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Strong convergence of a Halpern-type algorithm for common solutions of fixed point and equilibrium problems

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Abstract
In this article, fixed points of nonexpansive mappings and equilibrium problems based on a Halpern-type algorithm are investigated. Strong convergence theorems for common solutions of the two problems are obtained in the framework of real Hilbert spaces.

Keywords: nonexpansive mapping; equilibrium problem; fixed point; variational inequality

1 Introduction

The study of equilibrium problems is an important branch of optimization theory and nonlinear functional analysis. Numerous problems in physics, optimization, transportation, signal processing, and economics are reduced to find a solution to equilibrium problems, which cover fixed point problems, variational inequalities, saddle problems, inclusion problems, and so on. A closely related subject of current interest is the problem of finding common elements in the fixed point set of nonlinear operators and in the solution set of monotone variational inequalities; see [1–15] and the references therein. The motivation for this subject is mainly due to its possible applications to mathematical modeling of concrete complex problems. The aim of this paper is to investigate a common element problem based on a Halpern-type algorithm. Strong convergence of the algorithm is obtained in the framework of real Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a Halpern-type algorithm is proposed and analyzed. Strong convergence theorems for common solutions of two problems are established in the framework of Hilbert spaces. In Section 4, applications of the main results are provided.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and let Proj_C be the metric projection from H onto C.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. Recall that T is said to be *contractive* iff there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

For such a case, T is also said to be *α-contractive*. Recall that T is said to be *nonexpansive* iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is well known that the fixed point set of nonexpansive mappings is nonempty provided that the subset C is bounded, convex, and closed.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-inverse-strongly monotone*.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2.1}$$

In this paper, we use $\mathrm{VI}(C, A)$ to denote the solution set of (2.1). It is well known that $x \in C$ is a solution of the variational inequality (2.1) iff x is a solution of the fixed point equation $P_C(I - rA)x = x$, where $r > 0$ is a constant.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle > 0$. M is *maximal* iff the graph Graph(M) of R is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$, for all $(y, g) \in \mathrm{Graph}(M)$ implies $f \in Rx$.

For a maximal monotone operator M on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to D(M)$, where $D(M)$ denotes the domain of M. It is known that J_r is firmly nonexpansive, and $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$, and $M^{-1}(0) := \{x \in H : 0 \in Mx\}$.

Let $A : C \to H$ be a monotone mapping, and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2.2}$$

In this paper, we use $\mathrm{EP}(F, A)$ to denote the solution set of the generalized equilibrium problem (2.2).

Next, we give some special cases of the generalized equilibrium problem (2.2).

(I) If $F \equiv 0$, then problem (2.2) is reduced to the classical variational inequality (2.1).

(II) If $A \equiv 0$, the zero mapping, then problem (2.2) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \ge 0, \quad \forall y \in C. \tag{2.3}$$

In this paper, we use $\mathrm{EP}(F)$ to denote the solution set of the equilibrium problem (2.3).

To study the equilibrium problems, we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Recently, many authors have studied fixed point problems of nonexpansive mappings and solution problems of the equilibrium problems (2.2) and (2.3); for more details, see [16–25] and the references therein. In this paper, motivated and inspired by the research going on in this direction, we consider common element problems based on a mean iterative process. Strong convergence of the iterative process is obtained in the framework of real Hilbert spaces. The results presented in this paper improve and extend the corresponding results in Hao [1], Qin *et al.* [24], Chang *et al.* [25].

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [26] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2 [27] *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). *Then, for any* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : H \to C$ *as follows*:

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad x \in H,$$

then the following conclusions hold.
(1) T_r *is single-valued*;
(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = \mathrm{EP}(F)$;
(4) $\mathrm{EP}(F)$ *is closed and convex.*

Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I.
\end{aligned} \tag{2.4}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$; see [28] and the references therein.

Lemma 2.3 [28] *Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, where l is some real number, $\forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{\infty} F(S_i)$, for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists;*
(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{2.5}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Lemma 2.4 [29] *Let $B : C \to H$ be a mapping and let $M : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_r(I - rB)) = (B + M)^{-1}(0)$, where r is some positive real number.*

Lemma 2.5 [25] *Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$.

Lemma 2.6 [30] *Let $A : C \to H$ a Lipschitz monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$; that is, $N_C x = \{y \in H : \langle x - u, y\rangle, \forall u \in C\}$. Define*

$$Wx = \begin{cases} Ax + N_C x, & x \in C,\\ \emptyset & x \notin C.\end{cases}$$

Then W is maximal monotone and $0 \in Wx$ if and only if $x \in \mathrm{VI}(C, A)$.

