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Convergence theorems for common solutions of various problems with nonlinear mapping

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Abstract

In this paper, motivated and inspired by Zegeye and Shahzad (Nonlinear Anal. 70:2707-2716, 2009), Qin *et al.* (J. Comput. Appl. Math. 225(1):20-30, 2009) and Kimura and Takahashi (J. Math. Anal. Appl. 357:356-363, 2009), we introduce a new hybrid projection iterative scheme that converges strongly to a common element of the solution set of a generalized mixed equilibrium problem, the solution set of a variational inequality problem, and the set of common fixed points for a family of hemi-relatively nonexpansive mappings in a Banach space.

MSC: 47H10; 47J20; 49J40

Keywords: common fixed point; generalized mixed equilibrium problem; generalized projection; hemi-relatively nonexpansive; variational inequality problem

1 Introduction

A real Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* denote the dual space of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued, and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on a bounded subset of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective

and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [1]). We note that J is the identity mapping in a Hilbert space.

A mapping $A : D(A) \subset E \to E^*$ is said to be *monotone* if for each $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle \geq 0.$$

A mapping A is said to be γ-*inverse strongly monotone* if there exists a positive real number $\gamma > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \quad \forall x, y \in D(A).$$

If a mapping A is γ-inverse strongly monotone, then it is Lipschitz continuous with constant $\frac{1}{\gamma}$, *i.e.*,

$$\|Ax - Ay\| \leq \frac{1}{\gamma} \|x - y\|, \quad x, y \in D(A).$$

A mapping A is said to be *strongly monotone*, if for each $x, y \in D(A)$, there exists $k \in (0, 1)$ such that

$$\langle x - y, Ax - Ay \rangle \geq k \|x - y\|^2.$$

A monotone mapping A is said to be *maximal* if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone mapping. It is known that the monotone mapping A is maximal if and only if for $(x, x^*) \in E \times E^*$,

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for every $(y, y^*) \in G(A)$ implies that $x^* \in Ax$.

Let C be a nonempty, closed convex subset of a Banach space E. For a bifunction $\theta : C \times C \to \mathbb{R}$, we assume that θ satisfies the following conditions:

(E1) $\theta(x, x) = 0$ for all $x \in C$;

(E2) θ is monotone, *i.e.*, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(E3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$

(E4) for each $x \in C$, the function $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

The generalized mixed equilibrium problem is to find $x \in C$ such that

$$\theta(x, y) + \psi(y) - \psi(x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where ψ is a lower semicontinuous and convex function. The set of solutions of problem (1.1) is denoted by *GMEP*. Recently, Zhang [2] considered this problem. Some special cases of problem (1.1) are stated as follows.

If $A = 0$, then problem (1.1) reduces to the following mixed equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [3].

If $\psi = 0$, then problem (1.1) reduces to the following generalized equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C,$$

which was studied in [4].

If $\psi = 0$ and $A = 0$, then problem (1.1) reduces to the following equilibrium problem of finding $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of problem (1.2) is denoted by *EP*.

If $\theta = 0$ and $\psi = 0$, then problem (1.1) reduces to the following classical variational inequality problem of finding $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of problem (1.3) is denoted by *VI(C, A)*.

Equilibrium problems, which were introduced in [5] in 1994, have had great impact and influence on the development of several branches of pure and applied sciences. They include numerous problems in economics, finance, physics, network, elasticity, optimization, variational inequalities, minimax problems, and semigroups; see, for instance, [2–4, 6–17] and the references therein.

As well known, if C is a nonempty, closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not true to more general Banach spaces. In this connection, Alber [18] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) reduces to

$$\phi(x, y) = \|x - y\|^2 \quad \text{for } x, y \in H.$$

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns an arbitrary point $x \in E$ to the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x).$$

The existence and uniqueness of the mapping Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [18] and [19]). In a Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that:

(1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$.

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$.

(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \|Jx - Jy\| + \|y - x\| \|y\|$ for all $x, y \in E$.

(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y.$$

Remark 1.1 In (4), it is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. In fact, from (1) we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$. For more details, see [1].

Let C be a nonempty closed and convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ be the set of fixed points of T. A point p in C is said to be a *weak asymptotic fixed point* of T [20] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \to \infty} (Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* [21–23] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [21, 22] and [12].

A point p in C is said to be a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n \to \infty} (Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T is denoted by $\tilde{F}(T)$. A mapping T from C into itself is called *relatively weak nonexpansive* if $\tilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T is called *hemi-relatively nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.2 (1) It is obvious that a relatively nonexpansive mapping is a relatively weak nonexpansive mapping (see [14]). In fact, for any mapping $T : C \to C$, we have $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \tilde{F}(T) = \hat{F}(T)$.

