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Sharp bounds for the Neuman mean in terms of the quadratic and second Seiffert means

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Abstract

In this paper, we prove that $\alpha = 0$ and $\beta = \frac{\sqrt{3}\pi - 4\log(2+\sqrt{3})}{(\sqrt{2}\pi - 4)\log(2+\sqrt{3})} = 0.29758\cdots$ are the best possible constants such that the double inequality

$$\alpha Q(a,b) + (1-\alpha)T(a,b) < S_{CA}(a,b) < \beta Q(a,b) + (1-\beta)T(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, where $Q(a,b) = \sqrt{(a^2+b^2)/2}$,

$$S_{CA}(a,b) = \frac{(a-b)\sqrt{3(a^2+b^2)+2ab}}{2(a+b)\sinh^{-1}(\frac{(a-b)\sqrt{3(a^2+b^2)+2ab}}{(a+b)^2})}$$

and $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ are the quadratic, Neuman and second Seiffert means of a and b, respectively.

MSC: 26E60

Keywords: Neuman mean; quadratic mean; second Seiffert mean

1 Introduction

For $a, b > 0$ with $a \neq b$, the Neuman mean $S_{CA}(a,b)$ [1, 2] derived from the Schwab-Borchardt mean [3, 4], the quadratic mean $Q(a,b)$ and the second Seiffert mean $T(a,b)$ [5] are given by

$$S_{CA}(a,b) = \frac{(a-b)\sqrt{3(a^2+b^2)+2ab}}{2(a+b)\sinh^{-1}(\frac{(a-b)\sqrt{3(a^2+b^2)+2ab}}{(a+b)^2})}, \tag{1.1}$$

$$Q(a,b) = \sqrt{\frac{a^2+b^2}{2}} \tag{1.2}$$

and

$$T(a,b) = \frac{a-b}{2\arctan(\frac{a-b}{a+b})}, \tag{1.3}$$

respectively, where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function. Recently, the Neuman, quadratic and second Seiffert means have been the subject of intensive

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research. In particular, many remarkable inequalities for these means can be found in the literature [1–4, 6–15].

Let $A(a,b)=(a+b)/2$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the arithmetic and contraharmonic means of a and b, respectively. Then Neuman [1] proved that the inequalities

$$A(a,b) < T(a,b) < S_{CA}(a,b) < Q(a,b) < C(a,b) \tag{1.4}$$

hold for any $a,b>0$ with $a\neq b$.

In [1, 2], Neuman found that $\alpha_1=[\sqrt{3}-\log(2+\sqrt{3})]/\log(2+\sqrt{3})=0.315\cdots$, $\beta_1=1/3$, $\alpha_2=1/3$, $\beta_2=[\log 3-2\log(\log(2+\sqrt{3}))]/(2\log 2)=0.395\cdots$, $\alpha_3=2\log(2+\sqrt{3})/3-1=0.520\cdots$ and $\beta_3=2/3$ are the best possible constants such that the double inequalities

$$\alpha_1 C(a,b)+(1-\alpha_1)A(a,b) < S_{CA}(a,b) < \beta_1 C(a,b)+(1-\beta_1)A(a,b),$$
$$C^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < S_{CA}(a,b) < C^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$$

and

$$\frac{\alpha_3}{A(a,b)}+\frac{1-\alpha_3}{C(a,b)} < \frac{1}{S_{CA}(a,b)} < \frac{\beta_3}{A(a,b)}+\frac{1-\beta_3}{C(a,b)}$$

hold for any $a,b>0$ with $a\neq b$.

He *et al.* [16] proved that $\alpha=1/2+\sqrt{\sqrt{3}/\log(2+\sqrt{3})-1}/2$ and $\beta=1/2+\sqrt{3}/6$ are the best possible constants in $[1/2,1]$ such that the double inequality

$$C\bigl[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a\bigr] < S_{CA}(a,b) < C\bigl[\beta a+(1-\beta)b,\beta b+(1-\beta)a\bigr]$$

holds for any $a,b>0$ with $a\neq b$.

In [17, 18], the authors proved that the double inequalities

$$\begin{aligned}&\alpha\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\alpha)C^{1/3}(a,b)A^{2/3}(a,b)\\&\quad< S_{CA}(a,b) < \beta\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\beta)C^{1/3}(a,b)A^{2/3}(a,b)\end{aligned}$$

and

$$\lambda A(a,b)+(1-\lambda)Q(a,b) < S_{CA}(a,b) < \mu A(a,b)+(1-\mu)Q(a,b)$$

hold for any $a,b>0$ with $a\neq b$ if and only if $\alpha\leq\frac{3[\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}]}{(3\sqrt[3]{2}-4)\log(2+\sqrt{3})}=0.7528\cdots$, $\beta\geq 4/5$, $\lambda\geq 1/3$ and $\mu\leq\frac{\sqrt{2}\log(2+\sqrt{3})-\sqrt{3}}{(\sqrt{2}-1)\log(2+\sqrt{3})}=0.2390\cdots$.

The main purpose of this paper is to present the best possible constants α and β such that the double inequality

$$\alpha Q(a,b)+(1-\alpha)T(a,b) < S_{CA}(a,b) < \beta Q(a,b)+(1-\beta)T(a,b)$$

holds for any $a,b>0$ with $a\neq b$. All numerical computations are carried out using MATHEMATICA software.

