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On the range of the parameters for the grand Furuta inequality to be valid

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Abstract

In order to investigate the precise range D of the parameters p, s, t, and r for which the grand Furuta inequality is valid. We use the method of *reductio ad absurdum*. We find the following results: The area $1 < p$, $0 < s < 1$, $0 < t < 1 + r$, and $ts < r$ is contained in the complement of the range D. The condition $1 \leq s$ seems essential for the grand Furuta inequality.

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1 Introduction

A bounded linear operator T on a Hilbert space is said to be positive semidefinite (denoted by $0 \leq T$) if $0 \leq (Th, h)$ for all vectors h. We write $0 < T$ if T is positive semidefinite and invertible.

Furuta obtained an epoch-making extension of the Löwner-Heinz inequality [1, 2].

Theorem 1.1 [3] *Let $0 \leq p$, $1 \leq q$, and $0 \leq r$ with $p + r \leq (1 + r)q$. If $0 \leq B \leq A$ holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}.$$

It is well known that Theorem 1.1 is equivalent to the next theorem, which is often called the essential case of the Furuta inequality.

Theorem 1.2 *Let $1 \leq p$ and $0 \leq r$. If $0 \leq B \leq A$ holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \leq A^{1+r}.$$

The following result by Tanahashi is a full description of the best possibility of the range

$$p + r \leq (1 + r)q \quad \text{and} \quad 1 \leq q$$

as far as all parameters are positive. We would like to emphasize that the theorem can be divided into two cases.

Theorem 1.3 [4] *Let p, q, r be positive real numbers. If $(1+r)q < p+r$ or $0 < q < 1$, then there exist 2×2 matrices A, B with $0 < B \leq A$ that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}.$$

The next proposition is corresponding to the case $(1+r)q < p+r$ of the previous theorem by putting $q = \frac{p+r}{(1+r)\alpha}$.

Proposition 1.4 *Let $0 < p$, $0 \leq r$. If $1 < \alpha$, then there exist 2×2 matrices A, B with $0 < B \leq A$ that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}\alpha} \leq A^{(1+r)\alpha}.$$

On the other hand, the following 'α-free' proposition corresponds to the case $0 < q < 1$ of Theorem 1.3 by putting $q = \frac{p+r}{1+r}$.

Proposition 1.5 *Let $0 < p < 1$ and $0 < r$. Then there exist 2×2 matrices A, B with $0 < B \leq A$ that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \leq A^{1+r}.$$

Since the condition $q = \frac{p+r}{1+r}$ is the essential case for the Furuta inequality, our interest in Proposition 1.5 is not at all less than Proposition 1.4.

Furuta gave a unifying extension of both Theorem 1.1 and the Ando-Hiai inequality [5], which is often called the grand Furuta inequality.

Theorem 1.6 [6] *Let $1 \leq p$, $1 \leq s$, $0 \leq t \leq 1$, and $t \leq r$. If $0 \leq B \leq A$ with $0 < A$, then the following inequality holds*:

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}. \tag{1}$$

Again, Tanahashi showed that the outside powers in this theorem are the best possible.

Theorem 1.7 [7] *Let $1 \leq p$, $1 \leq s$, $0 \leq t \leq 1$, and $t \leq r$. If $1 < \alpha$, then there exist 2×2 matrices A, B with $0 < B \leq A$ that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \leq A^{(1-t+r)\alpha}.$$

Remark 1.8 In [7], Theorem 1.7 is originally stated as follows:

Let p, r, s, t be real numbers satisfying $1 < s$, $0 < t < 1$, $t \leq r$, $1 \leq p$. If

$$\frac{1-t+r}{(p-t)s+r} < \alpha,$$

then there exist invertible matrices A, B with $0 \leq B \leq A$ which do not satisfy

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\alpha} \leq A^{\{(p-t)s+r\}\alpha}.$$