Lemma 2.7 [31] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and*

$$\limsup_{n\to\infty}\bigl(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\bigr) \le 0.$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H and let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $M : H \rightrightarrows H$ be a maximal monotone operator. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $F := \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP}(F,B) \cap (A+M)^{-1}(0) \ne \emptyset$. Let $f : C \to C$ be a κ-contraction. Let $\{x_n\}$ be a sequence generated by the process: $x_1 \in C$ and*

$$\begin{cases} F(u_n,y) + \langle Bx_n, y-u_n\rangle + \frac{1}{s_n}\langle y-u_n, u_n-x_n\rangle \ge 0, & \forall y\in C,\\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n \operatorname{Proj}_C J_{r_n}(u_n - r_n A u_n), & \forall n\ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ is the sequence generated in (2.4), *$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n = 1$ and $\{r_n\}$ and $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions*:

(a) $0 < r \le r_n \le r' < 2\alpha$, $0 < r'' \le s_n \le r''' < 2\beta$;
(b) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(d) $\lim_{n\to\infty}|s_n - s_{n+1}| = \lim_{n\to\infty}|r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \operatorname{Proj}_F f(\bar{x})$.

Proof First, we show that $I - r_nA$ is nonexpansive. For $\forall x,y\in C$, we have

$$\begin{aligned} &\bigl\|(I-r_nA)x - (I-r_nA)y\bigr\|^2 \\ &\quad = \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax-Ay\|^2 \\ &\quad \le \|x-y\|^2 - 2r_n\alpha\|Ax-Ay\|^2 + r_n^2\|Ax-Ay\|^2 \\ &\quad = \|x-y\|^2 + r_n(r_n-2\alpha)\|Ax-Ay\|^2. \end{aligned}$$

Using restriction (a), we have $I - r_nA$ is nonexpansive, so is $I - s_nB$. Fix $x^* \in F$. It follows that $\|u_n - x^*\| \le \|(I-s_nB)x_n - (I-s_nB)x^*\| \le \|x_n - x^*\|$. Putting $y_n = J_{r_n}(u_n - r_nAu_n)$, one finds that $\|y_n - x^*\| \le \|u_n - x^*\| \le \|x_n - x^*\|$. It follows that

$$\begin{aligned} \|x_{n+1}-x^*\| &\le \alpha_n\|f(x_n)-x^*\| + \beta_n\|x_n-x^*\| + \gamma_n\|W_n\operatorname{Proj}_C y_n - x^*\| \\ &\le \alpha_n\|f(x_n)-f(x^*)\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| + \gamma_n\|y_n-x^*\| \\ &\le \bigl(1-\alpha_n(1-\kappa)\bigr)\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\|. \end{aligned}$$

Hence, we have $\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{\|f(x^*)-x^*\|}{1-\alpha}\}$. This yields the result that $\{x_n\}$ is bounded. Therefore, both $\{y_n\}$ and $\{u_n\}$ are also bounded. Next, without loss of generality, we assume that there exists a bounded set $K \subset C$ such that $x_n, y_n, u_n \in K$. Notice that $F(u_{n+1}, y) + \frac{1}{s_{n+1}}\langle y - u_{n+1}, u_{n+1} - (I - s_{n+1}B)x_{n+1}\rangle \geq 0$, $\forall y \in C$, and $F(u_n, y) + \frac{1}{s_n}\langle y - u_n, u_n - (I - s_nB)x_n\rangle \geq 0$, $\forall y \in C$. It follows that

$$\left\langle u_{n+1} - u_n, \frac{u_n - (I - s_nB)x_n}{s_n} - \frac{u_{n+1} - (I - s_{n+1}B)x_{n+1}}{s_{n+1}}\right\rangle \geq 0.$$

Hence, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 \leq{}& \Big\langle u_{n+1} - u_n, (I - s_{n+1}B)x_{n+1} - (I - s_nB)x_n \\ &+ \left(1 - \frac{s_n}{s_{n+1}}\right)\big(u_{n+1} - (I - s_{n+1}B)x_{n+1}\big)\Big\rangle \\ \leq{}& \|u_{n+1} - u_n\|\bigg(\big\|(I - s_{n+1}B)x_{n+1} - (I - s_nB)x_n\big\| \\ &+ \left|1 - \frac{s_n}{s_{n+1}}\right|\big\|u_{n+1} - (I - s_{n+1}B)x_{n+1}\big\|\bigg).\end{aligned}$$