(2) The class of hemi-relatively nonexpansive mappings is more general than the class of relatively weak nonexpansive mappings.

The converse of Remark 1.2 is not true. In order to explain this better, we give the following example.

Example 1.1 ([24]) Let E be any smooth Banach space, and let $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ as follows:

$$Tx = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0 & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0 \end{cases}$$

for $n = 1, 2, 3, \ldots$. Then T is a hemi-relatively nonexpansive mapping but not a relatively nonexpansive mapping.

Remark 1.3 There are other examples of hemi-relatively nonexpansive mappings such as the generalized projections (or projections) from a smooth, strictly convex and reflexive Banach space, and others; see [25].

A mapping $T : C \to C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

In 2009, Kimura and Takahashi [26] proposed the following hybrid iteration method with a generalized projection for a family of relatively nonexpansive mappings $\{T_\lambda\}$ in a Banach space E:

$$\begin{cases} x_1 = x \in C, \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT_\lambda x_n) \quad \text{for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup_{\lambda \in \Lambda} \phi(z, y_n(\lambda)) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x. \end{cases}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x \in C$, where $\mathcal{F} = \bigcap_{\lambda \in \Lambda} F(T_\lambda)$ is the set of common fixed points of T_λ, and Π_K is the generalized projection of E onto a nonempty closed convex subset K of E.

Recently, Zegeye and Shahzad [14] introduced the following iterative scheme for finding a common element of the solution set of a variational inequality problem and a fixed point of a relatively weak nonexpansive mapping with γ-inverse strongly monotone mapping satisfying $\|Ax\| \le \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$ (see, *e.g.*, [27]):

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily,} \\ y_n = \Pi_C(J^{-1}(Jx_n - \alpha_n Ax_n)), \\ z_n = Ty_n, \\ H_0 = \{v \in C : \phi(v, z_0) \le \phi(v, y_0) \le \phi(v, x_0)\}, \\ W_0 = C, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \ge 1. \end{cases}$$

On the other hand, Qin *et al.* [25] proposed the following hybrid iterative scheme:

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $T : C \to C$ is a closed hemi-relatively nonexpansive mapping. Under suitable conditions, they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap EP(f)} x_0$, where $EP(f)$ is the solution of an equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$.

In this paper, we introduce a new hybrid projection iterative scheme that converges strongly to a common element of the solution set of a generalized mixed equilibrium problem, the solution set of a variational inequality problem, and the set of common fixed points for a family of hemi-relatively nonexpansive mappings in a Banach space.

2 Preliminaries

Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, $\forall \tau > 0$ and E is called *uniformly smooth* if and only if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \varepsilon = \|x - y\| \right\}.$$

E is called *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. Let $p > 1$. Then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c \cdot \varepsilon^p$ for all $\varepsilon \in [0, 2]$. Observe that every p-uniformly convex space is uniformly convex.

It is well known (see, for example, [28]) that

$$L_p(l_p) \quad \text{or} \quad W_m^p \quad \text{is} \quad \begin{cases} p\text{-uniformly convex} & \text{if } p \geq 2; \\ 2\text{-uniformly convex} & \text{if } 1 < p \leq 2. \end{cases}$$

In the following, we shall need the following results.

Lemma 2.1 ([28]) *Let E be a* 2-*uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where J is the normalized duality mapping of E, and $\frac{1}{c}$ ($0 < c \leq 1$) is the 2-*uniformly convex constant of E.*

Lemma 2.2 ([18, 29]) *Let E be a real smooth, strictly convex and reflexive Banach space, and let C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(i) $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$, $\forall x \in E$, $y \in C$.

(ii) *Suppose $x \in E$ and $z \in C$. Then*

$$z = \Pi_C x \quad \Leftrightarrow \quad \langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 ([30]) *Let E be a strictly convex and smooth Banach space, C be a nonempty closed and convex subset of E and $T : C \to C$ be a hemi-relatively nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.4 ([29]) *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 ([31]) *Let E be a real smooth Banach space and $A : E \to E^*$ be a maximal monotone mapping. Then $A^{-1}(0)$ is a closed and convex subset of E.*

We denote the normal cone for C at a point $v \in C$ by $N_C(v)$, that is,

$$N_C(v) := \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}.$$

Lemma 2.6 ([32]) *Let C be a nonempty closed convex subset of a Banach space E, and let A be a monotone and hemicontinuous mapping of C into E^* with $C = D(A)$. Let $T \subset E \times E^*$ be a mapping defined as follows*:

$$Tv := \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}(0) = VI(C, A)$.

Remark 2.1 It is well known that the monotone and hemicontinuous mapping A with $D(A) = E$ is maximal (see, *e.g.*, [1]). Note that Lemma 2.6 is for the monotone and hemi-continuous mapping.