These are just a matter of rephrasing, although their α differs each other. Theorem 1.7 can be naturally considered as an extension of Proposition 1.4. Indeed, if we put $s = 1$, $t = 0$ in Theorem 1.7, then we obtain Proposition 1.4 restricted to $1 \le p$. On the other hand, being different from Theorem 1.3, even if all parameters are positive, Theorem 1.7 does not show that the range

$$1 \le p, \qquad 1 \le s, \qquad 0 \le t \le 1, \qquad t \le r$$

cannot be expanded anymore for the grand Furuta inequality to be valid. Thus the clarification of the best possibility of the grand Furuta inequality is less satisfactory than that of the Furuta inequality. So our problem is to determine the range:

$$\bigl\{(p,s,t,r) \in \mathbb{R}_+^4; \text{the inequality (1) holds whenever } 0 < B \le A\bigr\}. \tag{2}$$

Although it would be a nice theorem if one could precisely determine the range (2) all at once, it seems difficult to the author. Therefore, we should treat several main cases of the problem.

It is quite natural to expect an 'α-free' version which can be regarded as corresponding to Proposition 1.5. The following result obtained by Koizumi and the author is such an attempt.

Theorem 1.9 [8] *Let* $0 < p$, $0 < s$, $0 < t \le 1$, *and* $t \le r$. *Suppose that*

$$t < p \quad \textit{and} \quad \frac{1-t+r}{(p-t)s+r} \cdot sp < 1.$$

Then there exist 2×2 *matrices* A, B *with* $0 < B \le A$ *that do not satisfy the inequality* (1).

Remark 1.10 The quantity $\frac{1-t+r}{(p-t)s+r} \cdot sp$ in the above assumption has an essential meaning. It also appears in a certain functional inequality (*cf.* [9]).

(a) If $1 \le p$, $1 \le s$, $0 \le t \le 1$, and $t \le r$, then $\frac{1-t+r}{(p-t)s+r} \le 1 \le \frac{1-t+r}{(p-t)s+r} \cdot sp$.
(b) If $0 < p$, $0 < s < 1$, $sp < 1$, and $0 < t \le r$, then $\frac{1-t+r}{(p-t)s+r} \cdot sp < 1$.

Remark 1.11 The case (ii) of [8, Theorem 2.1] by Koizumi and the author treats the case $0 < p = t < 1$, $0 < s$, $t < r$. However, we have $A_1 < A_2$ by the notations in [8] and the proof for (i) is not applicable to (ii). It seems still open.

The main purpose of this article is to show that the area $1 < p$, $0 < s < 1$, $0 < t < 1 + r$, and $ts < r$ is contained in the complement of the range (2).

2 Preliminaries

In this section, we will outline Tanahashi's argument in [4] and [7] without proofs.

Let A, B be 2×2 matrices with $0 < B \le A$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right)$, and let U be a unitary which diagonalizes A as $U^*AU = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$. Assume A and B satisfy the grand Furuta inequality (1). Put $\alpha = 1 - t + r$ and $\psi = (p-t)s + r$. Then

$$\bigl\{U^*A^{\frac{r}{2}}U\bigl(U^*A^{-\frac{t}{2}}UU^*B^pUU^*A^{-\frac{t}{2}}U\bigr)^sU^*A^{\frac{r}{2}}U\bigr\}^{\frac{\alpha}{\psi}} \le U^*A^{\alpha}U,$$

hence we have

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\left[\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix}\right]^s \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}} \leq \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}. \tag{3}$$

Denote

$$\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} = k \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix},$$

where k is a positive scalar to be specified later.

Lemma 2.1 *Suppose that $A_1 < A_2$ and $A_3 < 0$. Let*

$$V = \frac{1}{\sqrt{-A_1 + A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{\varepsilon_1} & -\sqrt{-A_1 + A_2 + \varepsilon_1} \\ -\sqrt{-A_1 + A_2 + \varepsilon_1} & -\sqrt{\varepsilon_1} \end{pmatrix},$$

where

$$2\varepsilon_1 = A_1 - A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then $A_3 = -\sqrt{(-A_1 + A_2 + \varepsilon_1)\varepsilon_1}$, V is unitary and

$$V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_2 + \varepsilon_1 & 0 \\ 0 & A_1 - \varepsilon_1 \end{pmatrix}.$$

