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The boundedness of Fourier transform on the Herz type amalgams and Besov spaces

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Abstract

In this paper, using the Young inequality as regards the Herz space, the authors obtain the boundedness of the Fourier transform on the Herz type amalgams and Besov spaces. As corollaries, the authors get the estimates of the Fourier transform on the weighted amalgams and Besov spaces.

MSC: 42B10; 42B35

Keywords: Fourier transform; Herz-Besov space; Herz-amalgam space; weighted

1 Introduction

One of the central problems in classic harmonic analysis is the study of the boundedness of Fourier transform on a given function space. A basic result is the Hausdorff-Young inequality and its various extensions and generalizations. Here and below we define the Fourier transform of $f \in L^1 \cap L^p$ to be

$$\mathcal{F}f(x) = \int_{\mathbf{R}^n} f(y) e^{-iyx}\,dy.$$

When $1 \le p \le 2$, $\mathcal{F}$ is bounded from L^p to $L^{p'}$. If the spaces are weighted Lebesgue spaces or Herz spaces, the following results are proved (see [1–4]).

Theorem A *If $1 < p \le 2$ and $0 \le \alpha < \frac{1}{p'}$, then*

$$\left(\int_{\mathbf{R}^n} |\mathcal{F}f|^{p'} |x|^{-\alpha n p'}\,dx\right)^{\frac{1}{p'}} \le c\left(\int_{\mathbf{R}^n} |f|^p |x|^{\alpha n p}\,dx\right)^{\frac{1}{p}}.$$

Theorem B *If $1 < p \le 2$ and $0 \le \alpha < \frac{1}{p'}$, then*

$$\|\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}} \le c\|f\|_{\dot{K}_p^{\alpha,p}}.$$

Here $\dot{K}_q^{\alpha,p}$ denotes a Beurling-Herz space.

In [5], the authors give a result as regards the Young theorem for amalgams and Besov spaces.

Theorem C (1) *Let $1 \le p \le 2$, $0 < q \le \infty$ and $s \in \mathbf{R}$. Then*

$$\mathcal{F} : \left(L^p, l^q\left(\langle z\rangle^s\right)\right) \to B_{p',q}^{s-n(\frac{1}{p}-\frac{1}{q})_+}.$$

(2) *Let* $1 \le p \le 2$, $0 < q \le \infty$ *and* $s \in \mathbf{R}$. *Then*

$$\mathcal{F} : B^s_{p,q} \to \left(L^{p'}, l^q\left(\langle z\rangle^{s-n(\frac{1}{q}-\frac{1}{p'})_+}\right)\right).$$

Inspired by Theorem B and Theorem C, in this paper, we discuss the boundedness as regards the Fourier transform on the Herz type amalgams and the Besov spaces. In Section 2, we give the definition of function spaces. The main theorems and their proofs are contained in Section 3.

2 Function spaces

In this section, we give the definition of function spaces that we work on. Let $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ be a Schwartz space and its dual space.

Definition 2.1 (Weighted Lebesgue spaces) Let $\omega(x)$ be a nonnegative function, $L^p(\omega) = \{f(x) \mid \|f\|_{L^p(w)} < \infty\}$, where

$$\|f\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |f|^p \omega(x)\,dx\right)^{\frac{1}{p}}.$$

Now we give the definitions as regards Herz spaces.

Definition 2.2 Suppose $-\infty < \alpha < \infty$, $0 < p \le \infty$, $0 < q \le \infty$. The Herz space $\dot{K}^{\alpha,p}_q$ is defined by

$$\dot{K}^{\alpha,p}_q = \left\{f \in L^q_{\mathrm{loc}}\left(\mathbf{R}^n\backslash\{0\}\right) : \|f\|_{K^{\alpha,p}_q} < \infty\right\},$$

where $\mathcal{X}_{B_k}(x) = \mathcal{X}_{C_k - C_{k-1}}(x)$, $C_k = \{x \in \mathbf{R}^n : |x| \le 2^k\}$,

$$\|f\|_{\dot{K}^{\alpha,p}_q} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|f\chi_{B_k}\|^q_{L^p}\right\}^{1/q}.$$

The usual modification should be made when $p = \infty$ or $q = \infty$ (see [6]).

In order to define the Herz type Besov space, we need the following Littlewood-Paley function (see [5, 7]).

Definition 2.3 Let $\phi_0, \phi_1 \in \mathcal{S}$ be even functions satisfying the following condition:

$$\mathcal{X}_{[-2,2]^n} \le \phi_0 \le \mathcal{X}_{[-4,4]^n}, \qquad \mathcal{X}_{[-4,4]^n-[-2,2]^n} \le \phi_1 \le \mathcal{X}_{[-8,8]^n-[-1,1]^n}.$$

We set $\phi_j(x) = \phi_1(2^{-j+1}x)$ for $j \ge 2$.