Remark 2.2 Let C be a nonempty closed convex subset of a Banach space E, and let A be a monotone and hemicontinuous mapping from C into E^* with $C = D(A)$. Then

$$VI(C, A) = \{u \in C : \langle v - u, Av \rangle \geq 0 \text{ for all } v \in C\}.$$

It is obvious that the set $VI(C, A)$ is a closed convex subset of C and the set $A^{-1}0 = VI(E, A)$ is a closed convex subset of E (see [27]).

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$$

for all $x \in E$ and $x^* \in E^*$, which was studied by Alber [18]. That is,

$$V(x, x^*) = \phi(x, J^{-1}x^*)$$

for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.7 ([18]) *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.8 ([2]) *Let C be a closed subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (E1)-(E4). *Then, for $r > 0$ and $x \in E$, there exists $u \in C$ such that*

$$\theta(u, y) + \langle Bu, y - u \rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : E \to C$ by

$$T_r(x) = \left\{ u \in C : \theta(u, y) + \langle Bu, y - u \rangle + \psi(y) - \psi(u) \right.$$
$$\left. + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C \right\} \tag{2.1}$$

for all $x \in E$. Then the following conclusions hold:

  (i) T_r is single-valued;
 (ii) T_r is a firmly nonexpansive type mapping [33], *i.e.*, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(iii) $F(T_r) = GMEP = \hat{F}(T_r)$;
 (iv) $GMEP$ is a closed and convex subset of C;
  (v) $\phi(p, T_r z) + \phi(T_r z, z) \leq \phi(p, z), \forall p \in F(T_r), x \in E$.

Remark 2.3 ([2]) *The mapping $T_r : E \to C$ defined by* (2.1) *is a relatively nonexpansive mapping. Thus, it is a hemi-relatively nonexpansive mapping.*

3 An iterative scheme for a family of hemi-relatively nonexpansive mappings

In this section, we introduce a new hybrid iterative scheme for a common element of the solution set of problem (1.1), the solution set of problem (1.3) for an inverse strongly monotone mapping and the set of common fixed points of a family of hemi-relatively nonexpansive mappings.

Theorem 3.1 *Let E be a real uniformly smooth and 2-uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a γ-inverse strongly monotone mapping and $B : C \to E^*$ be a continuous and monotone mapping. Let $\psi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (E1)-(E4). *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of closed hemi-relatively nonexpansive mappings of C into itself having*

$$\Omega = \mathcal{F} \cap VI(C, A) \cap GMEP \neq \emptyset,$$

where $\mathcal{F} = \bigcap_{\lambda \in \Lambda} F(T_\lambda)$ *is the set of common fixed points of* $\{T_\lambda\}$. *Assume that* $\|Ax\| \leq \|Ax - Ap\|$ *for all* $x \in C$ *and* $p \in VI(C, A)$. *Suppose that* $0 < a < \mu_n < b = \frac{c^2 \gamma}{2}$, *where c is the constant in Lemma* 2.1. *Let* $\{r_n\} \subset [c^*, +\infty)$ *for some* $c^* > 0$. *Let* $\{x_n\}$ *be the sequence generated by*

$$
\begin{cases}
x_0 \in C, \quad \text{chosen arbitrarily,} \\
y_n = \Pi_C J^{-1}(Jx_n - \mu_n Ax_n), \\
z_{n,\lambda} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_\lambda y_n), \\
u_{n,\lambda} = T_{r_n,\lambda} z_{n,\lambda}, \\
H_{0,\lambda} = \{v \in C : \sup_{\lambda \in \Lambda} \phi(v, u_{0,\lambda}) \\
\qquad \leq \alpha_0 \phi(v, x_0) + (1 - \alpha_0)\phi(v, y_0) \leq \phi(v, x_0)\}, \\
H_{n,\lambda} = \{v \in H_{n-1,\lambda} \cap W_{n-1,\lambda} : \sup_{\lambda \in \Lambda} \phi(v, u_{n,\lambda}) \\
\qquad \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, y_n) \leq \phi(v, x_n)\}, \\
W_{0,\lambda} = C, \\
W_{n,\lambda} = \{v \in H_{n-1,\lambda} \cap W_{n-1,\lambda} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_{n,\lambda} \cap W_{n,\lambda}} x_0, \quad n \geq 0, \lambda \in \Lambda,
\end{cases}
\tag{3.1}
$$

where J is the normalized duality mapping, and $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *satisfying* $\liminf_{n \to \infty} (1 - \alpha_n) > 0$. *Then* $\{x_n\}$ *converges to* $\Pi_\Omega x_0$, *where* Π_Ω *is the generalized projection of E onto* Ω.