The formula (3) implies

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix} k^s V \begin{pmatrix} (A_2 + \varepsilon_1)^s & 0 \\ 0 & (A_1 - \varepsilon_1)^s \end{pmatrix} V^* \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}} \leq \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}.$$

Write the left-hand matrix as

$$k^{s\frac{\alpha}{\psi}} (-A_1 + A_2 + 2\varepsilon_1)^{-\frac{\alpha}{\psi}} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}^{\frac{\alpha}{\psi}},$$

where

$$\begin{aligned} B_1 &= d_1^r \{\varepsilon_1 (A_2 + \varepsilon_1)^s + (-A_1 + A_2 + \varepsilon_1)(A_1 - \varepsilon_1)^s\}, \\ B_2 &= d_2^r \{(-A_1 + A_2 + \varepsilon_1)(A_2 + \varepsilon_1)^s + \varepsilon_1 (A_1 - \varepsilon_1)^s\}, \\ B_3 &= -d_1^{\frac{r}{2}} d_2^{\frac{r}{2}} \sqrt{(-A_1 + A_2 + \varepsilon_1)\varepsilon_1} \{(A_2 + \varepsilon_1)^s - (A_1 - \varepsilon_1)^s\}. \end{aligned}$$

Lemma 2.2 *Keep the situation as above. Assume that* $B_2 < B_1$. *Then the following inequality holds*:

$$\begin{aligned} &\varepsilon_2\left\{\gamma d_1^\alpha - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma d_2^\alpha\right\} \\ &\quad \le (B_1 - B_2 + \varepsilon_2)\left\{\gamma d_1^\alpha - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{\gamma d_2^\alpha - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \varepsilon_2 &= \frac{1}{2}\left(-B_1 + B_2 + \sqrt{(B_1 - B_2)^2 + 4B_3^2}\right), \\ \gamma &= \left\{k^{-s}(-A_1 + A_2 + 2\varepsilon_1)\right\}^{\frac{\alpha}{\psi}}. \end{aligned}$$

3 Results

We begin with an elementary inequality for real numbers.

Lemma 3.1 *If* $1 < p$, *then*

$$-2^{p-1}p + 2^p - 1 < 0.$$

As Theorem 1.7 is regarded as an extension of Proposition 1.4, our main theorem can be considered as an extension of Proposition 1.5, which shows one of the largest pieces of the complement of the range (2). The advantage is that the assumptions on the parameters other than $0 < s < 1$ are very mild. Note that, if we change $0 < s < 1$ to $1 \le s$, then the inequality holds for $0 < t \le 1$.

Theorem 3.2 *Let* $1 < p$, $0 < s < 1$, $0 < t < 1 + r$, *and* $ts < r$. *Then there exist* 2×2 *matrices* A *and* B *with* $0 < B \le A$ *that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r}.$$

Yamazaki's simplified proof of Theorem 1.7 in [10] is not applicable in our context. The method of our proof is the same as Tanahashi's argument, whose outline is explained in the previous section. We would like to emphasize there are several branching points such that the conditions about parameters in the assumption are to be reflected to powers or coefficients in calculations. Moreover, we have to estimate all terms up to the order of x^{-2} in the sequel. On the other hand, in the existing literature, such as [7, 8] and [11], it is sufficient to estimate only main and second terms.

Proof As in the preliminaries, we set $\alpha = 1 - t + r$ and $\psi = (p - t)s + r$. Note that $0 < \alpha$ and $0 < \psi$. We will consider matrices

$$A = \begin{pmatrix} x^2 + 1 & x \\ x & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then we have $0 < B \le A$. The eigenvalues of A are $\frac{x^2+4\pm\sqrt{x^4+4}}{2}$. Let

$$c = \frac{x^2 - 2 + \sqrt{x^4 + 4}}{2x} \quad \text{and} \quad U = \frac{1}{\sqrt{c^2 + 1}}\begin{pmatrix} c & 1 \\ 1 & -c \end{pmatrix}.$$

Then U is unitary and $U^*AU = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, where

$$d_1 = \frac{x^2+4+\sqrt{x^4+4}}{2} \quad \text{and} \quad d_2 = \frac{x^2+4-\sqrt{x^4+4}}{2}.$$