This yields the result that

$$\begin{aligned}\|u_{n+1} - u_n\| \leq{}& \big\|(I - s_{n+1}B)x_{n+1} - (I - s_nB)x_n\big\| \\ &+ \frac{|s_{n+1} - s_n|}{s_{n+1}}\big\|u_{n+1} - (I - s_{n+1}B)x_{n+1}\big\| \\ ={}& \big\|(I - s_{n+1}B)x_{n+1} - (I - s_{n+1}B)x_n + (I - s_{n+1}B)x_n - (I - s_nB)x_n\big\| \\ &+ \frac{|s_{n+1} - s_n|}{s_{n+1}}\big\|u_{n+1} - (I - s_{n+1}B)x_{n+1}\big\| \\ \leq{}& \|x_{n+1} - x_n\| + |s_{n+1} - s_n|M_1,\end{aligned} \tag{3.1}$$

where M_1 is an appropriate constant such that

$$M_1 = \sup_{n\geq1}\left\{\|Bx_n\| + \frac{\|u_{n+1} - (I - s_{n+1}B)x_{n+1}\|}{\bar{a}}\right\}.$$

Since J_{r_n} is firmly nonexpansive, one sees that

$$\begin{aligned}&\|y_{n+1} - y_n\| \\ &\quad= \big\|J_{r_n}(u_{n+1} - r_{n+1}Au_{n+1}) - J_{r_n}(u_n - r_nAu_n)\big\| \\ &\quad\leq \big\|u_{n+1} - r_{n+1}Au_{n+1} - (u_n - r_nAu_n)\big\| \\ &\quad= \big\|(I - r_{n+1}A)u_{n+1} - (I - r_{n+1}A)u_n + (r_n - r_{n+1})Au_n\big\| \\ &\quad\leq \|u_{n+1} - u_n\| + |r_n - r_{n+1}|\|Au_n\|.\end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), one finds that

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \big(|s_{n+1} - s_n| + |r_n - r_{n+1}|\big)M_2, \tag{3.3}$$

where M_2 is an appropriate constant such that $M_2 = \max\{\sup_{n\geq 1}\{\|Au_n\|\}, M_1\}$. On the other hand, one has

$$\begin{aligned}
&\|W_{n+1}\operatorname{Proj}_C y_{n+1} - W_n \operatorname{Proj}_C y_n\| \\
&\quad\leq \|W_{n+1}y_{n+1} - W_n y_n\| \\
&\quad\leq \|W_{n+1}y_{n+1} - Wy_{n+1}\| + \|Wy_{n+1} - Wy_n\| + \|Wy_n - W_n y_n\| \\
&\quad\leq \sup_{x\in K}\bigl\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\bigr\} + \|y_{n+1} - y_n\|, \qquad (3.4)
\end{aligned}$$

where K is the bounded subset of C defined above. Combining (3.3) with (3.4), one finds

$$\begin{aligned}
&\|W_{n+1}\operatorname{Proj}_C y_{n+1} - W_n \operatorname{Proj}_C y_n\| \\
&\quad\leq \sup_{x\in K}\bigl\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\bigr\} + \|x_{n+1} - x_n\| \\
&\qquad + \bigl(|r_{n+1} - r_n| + |s_n - s_{n+1}|\bigr)M_2. \qquad (3.5)
\end{aligned}$$

Letting $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ we see that

$$\begin{aligned}
\|z_{n+1} - z_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\bigl\|f(x_{n+1}) - W_{n+1}y_{n+1}\bigr\| + \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - W_n y_n\bigr\| \\
&+ \|W_{n+1}\operatorname{Proj}_C y_{n+1} - W_n \operatorname{Proj}_C y_n\|. \qquad (3.6)
\end{aligned}$$

Substituting (3.5) into (3.6), we see that

$$\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
&\quad\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\bigl\|f(x_{n+1}) - W_{n+1}y_{n+1}\bigr\| + \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - W_n y_n\bigr\| \\
&\qquad + \sup_{x\in K}\bigl\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\bigr\} \\
&\qquad + \bigl(|r_{n+1} - r_n| + |s_n - s_{n+1}|\bigr)M_2.
\end{aligned}$$

It follows from restrictions (a), (c), and (d) that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Using Lemma 2.7, we find that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.7)$$