2 Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

Lemma 2.1 *The double inequality*

$$-\frac{2x}{3}+\frac{16x^3}{45}-\frac{2x^5}{7}<\frac{x}{(1+x^2)\arctan^2 x}-\frac{1}{\arctan x}<-\frac{2x}{3}+\frac{16x^3}{45} \tag{2.1}$$

holds for $x\in(0,0.6)$.

Proof Let

$$\phi_1(x)=x-\left(1+x^2\right)\arctan x+\left(\frac{2x}{3}-\frac{16x^3}{45}+\frac{2x^5}{7}\right)\left(1+x^2\right)\arctan^2 x, \tag{2.2}$$

$$\phi_2(x)=x-\left(1+x^2\right)\arctan x+\left(\frac{2x}{3}-\frac{16x^3}{45}\right)\left(1+x^2\right)\arctan^2 x. \tag{2.3}$$

Then we only need to show that $\phi_1(x)>0$ and $\phi_2(x)<0$ for $x\in(0,0.6)$.

Taking the differentiation of $\phi_1(x)$ yields

$$\phi_1(0)=0, \tag{2.4}$$

$$\phi_1'(x)=\frac{2\arctan x}{315}\phi_1^*(x), \tag{2.5}$$

where

$$\phi_1^*(x)=\left(105+147x^2-55x^4+315x^6\right)\arctan x-x\left(105+112x^2-90x^4\right), \tag{2.6}$$

$$\phi_1^*(0)=0, \tag{2.7}$$

$$\phi^{*\prime}{}_1(x)=\frac{x}{1+x^2}\phi_1^{**}(x), \tag{2.8}$$

where

$$\phi_1^{**}(x)=2\left(147+37x^2+835x^4+945x^6\right)\arctan x-x\left(294-59x^2-765x^4\right). \tag{2.9}$$

It is well known that the inequality

$$\arctan x>x-\frac{x^3}{3} \tag{2.10}$$

holds for all $x\in(0,1)$.

Equation (2.9) and inequality (2.10) lead to the conclusion that

$$\begin{aligned}\phi_1^{**}(x)&>2\left(147+37x^2+835x^4+945x^6\right)\left(x-\frac{x^3}{3}\right)-x\left(294-59x^2-765x^4\right)\\&=\frac{x^3}{3}\left[105+7{,}231x^2+2{,}110x^4+1{,}890x^4\left(1-x^2\right)\right]>0\end{aligned} \tag{2.11}$$

for $x\in(0,0.6)$.

Therefore, $\phi_1(x) > 0$ for $x \in (0, 0.6)$ follows easily from (2.4)-(2.8) and (2.11).

Differentiating $\phi_2(x)$ leads to

$$\phi_2(0) = 0, \tag{2.12}$$

$$\phi_2'(x) = -\frac{2\arctan x}{45}\phi_2^*(x), \tag{2.13}$$

where

$$\phi_2^*(x) = \left(15x + 16x^3\right) - \left(15 + 21x^2 - 40x^4\right)\arctan x. \tag{2.14}$$

It is well known that the inequality

$$\arctan x < x - \frac{x^3}{3} + \frac{x^5}{5} \tag{2.15}$$

holds for all $x \in (0, 1)$.

Equation (2.14) and inequality (2.15) lead to the conclusion that

$$\begin{aligned}\phi_2^*(x) &> \left(15x + 16x^3\right) - \left(15 + 21x^2 - 40x^4\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) \\ &= \frac{x^5}{15}\left(660 - 263x^2 + 120x^4\right) > 0\end{aligned} \tag{2.16}$$

for $x \in (0, 0.6)$.

Therefore, $\phi_2(x) < 0$ for $x \in (0, 0.6)$ follows from (2.12) and (2.13) together with (2.16). □

Lemma 2.2 *The double inequality*

$$\frac{x}{\sqrt{1+x^2}} + \frac{x}{(1+x^2)\arctan^2 x} - \frac{1}{\arctan x} > \frac{x}{3} - \frac{x^3}{6} \tag{2.17}$$

holds for $x \in (0, 0.6)$.

Proof A simple computation leads to

$$\begin{aligned}&\left(1 - \frac{x^2}{2} + \frac{x^4}{4}\right)^2\left(1 + x^2\right) \\ &\quad = 1 - \frac{x^4}{16}\left[8\left(\frac{\sqrt{2}}{2} + x\right)\left(\frac{\sqrt{2}}{2} - x\right) + 2x^4 + x^4\left(1 - x^2\right)\right] < 1\end{aligned}$$

for $x \in (0, 0.6)$. This implies

$$\frac{x}{\sqrt{1+x^2}} > x - \frac{x^3}{2} + \frac{x^5}{4} \tag{2.18}$$

for $x \in (0, 0.6)$.