Proof Step 1. We prove that $H_{n,\lambda}$ and $W_{n,\lambda}$ both are closed and convex subsets of C and $\Omega \subset H_{n,\lambda} \cap W_{n,\lambda}$ with $n \geq 0$, $\lambda \in \Lambda$. In fact, it is obvious that $W_{n,\lambda}$ is closed and convex, and $H_{n,\lambda}$ is closed for each $n \geq 0$, $\lambda \in \Lambda$. Since

$$
\phi(v, u_{n,\lambda}) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, y_n)
$$
$$
\Leftrightarrow \quad 2\langle v, \alpha_n Jx_n + (1 - \alpha_n)Jy_n - Ju_{n,\lambda} \rangle \leq (1 - \alpha_n)\|y_n\|^2 - \|u_{n,\lambda}\|^2 + \alpha_n \|x_n\|^2
$$

and

$$
\alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, y_n) \leq \phi(v, x_n)
$$
$$
\Leftrightarrow \quad \phi(v, y_n) \leq \phi(v, x_n),
$$

$H_{n,\lambda}$ is convex for each $n \geq 0$, $\lambda \in \Lambda$. Hence $H_{n,\lambda} \cap W_{n,\lambda}$ is closed and convex for all $n \geq 0$, $\lambda \in \Lambda$.

Step 2. For any given $p \in \Omega$, from (v) of Lemma 2.8 and that T_λ is hemi-relatively non-expansive, we have

$$
\begin{aligned}
\phi(p, u_{0,\lambda}) &= \phi(p, T_{r_0,\lambda} z_{0,\lambda}) \\
&\leq \phi(p, z_{0,\lambda}) = \phi\big(p, J^{-1}(\alpha_0 Jx_0 + (1 - \alpha_0)JT_\lambda y_0)\big) \\
&= \|p\|^2 - 2\langle p, \alpha_0 Jx_0 + (1 - \alpha_0)JT_\lambda y_0 \rangle \\
&\quad + \big\|\alpha_0 Jx_0 + (1 - \alpha_0)JT_\lambda y_0\big\|^2 \\
&\leq \|p\|^2 - 2\alpha_0 \langle p, Jx_0 \rangle - 2(1 - \alpha_0)\langle p, JT_\lambda y_0 \rangle
\end{aligned}
$$

$$+ \alpha_0 \|x_0\|^2 + (1 - \alpha_0)\|T_\lambda y_0\|^2$$

$$= \alpha_0 \phi(p, x_0) + (1 - \alpha_0)\phi(p, T_\lambda y_0)$$

$$\leq \alpha_0 \phi(p, x_0) + (1 - \alpha_0)\phi(p, y_0) \tag{3.2}$$

for each $\lambda \in \Lambda$. From Lemma 2.1, Lemma 2.7, and the assumption of A, we obtain

$$\begin{aligned}
\phi(p, y_0) &= \phi\big(p, \Pi_C J^{-1}(Jx_0 - \mu_0 Ax_0)\big) \\
&\leq \phi\big(p, J^{-1}(Jx_0 - \mu_0 Ax_0)\big) \\
&= V(p, Jx_0 - \mu_0 Ax_0) \\
&\leq V(p, Jx_0 - \mu_0 Ax_0 + \mu_0 Ax_0) \\
&\quad - 2\big\langle J^{-1}(Jx_0 - \mu_0 Ax_0) - p, \mu_0 Ax_0 \big\rangle \\
&= V(p, Jx_0) - 2\mu_0\big\langle J^{-1}(Jx_0 - \mu_0 Ax_0) - J^{-1}(Jx_0), Ax_0 \big\rangle \\
&\quad - 2\mu_0\langle x_0 - p, Ax_0 \rangle \\
&\leq V(p, Jx_0) - 2\mu_0\big\langle J^{-1}(Jx_0 - \mu_0 Ax_0) - J^{-1}(Jx_0), Ax_0 \big\rangle \\
&\quad - 2\mu_0\langle x_0 - p, Ax_0 - Ap \rangle - 2\mu_0\langle x_0 - p, Ap \rangle \\
&\leq \phi(p, x_0) + \frac{4\mu_0^2}{c^2}\|Ax_0\|^2 - 2\mu_0 \gamma\|Ax_0 - Ap\|^2 \\
&\leq \phi(p, x_0) + 2\mu_0\left(\frac{2\mu_0}{c^2} - \gamma\right)\|Ax_0 - Ap\|^2 \\
&\leq \phi(p, x_0). \tag{3.3}
\end{aligned}$$

