpha_n\|T_k x_{k,n} - x_{k,n}\| \\
&\leq \alpha_n\bigl(\bigl\|T_k x_{k,n} - x_k^*\bigr\| + \bigl\|x_{k,n} - x_k^*\bigr\|\bigr) \\
&= \alpha_n\bigl(a_k\bigl\|Tx_{k,n} - x_k^*\bigr\| + \bigl\|x_{k,n} - x_k^*\bigr\|\bigr) \\
&\leq \alpha_n\bigl(a_k L\bigl\|x_{k,n} - x_k^*\bigr\| + \bigl\|x_{k,n} - x_k^*\bigr\|\bigr) \\
&\leq \alpha_n(L + 1)\bigl\|x_{k,n} - x_k^*\bigr\|.
\end{aligned}$$

Therefore,

$$\begin{aligned} \|T_k x_{k,n+1} - T_k x_{k,n}\| &= a_k \|T x_{k,n+1} - T x_{k,n}\| \\ &\le a_k L \|x_{k,n+1} - x_{k,n}\| \\ &\le L \|x_{k,n+1} - x_{k,n}\| \\ &\le \alpha_n L(L+1) \|x_{k,n} - x_k^*\|. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we arrive at

$$\begin{aligned} \|x_{k,n} - x_k^*\| \ge{} & (1+\alpha_n)\|x_{k,n+1} - x_k^*\| - (1-\lambda)\alpha_n \|x_{k,n} - x_k^*\| \\ & - (2-\lambda)\alpha_n^2 \|x_{k,n} - T_k x_{k,n}\| - L(L+1)\alpha_n^2 \|x_{k,n} - x_k^*\|, \end{aligned}$$

which implies that

$$\begin{aligned} \alpha_n \|x_{k,n+1} - x_k^*\| \le{} & (1-\lambda)\alpha_n \|x_{k,n} - x_k^*\| + \alpha_n^2 \big[L(L+1)\|x_{k,n} - x_k^*\| \\ & + (2-\lambda)\|x_{k,n} - T_k x_{k,n}\|\big], \end{aligned}$$

and so

$$\begin{aligned} \|x_{k,n+1} - x_k^*\| \le{} & (1-\lambda)\|x_{k,n} - x_k^*\| + \alpha_n \big[L(L+1)\|x_{k,n} - x_k^*\| \\ & + (2-\lambda)\|x_{k,n} - T_k x_{k,n}\|\big]. \end{aligned} \tag{2.4}$$

Since C is bounded, there exists $M > 0$ such that

$$M = \max\Big\{L(L+1)\sup_{n\ge 0}\|x_{k,n} - x_k^*\|, (2-\lambda)\sup_{n\ge 0}\|x_{k,n} - T_k x_{k,n}\|\Big\}.$$

Hence, it follows from (2.4) that

$$\|x_{k,n+1} - x_k^*\| \le (1-\lambda)\|x_{k,n} - x_k^*\| + \alpha_n M.$$

Since $\lambda \in (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, it follows from Lemma 1.2 that

$$\lim_{n\to\infty} \|x_{k,n} - x_k^*\| = 0,$$

i.e., $x_{k,n} \to x_k^*$ as $n \to \infty$.

Proof of (II). We prove that $\{x_k^*\}_{k=0}^{\infty} = \{T_k x_k^*\}_{k=0}^{\infty}$ converges to some $x^* \in C$. For this purpose, we need only to prove that $\{x_k^*\}_0^{\infty}$ is a Cauchy sequence.

In fact, we have

$$\begin{aligned} \|x_l^* - x_m^*\|^2 &= \langle x_l^* - x_m^*, j(x_l^* - x_m^*)\rangle \\ &= \langle T_l x_l^* - T_m x_m^*, j(x_l^* - x_m^*)\rangle \\ &= \langle (1-a_l)\omega + a_l T x_l^* - (1-a_m)\omega - a_m T x_m^*, j(x_l^* - x_m^*)\rangle \end{aligned}$$

$$
\begin{aligned}
&= (a_m - a_l)\langle \omega, j(x_l^* - x_m^*)\rangle + a_l\langle Tx_l^* - Tx_m^*, j(x_l^* - x_m^*)\rangle \\
&\quad + (a_l - a_m)\langle Tx_m^*, j(x_l^* - x_m^*)\rangle \\
&\le |a_l - a_m|\bigl(\|\omega\|\|x_l^* - x_m^*\| + \|Tx_m^*\|\|x_l^* - x_m^*\|\bigr) \\
&\quad + a_l\langle Tx_l^* - Tx_m^*, j(x_l^* - x_m^*)\rangle \\
&\le |a_l - a_m|\bigl(\|\omega\| + \|Tx_m^*\|\bigr)\|x_l^* - x_m^*\| + a_l\lambda\|x_l^* - x_m^*\|^2 \\
&\le |a_l - a_m|\bigl(\|\omega\| + \|Tx_m^*\|\bigr)\|x_l^* - x_m^*\| + \lambda\|x_l^* - x_m^*\|^2,
\end{aligned}
$$

that is,

$$
\|x_l^* - x_m^*\| \le \bigl[|a_l - a_m|\bigl(\|\omega\| + \|Tx_m^*\|\bigr) + \lambda\|x_l^* - x_m^*\|\bigr],
$$

hence

$$
\|x_l^* - x_m^*\| \le 2\frac{|a_l - a_m|}{1-\lambda}d,
$$

where $d = \operatorname{diam} C$. If follows from condition (i) that

$$
\lim_{l,m\to\infty}\|x_l^* - x_m^*\| = 0.
$$

This completes step (II) of the proof.