For any $x^* \in F$, we see that

$$\begin{aligned}
\bigl\|x_{n+1} - x^*\bigr\|^2 &\leq \alpha_n\bigl\|f(x_n) - x^*\bigr\|^2 + \beta_n\bigl\|x_n - x^*\bigr\|^2 + \gamma_n\bigl\|W_n \operatorname{Proj}_C y_n - x^*\bigr\|^2 \\
&\leq \alpha_n\bigl\|f(x_n) - x^*\bigr\|^2 + \beta_n\bigl\|x_n - x^*\bigr\|^2 + \gamma_n\bigl\|y_n - x^*\bigr\|^2. \qquad (3.8)
\end{aligned}$$

Since

$$\begin{aligned}
\bigl\|y_n - x^*\bigr\|^2 &= \bigl\|J_{r_n}(u_n - r_n A u_n) - x^*\bigr\|^2 \\
&\leq \bigl\|(I - r_n A)u_n - (I - r_n A)x^*\bigr\|^2
\end{aligned}$$

$$\begin{aligned}&= \|u_n - x^*\|^2 - 2r_n\langle u_n - x^*, Au_n - Ax^*\rangle + r_n^2\|Au_n - Ax^*\|^2 \\ &\leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Au_n - Ax^*\|^2,\end{aligned}$$

we find from (3.8) that

$$\lim_{n\to\infty}\|Au_n - Ax^*\| = 0. \tag{3.9}$$

It also follows from (3.8) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^* - s_n(Bx_n - Bx^*)\|^2 \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + \gamma_n\big(\|x_n - x^*\|^2 + s_n^2\|Bx_n - Bx^*\|^2 - 2s_n\langle Bx_n - Bx^*, x_n - x^*\rangle\big) \\ &\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\ &\quad - s_n\gamma_n(2\beta - s_n)\|Bx_n - Bx^*\|^2.\end{aligned}$$

Using (3.7), one arrives at

$$\lim_{n\to\infty}\|Bx_n - Bx^*\| = 0. \tag{3.10}$$

Since T_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|u_n - x^*\|^2 &\leq \langle (I - s_nB)x_n - (I - s_nB)x^*, u_n - x^*\rangle \\ &\leq \frac{1}{2}\big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - s_n^2\|Bx_n - Bx^*\|^2 \\ &\quad + 2s_n\langle Bx_n - Bx^*, x_n - u_n\rangle\big),\end{aligned}$$

which implies that $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2s_n\|Bx_n - Bx^*\|\|x_n - u_n\|$. Hence

$$\begin{aligned}\gamma_n\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + \big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big)\|x_n - x_{n+1}\| \\ &\quad + 2s_n\|Bx_n - Bx^*\|\|x_n - u_n\|.\end{aligned}$$

Using (3.7) and (3.10), one has

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.11}$$

Similarly, one also has

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.12}$$

Since

$$\|W_ny_n - y_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| + \|x_n - W_ny_n\|,$$

we find from (3.11) and (3.12) that

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{3.13}$$

Now, we are in a position to show $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - z\rangle \le 0$, where $\bar{x} = \mathrm{Proj}_F f(\bar{x})$. To prove this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\bigr\rangle = \lim_{i\to\infty}\bigl\langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\bigr\rangle. \tag{3.14}$$

Since $\{x_{n_i}\}$ is bounded, without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$. Using (3.11) and (3.12), we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore, we see that $y_{n_i} \rightharpoonup q$. Now, we are in a position to prove $q \in (A+M)^{-1}(0)$. Notice that $\frac{u_n - y_n}{r_n} - Au_n \in My_n$. Let $\mu \in M\nu$. Since M is monotone, we find that $\langle \frac{u_n-y_n}{r_n} - Au_n - \mu, y_n - \nu\rangle \ge 0$. This implies that $\langle -Aq - \mu, q - \nu\rangle \ge 0$. This implies that $-Aq \in Mq$, that is, $q \in (A+M)^{-1}(0)$. Next, we show that $q \in \mathrm{EP}(F, B)$. Since $u_n = T_{s_n}(I - s_n B)x_n$, we find from (A2) that

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{s_{n_i}}\right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C. \tag{3.15}$$

Putting $y_t = ty + (1-t)q$ for any $t \in (0,1]$ and $y \in C$, we see that $y_t \in C$. Using (3.15), we find that

$$\begin{aligned}
&\langle y_t - u_{n_i}, By_t\rangle \\
&\quad \ge \langle y_t - u_{n_i}, By_t\rangle - \langle Bx_{n_i}, y_t - u_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{s_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\
&\quad = \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{s_{n_i}}\right\rangle \\
&\qquad + F(y_t, u_{n_i}).
\end{aligned}$$