Assume A and B satisfy the grand Furuta inequality (1). Then we have

$$\begin{aligned} &A_1 = d_1^{-t}\left(c^2+2^p\right), \qquad A_2 = d_2^{-t}\left(1+2^p c^2\right), \\ &A_3 = -d_1^{-\frac{t}{2}} d_2^{-\frac{t}{2}}\left(2^p-1\right)c, \qquad k = \frac{1}{c^2+1} \end{aligned}$$

by the notation of the previous section. It is easy to see that $d_1 \sim x^2$, $d_2 \sim 2$, $c \sim x$, $A_1 \sim x^{2-2t}$ and $A_2 \sim 2^{p-t}x^2$ as $x \to \infty$. Therefore, for sufficiently large x, we have $A_1 < A_2$. Further, we shall see later $B_1 \sim x^{2r+2+2(1-t)s}$, $B_2 \sim x^{2+2s}$ as $x \to \infty$. Since $ts < r$, we have $B_2 < B_1$ for sufficiently large x.

Now we estimate each term of the inequality (4) with respect to $x \to \infty$. The estimation of the factor $\gamma d_2^\alpha - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}$ in the right-hand side is a little delicate so that we need some calculation. Terms in other factors can be roughly estimated. As is a usual notation, $f(x) = o(x^\beta)$ means that

$$\frac{f(x)}{x^\beta} \to 0 \quad (x \to +\infty).$$

One can establish the following formulas:

$$\begin{aligned} &\sqrt{x^4+4} = x^2 + \frac{2}{x^2} + o\left(x^{-2}\right), \\ &d_1^\alpha = x^{2\alpha}\left(1 + \frac{2\alpha}{x^2} + o\left(x^{-2}\right)\right), \qquad d_2^\alpha = 2^\alpha\left(1 - \frac{\alpha}{2x^2} + o\left(x^{-2}\right)\right), \\ &c = x\left(1 - \frac{1}{x^2} + o\left(x^{-2}\right)\right), \qquad \left(c^2+1\right)^s = x^{2s}\left(1 - \frac{s}{x^2} + o\left(x^{-2}\right)\right). \end{aligned}$$

Next

$$\begin{aligned} &A_1 = x^{2-2t}\left(1 + \frac{2^p-2-2t}{x^2} + o\left(x^{-2}\right)\right), \\ &A_2 = 2^{p-t}x^2\left(1 + \frac{-2^{p+1}+2^{p-1}t+1}{2^p x^2} + o\left(x^{-2}\right)\right), \\ &A_3^2 = 2^{-t}\left(2^p-1\right)^2 x^{2-2t}\left(1 - \frac{3t+4}{2x^2} + o\left(x^{-2}\right)\right), \end{aligned}$$

hence

$$\begin{aligned} &A_2 - A_1 = 2^{p-t}x^2\left(1 - \frac{2^{-p+t}}{x^{2t}} + \frac{-2^{p+1}+2^{p-1}t+1}{2^p x^2} + o\left(x^{-2}\right)\right), \\ &(A_2 - A_1)\left(\frac{4A_3^2}{(A_1-A_2)^2}\right)^2 \sim \frac{1}{x^{2+4t}} = x^{-2t}o\left(x^{-2}\right) \end{aligned}$$

and

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}(A_2 - A_1)\left(-1 + \sqrt{1 + \frac{4A_3^2}{(A_1 - A_2)^2}}\right) \\
&= \frac{1}{2}(A_2 - A_1)\left\{\binom{\frac{1}{2}}{1}\frac{4A_3^2}{(A_1 - A_2)^2} + \binom{\frac{1}{2}}{2}\left(\frac{4A_3^2}{(A_1 - A_2)^2}\right)^2 + \cdots\right\} \\
&= \frac{A_3^2}{A_2 - A_1} + x^{-2t}o\left(x^{-2}\right) \\
&= 2^{-t}\left(2^p - 1\right)^2 x^{2-2t}\left(1 + o(1)\right)\left(2^{p-t}x^2\left(1 + o(1)\right)\right)^{-1} \\
&= 2^{-p}\left(2^p - 1\right)^2 x^{-2t}\left(1 + o(1)\right).
\end{aligned}$$