From (3.2) and (3.3),

$$\phi(p, u_{0,\lambda}) \leq \alpha_0 \phi(p, x_0) + (1 - \alpha_0)\phi(p, y_0) \leq \phi(p, x_0) \tag{3.4}$$

for each $\lambda \in \Lambda$. Thus

$$\sup_{\lambda \in \Lambda} \phi(p, u_{0,\lambda}) \leq \alpha_0 \phi(p, x_0) + (1 - \alpha_0)\phi(p, y_0) \leq \phi(p, x_0).$$

Therefore, $p \in H_{0,\lambda}$ and $p \in H_{0,\lambda} \cap W_{0,\lambda}$. Suppose that $\Omega \subset H_{n-1,\lambda} \cap W_{n-1,\lambda}$. Then, the methods in (3.2) and (3.3) imply that

$$\phi(p, u_{n,\lambda}) \leq \alpha_n \phi(p, x_n) + (1 - \alpha_n)\phi(p, y_n) \leq \phi(p, x_n), \quad \forall \lambda \in \Lambda,$$

which implies that $p \in H_{n,\lambda}$. Since $x_n = \Pi_{H_{n-1,\lambda} \cap W_{n-1,\lambda}} x_0$, it follows from Lemma 2.2 that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in H_{n-1,\lambda} \cap W_{n-1,\lambda}.$$

It implies that $\langle x_n - p, Jx_0 - Jx_n \rangle \geq 0$. Hence $p \in W_{n,\lambda}$. Therefore, $\Omega \subset H_{n,\lambda} \cap W_{n,\lambda}$. Then, by induction on n, $\Omega \subset H_{n,\lambda} \cap W_{n,\lambda}$ for all $n \geq 0$, $\lambda \in \Lambda$. From Lemma 2.5 and Lemma 2.6, we know that $VI(C, A)$ is closed and convex set. Therefore, Ω is closed and convex. The sequence $\{x_n\}$ generated by (3.1) is well defined.

Step 3. We prove that $\{x_n\}$ is a Cauchy sequence. Let $p \in \Omega$. From the definition of $H_{n,\lambda}$, $W_{n,\lambda}$ and Lemma 2.2, we have $x_n = \Pi_{H_{n-1,\lambda} \cap W_{n-1,\lambda}} x_0$ and

$$\phi(p, x_n) + \phi(x_n, x_0) \leq \phi(p, x_0).$$

Thus $\{x_n\}$ is bounded. Moreover, since

$$x_n = \Pi_{H_{n-1,\lambda} \cap W_{n-1,\lambda}} x_0, \qquad x_{n+1} = \Pi_{H_{n,\lambda} \cap W_{n,\lambda}} x_0 \in H_{n,\lambda} \cap W_{n,\lambda},$$

we have

$$\phi(x_{n+1}, x_n) + \phi(x_n, x_0) \leq \phi(x_{n+1}, x_0),$$

which implies that $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction $H_{n,\lambda} \cap W_{n,\lambda}$, one has that

$$H_{m,\lambda} \cap W_{m,\lambda} \subset H_{m-1,\lambda} \cap W_{m-1,\lambda},$$

$$x_m = \Pi_{H_{m,\lambda} \cap W_{m,\lambda}} x_0 \in H_{n,\lambda} \cap W_{n,\lambda}$$

for any positive integer $m \geq n$. From Lemma 2.2, we have

$$
\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{H_{n-1,\lambda} \cap W_{n-1,\lambda}} x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{H_{n,\lambda} \cap W_{n,\lambda}} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned}
\tag{3.5}
$$

Letting $m, n \to \infty$ in (3.5), we have

$$\phi(x_m, x_n) \to 0.$$

Thus, Lemma 2.4 implies that

$$\lim_{m,n \to \infty} \|x_m - x_n\| = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence.

Step 4. Now, we prove that $\lim_{n \to \infty} \|y_n - T_\lambda y_n\| = 0$ for each $\lambda \in \Lambda$ and $q \in \mathcal{F} = \bigcap_{\lambda \in \Lambda} F(T_\lambda)$. Since $x_{n+1} \in H_{n,\lambda}$, we obtain

$$\phi(x_{n+1}, u_{n,\lambda}) \leq \phi(x_{n+1}, x_n)$$

for all $\lambda \in \Lambda$ and

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

By (3.5) and Lemma 2.4, we have

$$\lim_{n \to \infty} \|x_{n+1} - u_{n,\lambda}\| = \lim_{n \to \infty} \|x_{n+1} - y_n\| = 0, \quad \forall \lambda \in \Lambda.$$