From Lemma 2.1 and (2.18) we clearly see that

$$\begin{aligned}
&\frac{x}{\sqrt{1+x^2}}+\frac{x}{(1+x^2)\arctan^2 x}-\frac{1}{\arctan x}\\
&\quad>\left(x-\frac{x^3}{2}+\frac{x^5}{4}\right)+\left(-\frac{2x}{3}+\frac{16x^3}{45}-\frac{2x^5}{7}\right)\\
&\quad=\frac{x}{3}-\frac{13x^3}{90}-\frac{x^5}{28}=\frac{x}{3}-\frac{x^3}{6}+\frac{x^3}{28}\left(\sqrt{\frac{28}{45}}+x\right)\left(\sqrt{\frac{28}{45}}-x\right)>\frac{x}{3}-\frac{x^3}{6}
\end{aligned}$$

for $x\in(0,0.6)$. □

Lemma 2.3 *The inequality*

$$\frac{x}{[\sinh^{-1}(x\sqrt{2+x^2})]^2}-\frac{1+x^2}{\sqrt{2+x^2}\sinh^{-1}(x\sqrt{2+x^2})}>-\frac{x}{3}+\frac{2x^3}{45}-\frac{x^5}{63} \tag{2.19}$$

holds for $x\in(0,1)$.

Proof Let

$$\begin{aligned}
\varphi(x)={}&x\sqrt{2+x^2}-\left(1+x^2\right)\sinh^{-1}\left(x\sqrt{2+x^2}\right)\\
&+\left(\frac{x}{3}-\frac{2x^3}{45}+\frac{x^5}{63}\right)\left[\sinh^{-1}\left(x\sqrt{2+x^2}\right)\right]^2\sqrt{2+x^2}.
\end{aligned} \tag{2.20}$$

Then we only need to show that $\varphi(x)>0$ for $x\in(0,1)$.
Differentiating (2.20) leads to

$$\varphi(0)=0, \tag{2.21}$$

$$\varphi'(x)=\frac{2x\sinh^{-1}(x\sqrt{2+x^2})}{315(1+x^2)}\varphi_1(x), \tag{2.22}$$

where

$$\begin{aligned}
\varphi_1(x)={}&-105-133x^2-18x^4+10x^6\\
&+3\left(35+56x^2+20x^4+4x^6+5x^8\right)\frac{\sinh^{-1}(x\sqrt{2+x^2})}{x\sqrt{2+x^2}}.
\end{aligned} \tag{2.23}$$

We claim that

$$\frac{\sinh^{-1}(x\sqrt{2+x^2})}{x\sqrt{2+x^2}}>1-\frac{x^2}{3}+\frac{2x^4}{15}-\frac{2x^6}{35} \tag{2.24}$$

for $x\in(0,1)$. Indeed, let

$$\omega(x)=\sinh^{-1}\left(x\sqrt{2+x^2}\right)-x\sqrt{2+x^2}\left(1-\frac{x^2}{3}+\frac{2x^4}{15}-\frac{2x^6}{35}\right),$$

then $\omega(x) > 0$ for $x \in (0,1)$ follows from the fact that

$$\omega(0) = 0, \qquad \omega'(x) = \frac{16x^8}{35\sqrt{2+x^2}} > 0.$$

It follows from (2.23) and (2.24) that

$$\begin{aligned}\varphi_1(x) &> -105 - 133x^2 - 18x^4 + 10x^6 \\ &\quad + 3\left(35 + 56x^2 + 20x^4 + 4x^6 + 5x^8\right)\left(1 - \frac{x^2}{3} + \frac{2x^4}{15} - \frac{2x^6}{35}\right) \\ &= \frac{x^6}{35}\left[644 + 90x^2 + 16x^6 + \left(1-x^2\right)\left(239x^2 + 30x^6\right)\right] > 0 \end{aligned} \tag{2.25}$$

for $x \in (0,1)$.

Therefore, $\varphi(x) > 0$ for $x \in (0,1)$ follows from (2.21) and (2.22) together with (2.25). □

Lemma 2.4 *The inequality*

$$\arctan x > \frac{\pi}{4} + \frac{x-1}{2} - \frac{2(x-1)^2}{7} > \frac{\pi}{4} + \frac{3(x-1)}{4} \tag{2.26}$$

holds for $x \in [0.55, 1)$.

Proof Let

$$\nu(x) = \arctan x - \left[\frac{\pi}{4} + \frac{x-1}{2} - \frac{2(x-1)^2}{7}\right]. \tag{2.27}$$

Then simple computations lead to

$$\nu(0.55) = 0.00030219\cdots, \qquad \nu(1) = 0, \tag{2.28}$$

$$\nu'(x) = \frac{\nu_1(x)}{14(1+x^2)}, \tag{2.29}$$

$$\nu_1(x) = -1 + 8x - 15x^2 + 8x^3, \tag{2.30}$$

$$\nu_1(0.55) = 0.1935, \qquad \nu_1(1) = 0, \tag{2.31}$$

$$\nu_1'(x) = 24\left(x - \frac{15-\sqrt{33}}{24}\right)\left(x - \frac{15+\sqrt{33}}{24}\right). \tag{2.32}$$

From (2.32) and $(15-\sqrt{33})/24 = 0.385643\cdots < 0.55$ together with $0.55 < (15+\sqrt{33})/24 = 0.864357\cdots < 1$, we clearly see that $\nu_1(x)$ is strictly decreasing on $[0.55, (15+\sqrt{33})/24]$ and strictly increasing on $[(15+\sqrt{33})/24, 1)$. This in conjunction with (2.31) implies that there exists $x_1 \in (0.55, 1)$ such that $\nu_1(x) > 0$ for $x \in [0.55, x_1)$ and $\nu_1(x) < 0$ for $x \in (x_1, 1)$. Then equation (2.29) leads to the conclusion that $\nu(x)$ is strictly increasing on $[0.55, x_1]$ and strictly decreasing on $[x_1, 1]$.