Since ε_1 is small, the estimations of $-A_1 + A_2 + 2\varepsilon_1$ and $-A_1 + A_2 + \varepsilon_1$ (resp. $A_2 + \varepsilon_1$) in these details are the same as $A_2 - A_1$ (resp. A_2), and the main term of $A_1 - \varepsilon_1$ is the same as that of A_1. Hence

$$\varepsilon_1(A_1 - \varepsilon_1)^s = 2^{-p}\left(2^p - 1\right)^2 x^{-2t+2(1-t)s}\left(1 + o(1)\right) = x^{2(s+1)}o\left(x^{-2}\right)$$

and

$$\begin{aligned}
&(-A_1 + A_2 + \varepsilon_1)(A_2 + \varepsilon_1)^s + \varepsilon_1(A_1 - \varepsilon_1)^s \\
&\quad = 2^{(p-t)(s+1)}x^{2(s+1)}\left(1 - \frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1)}{2^p x^2} + o\left(x^{-2}\right)\right).
\end{aligned}$$

Further, since $0 < s < 1$, $0 < t < 1 + r$, and $ts < r$,

$$\begin{aligned}
B_1 &= 2^{p-t}x^{2+2r+2(1-t)s}\left(1 + o(1)\right), \\
B_2 &= 2^{r+(p-t)(s+1)}x^{2(s+1)} \\
&\quad \cdot \left(1 - \frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2} + o\left(x^{-2}\right)\right), \\
B_3^2 &= 2^{r-t+2(p-t)s}\left(2^p - 1\right)^2 x^{2+2r-2t+4s}\left(1 + o(1)\right).
\end{aligned}$$

The main term of $B_1 - B_2$ is the same as B_1, hence

$$\begin{aligned}
&\frac{B_3^2}{(B_1 - B_2)^2} \sim \frac{1}{x^{2+2r+2t-4ts}}, \\
&(B_1 - B_2)\left(\frac{4B_3^2}{(B_1 - B_2)^2}\right)^2 \sim \frac{1}{x^{2+2r+4t-2s-6ts}} = x^{-2t+2s+2ts}o\left(x^{-2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2}(B_1 - B_2)\left(-1 + \sqrt{1 + \frac{4B_3^2}{(B_1 - B_2)^2}}\right) \\
&= \frac{1}{2}(B_1 - B_2)\left\{\binom{\frac{1}{2}}{1}\frac{4B_3^2}{(B_1 - B_2)^2} + \binom{\frac{1}{2}}{2}\left(\frac{4B_3^2}{(B_1 - B_2)^2}\right)^2 + \cdots\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{B_3^2}{B_1 - B_2} + x^{-2t+2s+2ts} o\left(x^{-2}\right) \\
&= 2^{r-t+2(p-t)s}\left(2^p - 1\right)^2 x^{2+2r-2t+4s}\left(1 + o(1)\right) \cdot \left(2^{p-t} x^{2+2r+2(1-t)s}\left(1 + o(1)\right)\right)^{-1} \\
&= 2^{r-p+2(p-t)s}\left(2^p - 1\right)^2 x^{-2t+2s+2ts}\left(1 + o(1)\right),
\end{aligned}
$$

so the estimation of $B_2 - \varepsilon_2$ is the same as B_2 in its detail, where the assumption $0 < s < 1$ is crucial. Hence