Hence,

$$\lim_{n\to\infty} \|x_n - u_{n,\lambda}\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|u_{n,\lambda} - y_n\| = 0 \tag{3.6}$$

for all $\lambda \in \Lambda$. By the methods in (3.2) and (3.4), we have

$$\phi(p, u_{n,\lambda}) \le \phi(p, z_{n,\lambda}) \le \phi(p, x_n) \tag{3.7}$$

for all $n \ge 0$, $\lambda \in \Lambda$. Since J is uniformly continuous on the bounded sets, it follows from Lemma 2.8(v), (3.2) and (3.7) that for any given $p \in \Omega$,

$$\begin{aligned}
\phi(u_{n,\lambda}, z_{n,\lambda}) &= \phi(T_{r_n,\lambda} z_{n,\lambda}, z_{n,\lambda}) \\
&\le \phi(p, z_{n,\lambda}) - \phi(p, T_{r_n,\lambda} z_{n,\lambda}) \\
&\le \phi(p, x_n) - \phi(p, u_{n,\lambda}) \\
&= \|x_n\|^2 - \|u_{n,\lambda}\|^2 - 2\langle p, Jx_n - Ju_{n,\lambda}\rangle \\
&\le \big(\|x_n\| - \|u_{n,\lambda}\|\big)\big(\|x_n\| + \|u_{n,\lambda}\|\big) + 2\|p\|\|Jx_n - Ju_{n,\lambda}\| \\
&\to 0, \quad \text{as } n \to \infty
\end{aligned}$$

for all $\lambda \in \Lambda$. From Lemma 2.4,

$$\lim_{n\to\infty} \|u_{n,\lambda} - z_{n,\lambda}\| = 0. \tag{3.8}$$

Thus $\lim_{n\to\infty} \|z_{n,\lambda} - x_n\| = 0$ for each $\lambda \in \Lambda$. Since J is uniformly continuous on bounded sets, we obtain

$$\begin{aligned}
(1 - \alpha_n)\|Jx_n - JT_\lambda y_n\| &= \big\|\alpha_n Jx_n + (1 - \alpha_n)JT_\lambda y_n - Jx_n\big\| \\
&= \|Jz_{n,\lambda} - Jx_n\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Since $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and J^{-1} is uniformly continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_n - T_\lambda y_n\| = 0 \tag{3.9}$$

for all $\lambda \in \Lambda$. It follows from (3.6) and (3.9) that

$$\|y_n - T_\lambda y_n\| \le \|y_n - x_n\| + \|x_n - T_\lambda y_n\| \to 0, \quad \text{as } n \to \infty \tag{3.10}$$

for all $\lambda \in \Lambda$. Since $\{x_n\}$ is a Cauchy sequence, there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$. It follows from (3.6) that $y_n \to q$. Since T_λ is closed, from (3.10) we get $q \in \mathcal{F} = \bigcap_{\lambda\in\Lambda} F(T_\lambda)$.

   Step 5. Now, we show that $q \in VI(C, A) \cap GMEP$. Let $S \subset E \times E^*$ be a mapping as follows:

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

By Lemma 2.6, S is maximal monotone, and $S^{-1}(0) = VI(C, A)$. Let $(v, w) \in G(S)$ (graph of S). Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. Moreover, $y_n \in C$ implies that

$$\langle v - y_n, w - Av \rangle \geq 0. \tag{3.11}$$

On the other hand, from $y_n = \Pi_C J^{-1}(Jx_n - \mu_n Ax_n)$ and Lemma 2.2, we obtain that

$$\langle v - y_n, Jy_n - (Jx_n - \mu_n Ax_n) \rangle \geq 0.$$

Hence,

$$\left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu_n} - Ax_n \right\rangle \leq 0. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$
\begin{aligned}
\langle v - y_n, w \rangle &\geq \langle v - y_n, Av \rangle \\
&\geq \langle v - y_n, Av \rangle + \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu_n} - Ax_n \right\rangle \\
&= \left\langle v - y_n, Av - Ax_n + \frac{Jx_n - Jy_n}{\mu_n} \right\rangle \\
&= \langle v - y_n, Av - Ay_n \rangle + \langle v - y_n, Ay_n - Ax_n \rangle + \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu_n} \right\rangle \\
&\geq -\|v - y_n\| \|Ay_n - Ax_n\| - \|v - y_n\| \left\| \frac{Jx_n - Jy_n}{\mu_n} \right\| \\
&\geq -\frac{1}{\gamma} \|v - y_n\| \|y_n - x_n\| - \frac{1}{\mu_n} \|v - y_n\| \|Jx_n - Jy_n\|.
\end{aligned}
$$

Since J is uniformly continuous on bounded sets, by (3.6) we have

$$\lim_{n \to \infty} \langle v - y_n, w \rangle = \langle v - q, w \rangle \geq 0.$$