Therefore, $\nu(x) > 0$ for $x \in [0.55, 1)$ follows from (2.28) and the piecewise monotonicity of $\nu(x)$. Moreover, the second inequality in (2.26) follows from

$$\frac{x-1}{2} - \frac{2(x-1)^2}{7} > \frac{3(x-1)}{4} + \frac{(1-x)(8x-1)}{28} > \frac{3(x-1)}{4}. \qquad \square$$

Lemma 2.5 *The inequality*

$$x - \arctan x < \frac{7}{20} x \arctan^2 x \tag{2.33}$$

holds for $x \in [0.55, 1)$.

Proof Let

$$\mu(x) = x - \arctan x - \frac{7}{20} x \arctan^2 x. \tag{2.34}$$

Then it suffices to show $\mu(x) < 0$ for $x \in [0.55, 1)$.

Differentiating $\mu(x)$ yields

$$\mu'(x) = \frac{\mu_1(x)}{20(1+x^2)}, \tag{2.35}$$

where

$$\mu_1(x) = 20x^2 - 14x \arctan x - 7\arctan^2 x - 7x^2 \arctan^2 x. \tag{2.36}$$

It is well known that

$$\arctan x > x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \tag{2.37}$$

for $x \in (0, 1)$.

For $x \in [0.55, 0.7]$, it follows from (2.36) and (2.37) that

$$\begin{aligned} \mu_1(x) &< 20x^2 - 14x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) - 7\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right)^2 \\ &\quad - 7x^2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right)^2 = \frac{x^2}{1{,}575}\mu^*\left(x^2\right), \end{aligned} \tag{2.38}$$

where

$$\begin{aligned} \mu^*(x) &= -1{,}575 + 3{,}675x - 2{,}695x^2 + 2{,}135x^3 \\ &\quad + 3{,}129x^4 - 861x^5 + 405x^6 - 225x^7, \end{aligned} \tag{2.39}$$

$$\mu^*(0.49) = -9.99966\cdots. \tag{2.40}$$

Differentiating $\mu^*(x)$ yields

$$\begin{aligned} \mu^{*\prime}(x) &= \left(3{,}675 - 5{,}390x + 6{,}405x^2\right) + \left(12{,}516x^3 - 4{,}305x^4\right) \\ &\quad + \left(2{,}430x^5 - 1{,}575x^6\right) > 0 \end{aligned} \tag{2.41}$$

for $x \in [0.3025, 0.49]$.

Therefore, $\mu^*(x) < 0$ for $x \in [0.3025, 0.49]$ follows from (2.40) and (2.41). This in conjunction with (2.35) and (2.38) implies that $\mu(x)$ is strictly decreasing on $[0.55, 0.7]$. Therefore, we get $\mu(x) \le \mu(0.55) = -0.00151709\cdots < 0$ for $x \in [0.55, 0.7]$.

It follows from Lemma 2.4 that

$$\mu(x) < x - \left[\frac{\pi}{4} + \frac{x-1}{2} - \frac{2(x-1)^2}{7}\right] - \frac{7}{20}\left[\frac{\pi}{4} + \frac{x-1}{2} - \frac{2(x-1)^2}{7}\right]^2 = \frac{\mu_2(x)}{2{,}240} \tag{2.42}$$

for $x \in (0.7, 1)$, where

$$\begin{aligned}\mu_2(x) = {}& (1{,}760 - 560\pi) + \left(308\pi - 49\pi^2 - 644\right)x \\ & + (1{,}960 - 420\pi)x^2 + (112\pi - 1{,}252)x^3 + 480x^4 - 64x^5.\end{aligned} \tag{2.43}$$

Differentiating $\mu_2(x)$ yields

$$\mu_2(0.7) = -1.68877\cdots, \qquad \mu_2(1) = -2.9025\cdots, \tag{2.44}$$

$$\begin{aligned}\mu_2'(x) = {}& \left(-644 + 308\pi - 49\pi^2\right) + (3{,}920 - 840\pi)x + (336\pi - 3{,}756)x^2 \\ & + 1{,}920x^3 - 320x^4,\end{aligned} \tag{2.45}$$

$$\mu_2'(0.7) = -4.73674\cdots, \qquad \mu_2'(1) = 20.6372\cdots, \tag{2.46}$$

$$\mu_2''(x) = 8\left(490 - 105\pi - 939x + 84\pi x + 720x^2 - 160x^3\right), \tag{2.47}$$

$$\mu_2''(0.7) = -116.173\cdots, \qquad \mu_2''(1) = 360.212\cdots, \tag{2.48}$$

$$\begin{aligned}\mu_2'''(x) &= 24\left(28\pi - 313 + 480x - 160x^2\right) \\ &> 24\left(28\pi - 313 + 480 \times 0.7 - 160 \times (0.7)^2\right) = 781.55\cdots > 0.\end{aligned} \tag{2.49}$$