$$
\begin{aligned}
&(B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \\
&= \left\{2^{r+(p-t)(s+1)} x^{2(s+1)}\left(1 - \frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2} + o\left(x^{-2}\right)\right)\right\}^{\frac{\alpha}{\psi}} \\
&= 2^{\{r+(p-t)(s+1)\}\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}} \\
&\quad \cdot \left\{1 + \frac{\alpha}{\psi}\left(-\frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2}\right)\right. \\
&\quad + \binom{\frac{\alpha}{\psi}}{2}\left(-\frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2}\right)^2 \\
&\quad \left. + \binom{\frac{\alpha}{\psi}}{3}\left(-\frac{2^{-p+t}}{x^{2t}} + \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2}\right)^3 + \cdots + o\left(x^{-2}\right)\right\} \\
&= 2^{\{r+(p-t)(s+1)\}\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}} \\
&\quad \cdot \left\{1 - \frac{\alpha}{\psi} \cdot \frac{2^{-p+t}}{x^{2t}} + \binom{\frac{\alpha}{\psi}}{2} \cdot \frac{2^{2(-p+t)}}{x^{4t}} - \binom{\frac{\alpha}{\psi}}{3} \cdot \frac{2^{3(-p+t)}}{x^{6t}} + \cdots + (-1)^k \binom{\frac{\alpha}{\psi}}{k} \cdot \frac{2^{k(-p+t)}}{x^{2kt}}\right. \\
&\quad \left. + \frac{\alpha}{\psi} \cdot \frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2} + o\left(x^{-2}\right)\right\},
\end{aligned}
$$

where k is the nonnegative integer determined by $kt \le 1 < (k+1)t$.

The main term of $B_1 + \varepsilon_2$ is obviously the same as that of B_1 and

$$
(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} = \left(2^{p-t} x^{2+2r+2(1-t)s}\right)^{\frac{\alpha}{\psi}}\left(1 + o(1)\right).
$$

Furthermore,

$$
\begin{aligned}
\gamma &= \left\{x^{2s}\left(1 - \frac{s}{x^2} + o\left(x^{-2}\right)\right) \cdot 2^{p-t} x^2\left(1 - \frac{2^{-p+t}}{x^{2t}} + \frac{-2^{p+1} + 2^{p-1}t + 1}{2^p x^2} + o\left(x^{-2}\right)\right)\right\}^{\frac{\alpha}{\psi}} \\
&= 2^{(p-t)\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}} \\
&\quad \cdot \left\{1 - \frac{\alpha}{\psi} \cdot \frac{2^{-p+t}}{x^{2t}} + \binom{\frac{\alpha}{\psi}}{2} \cdot \frac{2^{2(-p+t)}}{x^{4t}} - \binom{\frac{\alpha}{\psi}}{3} \cdot \frac{2^{3(-p+t)}}{x^{6t}} + \cdots + (-1)^k \binom{\frac{\alpha}{\psi}}{k} \cdot \frac{2^{k(-p+t)}}{x^{2kt}}\right. \\
&\quad \left. + \frac{\alpha}{\psi} \cdot \frac{-2^{p+1} + 2^{p-1}t + 1 - 2^p s}{2^p x^2} + o\left(x^{-2}\right)\right\}.
\end{aligned}
$$

For the following four factors in the formula (4), it is sufficient to estimate their main terms only,

$$
\begin{aligned}
&\gamma d_1^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} = 2^{(p-t)\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi} + 2\alpha}\left(1 + o(1)\right), \\
&(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma d_2^{\alpha} = \left(2^{p-t} x^{2+2r+2(1-t)s}\right)^{\frac{\alpha}{\psi}}\left(1 + o(1)\right),
\end{aligned}
$$

$$B_1 - B_2 + \varepsilon_2 = 2^{p-t} x^{2+2r+2(1-t)s}\bigl(1 + o(1)\bigr),$$
$$\gamma d_1^\alpha - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} = 2^{(p-t)\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}+2\alpha}\bigl(1 + o(1)\bigr).$$

Now we have the estimation of the most delicate factor in the formula (4), whose main terms are canceled by subtraction. We have