Thus, since S is maximal monotone, we have $q \in S^{-1}(0)$ and $q \in VI(C, A)$. Next, we show that $q \in GMEP = F(T_{r,\lambda})$. Let

$$H_\lambda(u_{n,\lambda}, y) = \theta_\lambda(u_{n,\lambda}, y) + \langle Bu_{n,\lambda}, y - u_{n,\lambda} \rangle + \psi(y) - \psi(u_{n,\lambda}), \quad \forall y \in C.$$

From (3.6) and (3.8), we obtain $\lim_{n \to \infty} u_{n,\lambda} = q$ and $\lim_{n \to \infty} z_{n,\lambda} = q$ for all $\lambda \in \Lambda$. Since J is uniformly continuous, from (3.8) we have $\lim_{n \to \infty} \|Ju_{n,\lambda} - Jz_{n,\lambda}\| = 0$. Therefore, it follows from $r_n \in [c^*, \infty)$ for some $c^* > 0$ that $\lim_{n \to \infty} \frac{\|Ju_{n,\lambda} - Jz_{n,\lambda}\|}{r_n} = 0$. Since $u_{n,\lambda} = T_{r_n,\lambda} z_{n,\lambda}$, we have

$$H_\lambda(u_{n,\lambda}, y) + \frac{1}{r_n} \langle y - u_{n,\lambda}, Ju_{n,\lambda} - Jz_{n,\lambda} \rangle \geq 0, \quad \forall y \in C, \lambda \in \Lambda.$$

Combining the above inequality and (E2), we get

$$
\begin{aligned}
\|y - u_{n,\lambda}\| \cdot \frac{\|Ju_{n,\lambda} - Jz_{n,\lambda}\|}{r_n} &\geq \frac{1}{r_n} \langle y - u_{n,\lambda}, Ju_{n,\lambda} - Jz_{n,\lambda} \rangle = -H_\lambda(u_{n,\lambda}, y) \\
&\geq H_\lambda(y, u_{n,\lambda}), \quad \forall y \in C, \lambda \in \Lambda.
\end{aligned}
$$

Taking the limit as $n \to \infty$ in the above inequality and by (E4), we have $H_\lambda(y, q) \leq 0$ for all $y \in C$, $\lambda \in \Lambda$. For any $t \in (0, 1)$ and $y \in C$, define

$$y_t = ty + (1 - t)q \in C.$$

Then $H_\lambda(y_t, q) \leq 0$. From (E1) and (E4), we have

$$
\begin{aligned}
0 &= H_\lambda(y_t, y_t) = H_\lambda\big(y_t, ty + (1 - t)q\big) \\
&\leq t H_\lambda(y_t, y) + (1 - t)H_\lambda(y_t, q) \\
&\leq t H_\lambda(y_t, y),
\end{aligned}
$$

i.e., $H_\lambda(y_t, y) \geq 0$, for all $\lambda \in \Lambda$. Thus, from (E3) and let $t \downarrow 0$, we have $H_\lambda(q, y) \geq 0$ for all $y \in C$, $\lambda \in \Lambda$. This implies that $q \in GMEP$. Therefore $q \in \Omega$.

Step 6. Finally, we prove that $q = \Pi_\Omega x_0$. Since $x_{n+1} = \Pi_{H_{n,\lambda} \cap W_{n,\lambda}} x_0$ and by Lemma 2.2, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad \forall z \in H_{n,\lambda} \cap W_{n,\lambda}. \tag{3.13}$$

Taking the limit in (3.13) and from $\Omega \subset H_{n,\lambda} \cap W_{n,\lambda}$ for all $n \geq 0$, $\lambda \in \Lambda$, we obtain

$$\langle q - z, Jx_0 - Jq \rangle \geq 0, \quad \forall z \in \Omega.$$

Therefore, from Lemma 2.2, we have $q = \Pi_\Omega x_0$. $\qquad\square$

Remark 3.1 An iterative scheme for finding a solution of the variational inequality problem for a mapping A that satisfies the following conditions in a 2-uniformly and uniformly smooth Banach space E:

(1) A is inverse strongly monotone,
(2) $VI(C, A) \neq \emptyset$,
(3) $\|Ax\| \leq \|Ax - Au\|$ for all $x \in C$ and $u \in VI(C, A)$.

If condition (3) holds, then we can prove a convergence theorem for variational inequality problems. To consider the general variational inequality problem for inverse strongly monotone mappings, we have to assume condition (3) (see [27]).

For a practical case, we may apply this theorem to a finite number of mappings $\{T_1, T_2, \ldots, T_m\}$ as follows.