It follows from (2.48) and (2.49) that there exists $x_2 \in (0.7, 1)$ such that $\mu_2'(x)$ is strictly decreasing on $(0.7, x_2]$ and strictly increasing on $[x_2, 1)$. This in conjunction with (2.46) implies that there exists $x_3 \in (0.7, 1)$ such that $\mu_2(x)$ is strictly decreasing on $(0.7, x_3]$ and strictly increasing on $[x_3, 1)$. From (2.44) and the piecewise monotonicity of $\mu_2(x)$, we know that $\mu_2(x) < 0$ for $x \in (0.7, 1)$; this in conjunction with (2.42) implies $\mu(x) < 0$ for $x \in (0.7, 1)$. $\square$

Lemma 2.6 *The function*

$$\sigma(x) = \frac{\sqrt{1+x^2}\arctan^3 x - 2(x - \arctan x)}{(1+x^2)^2\arctan^3 x}$$

is strictly decreasing on $[0.55, 1)$. *Moreover,* $\sigma(x) < 0.236$ *for* $x \in [0.55, 1)$.

Proof Differentiating $\sigma(x)$ yields

$$\sigma'(x) = \frac{\sigma_1(x)}{(1+x^2)^3\arctan^4 x}, \tag{2.50}$$

where

$$\sigma_1(x) = 6(x - \arctan x) + 6x^2\arctan x - 8x\arctan^2 x - 3x\sqrt{1+x^2}\arctan^4 x. \tag{2.51}$$

From Lemma 2.5 and (2.51) we clearly see that

$$\sigma_1(x) < 6x^2 \arctan x - \frac{59}{10} x \arctan^2 x - 3x \arctan^4 x = x \arctan x \sigma_2(x) \tag{2.52}$$

for $x \in [0.55, 1)$, where

$$\sigma_2(x) = 6x - \frac{59}{10} \arctan x - 3 \arctan^3 x. \tag{2.53}$$

Differentiating $\sigma_2(x)$ leads to

$$\sigma_2(0.55) = -0.0482086\cdots, \qquad \sigma_2(1) = -0.0872684\cdots, \tag{2.54}$$

$$\sigma_2'(x) = \frac{\sigma_3(x)}{10(1+x^2)}, \tag{2.55}$$

$$\sigma_3(x) = 1 + 60x^2 - 90 \arctan^2 x, \tag{2.56}$$

$$\sigma_3(0.55) = -3.60662\cdots, \qquad \sigma_3(1) = 5.48348\cdots, \tag{2.57}$$

$$\sigma_3'(x) = \frac{60\sigma_4(x)}{1+x^2}, \tag{2.58}$$

$$\sigma_4(x) = 2x + 2x^3 - 3 \arctan x, \tag{2.59}$$

$$\sigma_4(0.55) = -0.0757796\cdots, \qquad \sigma_4(1) = 1.64381\cdots, \tag{2.60}$$

$$\sigma_4'(x) = \frac{-1 + 8x^2 + 6x^4}{1+x^2} > 0. \tag{2.61}$$

It follows from (2.58)-(2.61) that there exists $x_4 \in (0.55, 1)$ such that $\sigma_3(x)$ is strictly decreasing on $(0.55, x_4]$ and strictly increasing on $[x_4, 1)$. This in conjunction with (2.55)-(2.57) implies that there exists $x_5 \in (0.55, 1)$ such that $\sigma_2(x)$ is strictly decreasing on $(0.55, x_5]$ and strictly increasing on $[x_5, 1)$. Then from (2.54) we clearly see that $\sigma_2(x) < 0$ for $x \in (0.55, 1)$.

Therefore, it follows from (2.50) and (2.52) that $\sigma(x)$ is strictly decreasing on $[0.55, 1)$. Moreover, $\sigma(x) \le \sigma(0.55) = 0.235477\cdots < 0.236$ for $x \in [0.55, 1)$. □

Lemma 2.7 *The function*

$$\kappa(x) = \frac{2(4+3x^2)\sinh^{-1}(x\sqrt{2+x^2}) - 8x\sqrt{2+x^2}}{(2+x^2)[\sinh^{-1}(x\sqrt{2+x^2})]^3}$$

is strictly decreasing on $[0.55, 1)$. *Moreover,* $\kappa(x) < 0.771$ *for* $x \in [0.55, 1)$.

Proof Simple computations lead to

$$\kappa(0.55) = 0.770758\cdots, \tag{2.62}$$

$$\kappa'(x) = \frac{8\kappa_1(x)}{(2+x^2)^2[\sinh^{-1}(x\sqrt{2+x^2})]^4}, \tag{2.63}$$

where

$$\kappa_1(x) = 6x(2+x^2) - 3(2+x^2)^{3/2} \sinh^{-1}(x\sqrt{2+x^2}) + x[\sinh^{-1}(x\sqrt{2+x^2})]^2. \tag{2.64}$$