Proof of (III). In order to accomplish step (III), we first have to prove that $\{x_k^*\}_{k=0}^{\infty}$ is an approximate fixed point sequence for T. In fact, from $T_k x_k^* = (1-a_k)\omega + a_k Tx_k^*$, we have

$$
\begin{aligned}
\|x_k^* - Tx_k^*\| &= \left\| x_k^* - \frac{1}{a_k}T_k x_k^* + \frac{1-a_k}{a_k}\omega \right\| \\
&= \left\| x_k^* - \frac{1}{a_k}x_k^* + \frac{1-a_k}{a_k}\omega \right\| \\
&= \left\| \frac{1-a_k}{a_k}\bigl(\omega - x_k^*\bigr) \right\| \\
&\le \frac{1-a_k}{a_k}\bigl(\|\omega\| + \|x_k^*\|\bigr) \\
&\le \frac{1-a_k}{a_k}\cdot 2d,
\end{aligned}
$$

where $d = \operatorname{diam} C$. Hence $\lim_{k\to\infty}\|x_k^* - Tx_k^*\| = 0$. Since $x_k^* \to x^*$ as $k\to\infty$, T is continuous and using continuity of the norm, we get $\lim_{k\to\infty}\|x^* - Tx^*\| = 0$, *i.e.*, $x^* = Tx^*$. This completes the proof. □

Corollary 2.2 *Let C be a bounded closed convex subset of a Banach space E and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\ge 0}, \{a_k\}_{k\ge 0} \subset (0,1)$ be real sequences satisfying conditions* (i)-(ii) *in Theorem* 2.1. *For an arbitrary but fixed $\omega \in C$, and for each $k \ge 0$, define $T_k : C \to C$ by $T_k x = (1-a_k)\omega + a_k Tx$, $\forall x \in C$. Then the double sequence*

$\{x_{k,n}\}_{k\geq 0,n\geq 0}$ *generated from an arbitrary* $x_{0,0}\in C$ *by*

$$x_{k,n+1}=(1-\alpha_n)x_{k,n}+\alpha_n T_k x_{k,n},\quad k,n\geq 0,$$

converges strongly to a fixed point of T *in* C.

Proof Obvious, observing the fact that every nonexpansive mapping is Lipschitz and pseudocontractive. □

The following corollary follows from Theorem 2.1 on setting $\omega = 0 \in C$.

Corollary 2.3 *Let* C, E, T, $\{\alpha_n\}_{n=0}^{\infty}$, $\{a_k\}_{k=0}^{\infty}$ *be as in Theorem* 2.1. *For an arbitrary but fixed* $\omega\in C$, *and for each* $k\geq 0$, *define* $T_k: C\to C$ *by* $T_k x = a_k Tx$ *for all* $x\in C$. *Then the double sequence* $\{x_{k,n}\}_{k\geq 0,n\geq 0}$ *generated from an arbitrary* $x_{0,0}\in C$ *by*

$$x_{k,n+1}=(1-\alpha_n)x_{k,n}+\alpha_n T_k x_{k,n},\quad k,n\geq 0,$$

converges strongly to a fixed point of T *in* C.

Remark 2.1 Theorem 2.1 improves and extends Theorem 3.1 of Moore [3] in three respects:

(1) It abolishes the condition that $\lim_{r,k\to\infty}\frac{a_k-a_r}{1-a_k}=0$.
(2) It abolishes the condition that $\sum_{n=1}^{\infty}\alpha_n=\infty$.
(3) The ambient space is no longer required to be a Hilbert space and is taken to be the more general Banach space instead.

Remark 2.2

(1) Whereas the Ishikawa iteration process was proved to converge to a fixed point of a Lipschitz pseudocontractive mapping in compact convex subsets of a Hilbert space, we imposed no compactness conditions to obtain the strong convergence of the double sequence iteration process to a fixed point of a Lipschitz pseudocontraction.
(2) Our results may easily be extended to the slightly more general classes of Lipschitz hemicontractive and Lipschitz quasi-nonexpansive mappings.
(3) Prototypes of the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ are

$$a_k=\frac{k}{1+k}\quad\text{and}\quad \alpha_n=\frac{1}{(n+1)^2}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt. [2]Institute of Applied Mathematics and Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China.

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