$$\begin{aligned}
&\gamma d_2^\alpha - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \\
&= 2^{(p-t)\frac{\alpha}{\psi}+\alpha} x^{2(s+1)\frac{\alpha}{\psi}} \\
&\quad \cdot \left\{1 - \frac{\alpha}{\psi}\cdot\frac{2^{-p+t}}{x^{2t}} + \binom{\frac{\alpha}{\psi}}{2}\cdot\frac{2^{2(-p+t)}}{x^{4t}} - \binom{\frac{\alpha}{\psi}}{3}\cdot\frac{2^{3(-p+t)}}{x^{6t}} + \cdots + (-1)^k\binom{\frac{\alpha}{\psi}}{k}\cdot\frac{2^{k(-p+t)}}{x^{2kt}}\right. \\
&\quad \left. + \frac{\alpha}{\psi}\cdot\frac{-2^{p+1} + 2^{p-1}t + 1 - 2^p s - 2^{p-1}\psi}{2^p x^2} + o\bigl(x^{-2}\bigr)\right\} \\
&\quad - 2^{\{r+(p-t)(s+1)\}\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}} \\
&\quad \cdot \left\{1 - \frac{\alpha}{\psi}\cdot\frac{2^{-p+t}}{x^{2t}} + \binom{\frac{\alpha}{\psi}}{2}\cdot\frac{2^{2(-p+t)}}{x^{4t}} - \binom{\frac{\alpha}{\psi}}{3}\cdot\frac{2^{3(-p+t)}}{x^{6t}} + \cdots + (-1)^k\binom{\frac{\alpha}{\psi}}{k}\cdot\frac{2^{k(-p+t)}}{x^{2kt}}\right. \\
&\quad \left. + \frac{\alpha}{\psi}\cdot\frac{(s+1)(-2^{p+1} + 2^{p-1}t + 1) - 2^{p-1}r}{2^p x^2} + o\bigl(x^{-2}\bigr)\right\} \\
&= 2^{(p-t)\frac{\alpha}{\psi}+\alpha} x^{2(s+1)\frac{\alpha}{\psi}} \frac{\alpha}{\psi}\left\{\frac{s(-2^{p-1}p + 2^p - 1)}{2^p x^2} + o\bigl(x^{-2}\bigr)\right\}.
\end{aligned}$$

Applying these estimations to the inequality (4), we can obtain

$$\begin{aligned}
&2^{r-p+2(p-t)s}\bigl(2^p - 1\bigr)^2 x^{-2t+2s+2ts}\cdot 2^{(p-t)\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}+2\alpha} \\
&\quad \cdot \bigl(2^{p-t} x^{2+2r+2(1-t)s}\bigr)^{\frac{\alpha}{\psi}}\bigl(1 + o(1)\bigr) \\
&\leq 2^{p-t} x^{2+2r+2(1-t)s}\cdot 2^{(p-t)\frac{\alpha}{\psi}} x^{2(s+1)\frac{\alpha}{\psi}+2\alpha} \\
&\quad \cdot 2^{(p-t)\frac{\alpha}{\psi}+\alpha} x^{2(s+1)\frac{\alpha}{\psi}} \frac{\alpha}{\psi}\left\{\frac{s(-2^{p-1}p + 2^p - 1)}{2^p x^2} + o\bigl(x^{-2}\bigr)\right\},
\end{aligned}$$

and hence

$$\begin{aligned}
&2^{r-p+2(p-t)s}\bigl(2^p - 1\bigr)^2\bigl(1 + o(1)\bigr) \\
&\quad \leq 2^{p-t} 2^\alpha x^{2r+2(1-t)s} x^{2(s+1)\frac{\alpha}{\psi}} x^{2t-2s-2ts} x^{-\{2+2r+2(1-t)s\}\frac{\alpha}{\psi}} \\
&\quad \cdot \frac{\alpha s(-2^{p-1}p + 2^p - 1)}{2^p\psi}\bigl(1 + o(1)\bigr).
\end{aligned} \tag{5}$$

The power of x in the right-hand may be positive, 0 or negative. However, by Lemma 3.1, the assumption $1 < p$ implies that $-2^{p-1}p + 2^p - 1 < 0$. By letting $x \to \infty$ in (5), we have

$$0 < 2^{r-p+2(p-t)s}\bigl(2^p - 1\bigr)^2 < -\infty$$

or

$$0 < 2^{r-p+2(p-t)s}\bigl(2^p - 1\bigr)^2 \leq 0.$$

This is a contradiction and completes the proof of Theorem 3.2. □

Corollary 3.3 *Let $1 < p$, $0 < s < 1$, $0 < t \leq 1$, and $t \leq r$. Then there exist 2×2 matrices A, B with $0 < B \leq A$ that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}.$$

Proof It is obvious that $0 < t < 1 + r$ and $ts < r$. □

Competing interests
The author declares that he has no competing interests.

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