We claim that

$$\sqrt{2}x - \frac{x^3}{6\sqrt{2}} < \sinh^{-1}\bigl(x\sqrt{2+x^2}\bigr) < \sqrt{2}x \tag{2.65}$$

for $x \in (0,1)$. Indeed, let

$$\eta_1(x) = \sinh^{-1}\bigl(x\sqrt{2+x^2}\bigr) - \sqrt{2}x + \frac{x^3}{6\sqrt{2}}, \tag{2.66}$$

$$\eta_2(x) = \sinh^{-1}\bigl(x\sqrt{2+x^2}\bigr) - \sqrt{2}x. \tag{2.67}$$

Then we clearly see that

$$\eta_1(0) = \eta_2(0) = 0, \tag{2.68}$$

$$\eta_1'(x) = \frac{2}{\sqrt{2+x^2}} + \frac{\sqrt{2}}{4}x^2 - \sqrt{2}, \tag{2.69}$$

$$\eta_2'(x) = \frac{2}{\sqrt{2+x^2}} - \sqrt{2} < 0, \tag{2.70}$$

$$\eta_1'(0) = 0, \tag{2.71}$$

$$\eta_1''(x) = x\left(\frac{1}{\sqrt{2}} - \frac{2}{(2+x^2)^{3/2}}\right) > 0. \tag{2.72}$$

Therefore, the double inequality (2.65) follows easily from (2.68)-(2.72).

Equation (2.64) and inequality (2.65) imply that

$$\kappa_1(x) < 6x\bigl(2+x^2\bigr) - 3\bigl(2+x^2\bigr)^{3/2}\left(\sqrt{2}x - \frac{x^3}{6\sqrt{2}}\right) + x(\sqrt{2}x)^2 = \frac{x}{4}\kappa_2(x), \tag{2.73}$$

where

$$\kappa_2(x) = 16\bigl(3+2x^2\bigr) - \sqrt{2}\bigl(12-x^2\bigr)\bigl(2+x^2\bigr)^{3/2}. \tag{2.74}$$

Let $u = \sqrt{2+x^2}$, then $x^2 = u^2 - 2$, $\sqrt{2} < u < \sqrt{3}$ and $\kappa_2(x)$ becomes

$$\tilde{\kappa}(u) = -16 + 32u^2 - 14\sqrt{2}u^3 + \sqrt{2}u^5. \tag{2.75}$$

Equation (2.75) leads to

$$\tilde{\kappa}(\sqrt{2}) = 0, \tag{2.76}$$

$$\tilde{\kappa}'(u) = u\bigl(64 - 42\sqrt{2}u + 5\sqrt{2}u^3\bigr) = u\tilde{\kappa}_1(u), \tag{2.77}$$

$$\tilde{\kappa}_1(u) = 64 - 42\sqrt{2}u + 5\sqrt{2}u^3, \qquad \tilde{\kappa}_1(\sqrt{2}) = 0, \qquad \tilde{\kappa}_1(\sqrt{3}) = -2.1362\cdots, \tag{2.78}$$

$$\tilde{\kappa}_1'(u) = 15\sqrt{2}\left(u - \sqrt{\frac{14}{5}}\right)\left(u + \sqrt{\frac{14}{5}}\right). \tag{2.79}$$

From (2.79) we clearly see that $\tilde{\kappa}_1'(u) < 0$ for $u \in (\sqrt{2}, \sqrt{14/5})$ and $\tilde{\kappa}_1'(u) > 0$ for $u \in (\sqrt{14/5}, \sqrt{3})$. This in conjunction with (2.77) implies that $\tilde{\kappa}'(u)$ is strictly decreasing on

$(\sqrt{2}, \sqrt{14/5}]$ and strictly increasing on $[\sqrt{14/5}, \sqrt{3})$. Thus $\tilde{\kappa}'(u) < 0$ for $u \in (\sqrt{2}, \sqrt{3})$ follows from (2.78) and the piecewise monotonicity of $\tilde{\kappa}'(u)$.

Therefore, $\kappa_2(x) = \tilde{\kappa}(u) < 0$ follows from (2.76). This in conjunction with (2.63) and (2.73) implies that $\kappa(x)$ is strictly decreasing on $[0.55, 1)$. Moreover, it follows from (2.62) that $\kappa(x) \leq \kappa(0.55) = 0.770758\cdots < 0.771$ for $x \in [0.55, 1)$. □

Lemma 2.8 *The function*

$$\tau(x) = \frac{2(x - \arctan x)}{(1 + x^2)^2 \arctan^3 x} - \frac{2x(3 + x^2)}{(2 + x^2)^{3/2} \sinh^{-1}(x\sqrt{2 + x^2})} < -0.88$$

for $x \in [0.55, 1)$.

Proof We first prove

$$\sqrt{2 + x^2} \sinh^{-1}\bigl(x\sqrt{2 + x^2}\bigr) < 2x + \frac{x^3}{3} \tag{2.80}$$

for $x \in (0, 1)$. Let

$$\varepsilon(x) = \sqrt{2 + x^2} \sinh^{-1}\bigl(x\sqrt{2 + x^2}\bigr) - \left(2x + \frac{x^3}{3}\right).$$

Then $\varepsilon(x) < 0$ follows from $\varepsilon(0) = 0$ and the fact that

$$\varepsilon'(x) = \frac{x}{\sqrt{2 + x^2}}\Bigl(\sinh^{-1}\bigl(x\sqrt{2 + x^2}\bigr) - x\sqrt{2 + x^2}\Bigr) < \frac{x}{\sqrt{2 + x^2}}\bigl(\sqrt{2}x - x\sqrt{2 + x^2}\bigr) < 0,$$

where the second term follows from (2.65).