Corollary 3.1 *Let E be a real uniformly smooth and 2-uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a γ-inverse strongly monotone mapping and $B : C \to E^*$ be a continuous and monotone mapping. Let $\psi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (E1)-(E4). Let $\{T_1, T_2, \ldots, T_m\}$ be a finite family of closed hemi-relatively nonexpansive mappings of C into itself having*

$$\Omega = \mathcal{F} \cap VI(C, A) \cap GMEP \neq \emptyset,$$

where $\mathcal{F} = \bigcap_{k=1}^{m} F(T_k)$ is the set of common fixed points. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$. Suppose that $0 < a < \mu_n < b = \frac{c^2 \gamma}{2}$, where c is the constant in

Lemma 2.1. *Let* $\{r_n\} \subset [c^*, +\infty)$ *for some* $c^* > 0$. *Let* $\{x_n\}$ *be the sequence generated by*

$$
\begin{cases}
x_0 \in C, \quad \text{chosen arbitrarily,} \\
y_n = \Pi_C J^{-1}(Jx_n - \mu_n Ax_n), \\
z_{n,k} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_k y_n), \\
u_{n,k} = T_{r_{n,k}} z_{n,k}, \\
H_{0,k} = \{v \in C : \max_{k=1,2,\ldots,m} \phi(v, u_{0,k}) \\
\qquad \leq \alpha_0 \phi(v, x_0) + (1 - \alpha_0)\phi(v, y_0) \leq \phi(v, x_0)\}, \\
H_{n,k} = \{v \in H_{n-1,k} \cap W_{n-1,k} : \max_{k=1,2,\ldots,m} \phi(v, u_{n,k}) \\
\qquad \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, y_n) \leq \phi(v, x_n)\}, \\
W_{0,k} = C, \\
W_{n,k} = \{v \in W_{n-1,k} \cap H_{n-1,k} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{H_{n,k} \cap W_{n,k}} x_0, \quad n \geq 0, k = 1, 2, \ldots, m,
\end{cases}
$$

where J *is the normalized duality mapping, and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *satisfying* $\liminf_{n \to \infty}(1 - \alpha_n) > 0$. *Then* $\{x_n\}$ *converges to* $\Pi_\Omega x_0$, *where* Π_Ω *is the generalized projection of* E *onto* Ω.

If $E = H$ is the Hilbert space, then $J = J^{-1} = I$ is the identity mapping on H. Then Theorem 3.1 reduces to the following corollary.

Corollary 3.2 *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $A : C \to H$ be a continuous and monotone mapping. Let $\psi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (E1)-(E4). *Let $\{T_\lambda : C \to C : \lambda \in \Lambda\}$ be a family of closed hemi-relatively nonexpansive mappings with*

$$
\Omega = \mathcal{F} \cap VI(C, A) \cap GMEP \neq \emptyset,
$$

where $\mathcal{F} = \bigcap_{\lambda \in \Lambda} F(T_\lambda)$ is the set of common fixed points of $\{T_\lambda\}$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$. Suppose that $0 < a < \mu_n < b = \frac{c^2 \gamma}{2}$, where c is the constant in Lemma 2.1. Let $\{r_n\} \subset [c^*, +\infty)$ for some $c^* > 0$. Let $\{x_n\}$ be the sequence generated by

$$
\begin{cases}
x_0 \in C, \quad \text{chosen arbitrarily,} \\
y_n = P_C(x_n - \mu_n Ax_n), \\
z_{n,\lambda} = \alpha_n x_n + (1 - \alpha_n)T_\lambda y_n, \\
u_{n,\lambda} = T_{r_{n,\lambda}} z_{n,\lambda}, \\
H_{0,\lambda} = \{v \in C : \sup_{\lambda \in \Lambda} \|v - u_{0,\lambda}\|^2 \\
\qquad \leq \alpha_0 \|v - x_0\|^2 + (1 - \alpha_0)\|v - y_0\|^2 \leq \|v - x_0\|^2\}, \\
H_{n,\lambda} = \{v \in H_{n-1,\lambda} \cap W_{n-1,\lambda} : \sup_{\lambda \in \Lambda} \|v - u_{n,\lambda}\|^2 \\
\qquad \leq \alpha_n \|v - x_n\|^2 + (1 - \alpha_n)\|v - y_n\|^2 \leq \|v - x_n\|^2\}, \\
W_{0,\lambda} = C, \\
W_{n,\lambda} = \{v \in W_{n-1,\lambda} \cap H_{n-1,\lambda} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = P_{H_{n,\lambda} \cap W_{n,\lambda}} x_0, \quad n \geq 0, \lambda \in \Lambda,
\end{cases}
$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\liminf_{n\to\infty}(1-\alpha_n)>0$. Then $\{x_n\}$ converges to $P_\Omega x_0$, where P_Ω is the metric projection of H onto Ω.

Authors' contributions
The main idea of this paper was proposed by KSK. KSK prepared the manuscript initially, JKK and WHL confirmed all the steps of proofs in this research. All authors read and approved the final manuscript.

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