Since B is monotone, we obtain from (A4) that $\langle y_t - w, By_t\rangle \ge F(y_t, w)$. Using (A1) and (A4), we find that

$$\begin{aligned}
0 &= F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, w) \\
&\le tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle \\
&= tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle.
\end{aligned}$$

Hence, $0 \le F(y_t, y) + (1-t)\langle y - w, By_t\rangle$, $\forall y \in C$. It follows from (A3) that $w \in \mathrm{EP}(F, B)$. Next, we prove that $q \in \bigcap_{i=1}^\infty F(S_i)$. Suppose to the contrary, $q \notin \bigcap_{i=1}^\infty F(S_i)$, *i.e.*, $Wq \ne q$. Since $y_{n_i} \rightharpoonup q$ and the space satisfies Opial's condition, one has

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i} - q\| &< \liminf_{i\to\infty}\|y_{n_i} - Wq\| \\
&\le \liminf_{i\to\infty}\bigl\{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Wq\|\bigr\} \\
&\le \liminf_{i\to\infty}\bigl\{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - q\|\bigr\}.
\end{aligned} \tag{3.16}$$

Since $\|Wy_n - y_n\| \leq \sup_{x\in K}\|Wx - W_nx\| + \|W_ny_n - y_n\|$, we find from Lemma 2.5 that $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. It follows that $\liminf_{i\to\infty}\|y_{n_i} - q\| < \liminf_{i\to\infty}\|y_{n_i} - q\|$. This leads to a contradiction. Thus, we have $q \in \bigcap_{i=1}^{\infty} F(S_i)$. This proves that $q \in F$. Therefore, one has

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0.$$

Finally, we show that $x_n \to \bar{x}$, as $n \to \infty$. Note that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&\quad\leq \alpha_n\langle f(x_n) - f(\bar{x}), x_{n+1} - \bar{x}\rangle + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\qquad + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n\|y_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\quad\leq \frac{\kappa}{2}\alpha_n\big(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\big) + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\qquad + (1 - \alpha_n)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\quad\leq \frac{1 - \alpha_n(1-\kappa)}{2}\|x_n - \bar{x}\|^2 + \frac{1}{2}\|x_{n+1} - \bar{x}\|^2 + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle.$$

Using Lemma 2.1, we find that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof. □

4 Applications

Recall that a mapping $T : C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2$$

for all $x, y \in C$. Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings. Put $A = I - T$, where $T : C \to C$ is a k-strict pseudo-contraction. Then A is $\frac{1-k}{2}$-inverse-strongly monotone. Now, we are in a position to state a results on fixed points of strict pseudo-contractions.

Theorem 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $T : C \to H$ be a k-strict pseudo-contraction, $B : C \to H$ be a β-inverse-strongly monotone mapping, and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $F := \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP}(F,B) \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a κ-contraction. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases}
F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{s_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = (1 - r_n)u_n + r_nTu_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1,
\end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence generated in* (2.4), $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{r_n\}$, *and* $\{s_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*:

(a) $0 < r \leq s_n \leq r' < 2\beta$, $0 < r'' \leq r_n \leq r''' < 1 - k$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} = \mathrm{Proj}_F f(\bar{x})$.

Proof Taking $A = I - T$, wee see that $A : C \to H$ is a α-strict pseudo-contraction with $\alpha = \frac{1-k}{2}$ and $F(T) = \mathrm{VI}(C, A)$. Using Theorem 3.1, we find the desired conclusion immediately. □

Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂g of g is defined as follows:

$$\partial fg(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle, z \in H\}, \quad \forall x \in H.$$

From Rockafellar [30], we know that ∂g is maximal monotone. It is not hard to verify that $0 \in \partial g(x)$ if and only if $g(x) = \min_{y\in H} g(y)$.

Let I_C be the indicator function of C, *i.e.*,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semi-continuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clear that $J_r x = P_C x$, $\forall x \in H$. Notice that $(A + \partial I_C)^{-1}(0) = \mathrm{VI}(C, A_1)$. Now, we are in a position to state the result on variational inequalities.

Theorem 4.2 *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping,* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping, and* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings. Assume that* $F := \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP}(F, B) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Let* $f : C \to C$ *be a* κ*-contraction. Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{s_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(u_n - s_n A u_n), & \forall n \geq 1, \end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence generated in* (2.4), $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{r_n\}$, *and* $\{s_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*:

(a) $0 < r \leq s_n \leq r' < 2\beta$, $0 < r'' \leq r_n \leq r''' < 2\alpha$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} = \mathrm{Proj}_F f(\bar{x})$.

Competing interests

The author declares to have no competing interests.

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