From Lemma 2.5 and (2.10) we clearly see that

$$\frac{x - \arctan x}{\arctan^3 x} < \frac{7x}{20 \arctan x} < \frac{21}{20(3 - x^2)} \tag{2.81}$$

for $x \in [0.55, 1)$.

It follows from (2.80) and (2.81) that

$$\tau(x) < \frac{21}{10(1 + x^2)^2(3 - x^2)} - \frac{6(3 + x^2)}{(2 + x^2)(6 + x^2)} =: \tau_1(x) \tag{2.82}$$

for $x \in [0.55, 1)$.

Simple computation yields

$$\tau_1(0.55) = -0.906585\cdots, \qquad \tau_1(1) = -0.880357\cdots, \tag{2.83}$$

$$\tau_1'(x) = \frac{3x}{5(x^2 - 3)^2(1 + x^2)^3(2 + x^2)^2(6 + x^2)^2}\tilde{\tau}(x), \tag{2.84}$$

where

$$\begin{aligned}\tilde{\tau}(x) = {}& -2{,}880 + 2{,}424x^2 + 6{,}052x^4 + 1{,}468x^6 \\ & - 939x^8 - 219x^{10} + 60x^{12} + 20x^{14},\end{aligned} \tag{2.85}$$

$$\tilde{\tau}(0.55) = -1{,}560.68\cdots, \qquad \tilde{\tau}(1) = 5{,}986, \tag{2.86}$$

$$\begin{aligned} \tilde{\tau}'(x) &= 2x\bigl(2{,}424 + 12{,}104x^2 + 4{,}404x^4 - 3{,}756x^6 \\ &\quad - 1{,}095x^8 + 360x^{10} + 140x^{12}\bigr) > 0. \end{aligned} \tag{2.87}$$

From (2.85)-(2.87) we know that there exists $x_6 \in (0.55, 1)$ such that $\tilde{\tau}(x) < 0$ for $x \in (0.55, x_6)$ and $\tilde{\tau}(x) > 0$ for $x \in (x_6, 1)$. This in conjunction with (2.84) implies that $\tau_1(x)$ is strictly decreasing on $[0.55, x_6)$ and strictly increasing on $[x_6, 1)$.

Therefore, $\tau(x) < \tau_1(x) \leq \max\{\tau_1(0.55), \tau_1(1)\} = -0.880357\cdots < -0.88$ follows from (2.83) and the piecewise monotonicity of $\tau_1(x)$. □

3 Main result

Theorem 3.1 *The double inequality*

$$\alpha Q(a,b) + (1-\alpha)T(a,b) < S_{CA}(a,b) < \beta Q(a,b) + (1-\beta)T(a,b) \tag{3.1}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 0$ and $\beta \geq \beta_0 = \frac{\sqrt{3}\pi - 4\log(2+\sqrt{3})}{(\sqrt{2}\pi - 4)\log(2+\sqrt{3})} = 0.29758\cdots$.

Proof Since the Neuman mean $S_{CA}(a,b)$, the quadratic mean $Q(a,b)$ and the second Seiffert mean $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$, then from (1.1)-(1.3) one has

$$S_{CA}(a,b) = A(a,b)\frac{v\sqrt{2+v^2}}{\sinh^{-1}(v\sqrt{2+v^2})}, \tag{3.2}$$

$$T(a,b) = A(a,b)\frac{v}{\arctan(v)}, \qquad Q(a,b) = A(a,b)\sqrt{1+v^2}. \tag{3.3}$$

Equations (3.2) and (3.3) lead to

$$\frac{S_{CA}(a,b) - T(a,b)}{Q(a,b) - T(a,b)} = \frac{\frac{v\sqrt{2+v^2}}{\sinh^{-1}(v\sqrt{2+v^2})} - \frac{v}{\arctan(v)}}{\sqrt{1+v^2} - \frac{v}{\arctan(v)}}. \tag{3.4}$$

It is easy to find that

$$\lim_{v\to 0^+} \frac{\frac{v\sqrt{2+v^2}}{\sinh^{-1}(v\sqrt{2+v^2})} - \frac{v}{\arctan(v)}}{\sqrt{1+v^2} - \frac{v}{\arctan(v)}} = 0, \tag{3.5}$$

$$\lim_{v\to 1^-} \frac{\frac{v\sqrt{2+v^2}}{\sinh^{-1}(v\sqrt{2+v^2})} - \frac{v}{\arctan(v)}}{\sqrt{1+v^2} - \frac{v}{\arctan(v)}} = \beta_0. \tag{3.6}$$

We investigate the difference between the convex combination of $Q(a,b)$, $T(a,b)$ and $S_{CA}(a,b)$ as follows:

$$\begin{aligned} &pQ(a,b) + (1-p)T(a,b) - S_{CA}(a,b) \\ &\quad = A(a,b)\left[p\sqrt{1+v^2} + (1-p)\frac{v}{\arctan(v)} - \frac{v\sqrt{2+v^2}}{\sinh^{-1}(v\sqrt{2+v^2})}\right]. \end{aligned} \tag{3.7}$$

Let

$$D_p(\nu) = p\sqrt{1+\nu^2} + (1-p)\frac{\nu}{\arctan(\nu)} - \frac{\nu\sqrt{2+\nu^2}}{\sinh^{-1}(\nu\sqrt{2+\nu^2})}. \tag{3.8}$$

Then simple computations lead to

$$D_p(0^+) = 0, \qquad D_p(1^-) = p\left(\sqrt{2} - \frac{4}{\pi}\right) + \frac{4}{\pi} - \frac{\sqrt{3}}{\log(2+\sqrt{3})}, \qquad D_{\beta_0}(1^-) = 0, \tag{3.9}$$

$$\begin{aligned} D_p'(\nu) &= p\left[\frac{\nu}{\sqrt{1+\nu^2}} + \frac{\nu}{(1+\nu^2)\arctan^2\nu} - \frac{1}{\arctan\nu}\right] + \frac{\nu}{(\sinh^{-1}(\nu\sqrt{2+\nu^2}))^2} \\ &\quad - \frac{1+\nu^2}{\sqrt{2+\nu^2}\sinh^{-1}(\nu\sqrt{2+\nu^2})} - \frac{\nu}{(1+\nu^2)\arctan^2\nu} + \frac{1}{\arctan\nu}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} D_p''(\nu) &= p\frac{\sqrt{1+\nu^2}\arctan^3\nu - 2(\nu - \arctan\nu)}{(1+\nu^2)^2\arctan^3\nu} \\ &\quad + \frac{2(4+3\nu^2)\sinh^{-1}(\nu\sqrt{2+\nu^2}) - 8\nu\sqrt{2+\nu^2}}{(2+\nu^2)(\sinh^{-1}(\nu\sqrt{2+\nu^2}))^3} \\ &\quad + \frac{2(\nu - \arctan\nu)}{(1+\nu^2)^2\arctan^3\nu} - \frac{2\nu(3+\nu^2)}{(2+\nu^2)^{3/2}\sinh^{-1}(\nu\sqrt{2+\nu^2})} \\ &= p\sigma(\nu) + \kappa(\nu) + \tau(\nu), \end{aligned} \tag{3.11}$$

where $\sigma(x)$, $\kappa(x)$ and $\tau(x)$ are defined as in Lemmas 2.6, 2.7 and 2.8, respectively.

From Lemmas 2.1-2.3 and (3.10) we clearly see that

$$\begin{aligned} D_{\beta_0}'(\nu) &> \beta_0\left(\frac{\nu}{3} - \frac{\nu^3}{6}\right) - \frac{\nu}{3} + \frac{2\nu^3}{45} - \frac{\nu^5}{63} + \frac{2\nu}{3} - \frac{16\nu^3}{45} \\ &= \frac{\nu}{630}\left[210(1+\beta_0) - 7(28+15\beta_0)\nu^2 - 10\nu^4\right] \\ &> \frac{\nu}{630}\left[210(1+0.29758) - 7(28+15\times 0.29759)\times(0.55)^2 - 10\times(0.55)^4\right] \\ &= \frac{\nu}{630}\times 202.83\cdots > 0 \end{aligned} \tag{3.12}$$

for $\nu \in (0, 0.55]$.

It follows from Lemmas 2.6-2.8 and (3.11) that

$$D_{\beta_0}''(\nu) = \beta_0\sigma(\nu) + \kappa(\nu) + \tau(\nu) < 0.236\beta_0 + 0.771 - 0.88 = -0.0387709\cdots \tag{3.13}$$

for $\nu \in [0.55, 1)$. Then from $D_{\beta_0}'(0.55) = 0.0139552\cdots$ and $D_{\beta_0}'(1) = -0.0650268\cdots$ we know that there exists $\nu_0 \in (0.55, 1)$ such that $D_{\beta_0}'(\nu) > 0$ for $\nu \in [0.55, \nu_0)$ and $D_{\beta_0}'(x) < 0$ for $\nu \in (\nu_0, 1)$. This in conjunction with (3.13) leads to the conclusion that $D_{\beta_0}(\nu)$ is strictly increasing on $[0.55, \nu_0]$ and strictly decreasing on $[\nu_0, 1)$.

Therefore, $D_{\beta_0}(\nu) > 0$ for $\nu \in (0, 1)$ follows from (3.9) and the monotonicity of $D_{\beta_0}(\nu)$. In other words, we obtain

$$\beta_0 Q(a,b) + (1-\beta_0)T(a,b) > S_{CA}(a,b) \tag{3.14}$$

for $a, b > 0$ with $a \neq b$.

Obviously, if $\alpha = 0$, then (1.4) gives

$$T(a,b) < S_{CA}(a,b) \tag{3.15}$$

for $a, b > 0$ with $a \neq b$.

Therefore, Theorem 3.1 follows from (3.14) and (3.15) together with the following statements:

- If $\alpha > 0$, then (3.4) and (3.5) imply that there exists $\delta_1 \in (0,1)$ such that $S_{CA}(a,b) < \alpha Q(a,b) + (1-\alpha)T(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.
- If $\beta < \beta_0$, then (3.4) and (3.6) imply that there exists $\delta_2 \in (0,1)$ such that $S_{CA}(a,b) > \beta Q(a,b) + (1-\beta)T(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Y-MC provided the main idea and carried out the proof of Theorem 3.1. HW carried out the proof of Lemmas 2.1-2.4. T-HZ carried out the proof of Lemmas 2.5-2.8 and drafted the manuscript. All authors read and approved the final manuscript.

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