For $f \in \mathcal{S}'$, we denote $\phi_j(D)f = \mathcal{F}^{-1}(\phi_j\mathcal{F}f)$. Now let us introduce the Herz type Besov spaces (see [8]).

Definition 2.4 Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $1 \le q < \infty$, and $-\infty < s < \infty$, $0 < r \le \infty$; then we define

$$\dot{K}^{\alpha,p}_q B^s_r = \left\{f \in S' \,\middle|\, \|f\|_{\dot{K}^{\alpha,p}_q B^s_r} = \left\{\sum_{i=0}^{\infty} 2^{isr}\left\|\phi_j(D)f\right\|^r_{\dot{K}^{\alpha,p}_q}\right\}^{1/r} < \infty\right\}.$$

Obviously, the definition of the Herz-Besov space is independent of the choice of ϕ_0, ϕ_1 and $\dot{K}_p^{0,p}B_r^s = B_{p,r}^s$, standard Besov spaces. More information as regards the Besov space and Herz type Besov spaces can be found in [7, 9–12].

Definition 2.5 (Herz type amalgam space) Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $1 \le q < \infty$, $s \in \mathbf{R}$, and $0 < r \le \infty$. Set $Q_z = z + [0,1]^n$, the translation of the unit cube. For a Lebesgue locally integrable function f we define

$$\|f\|_{(\dot{K}_q^{\alpha,p}, l^r(\langle z\rangle^s))} = \bigl\| \bigl\{ \langle z\rangle^s \|f\mathcal{X}_{Q_z}\|_{\dot{K}_q^{\alpha,p}} \bigr\} \bigr\|_{l^r},$$

where $\langle z\rangle = \sqrt{|z|^2+1}$. $(\dot{K}_q^{\alpha,p}, l^r(\langle z\rangle^s))$ is the set of all locally integrable functions f for which the quasi-norm $\|f\|_{(\dot{K}_q^{\alpha,p}, l^r(\langle z\rangle^s))} < \infty$.

If $\alpha = 0$ and $p = q$ in Definition 2.5, then the Herz type amalgam space is also an amalgam space $(L^p, l^r(\langle z\rangle^s))$ (see [5, 13, 14]).

3 Main theorems and proofs

In this section we formulate our main theorems and proofs.

Theorem 3.1 *Let $1 < p \le 2$, $0 < q \le \infty$, $0 \le \alpha < \frac{1}{p'}$ and $s \in \mathbf{R}$. Then the Fourier transform is bounded from the Herz type amalgam space $(\dot{K}_p^{\alpha,p}, l^q(\langle z\rangle^s))$ to the Herz type Besov space $\dot{K}_2^{-\alpha,p'}B_q^{s-n(\frac{1}{p}-\frac{1}{q})_+}$. Here and below, for $a \in \mathbf{R}$ we write $a_+ = \max\{a, 0\}$.*

Proof of Theorem 3.1 If we note that the lift operator $(I - \triangle)^{\frac{t}{2}}$ is bounded from $\dot{K}_u^{\alpha,p}B_q^s$ to $\dot{K}_u^{\alpha,p}B_q^{s-t}$ (see [8]) and the following multiplication operator is an isomorphism:

$$f \in \bigl(\dot{K}_q^{\alpha,p}, l^r\bigl(\langle z\rangle^s\bigr)\bigr) \mapsto \langle\cdot\rangle^t \cdot f \in \bigl(\dot{K}_q^{\alpha,p}, l^r\bigl(\langle z\rangle^{s-t}\bigr)\bigr),$$

we can assume $s = 0$.

Let $A_j = \operatorname{supp}(\phi_j)$, $j \in \mathbf{N}_0$, where ϕ_j is the Littlewood-Paley function. Using the Young inequality $\|\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}} \le c\|f\|_{\dot{K}_p^{\alpha,p}}$, we obtain

$$\begin{aligned}
\bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\phi_j(D)\mathcal{F}f\bigr\|_{\dot{K}_2^{-\alpha,p'}} &\le \bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\phi_j\mathcal{F}\mathcal{F}f\bigr\|_{\dot{K}_p^{\alpha,p}} \\
&\le c\bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\phi_j f(-x)\bigr\|_{\dot{K}_p^{\alpha,p}} \\
&\le c\bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\mathcal{X}_{A_j}f\bigr\|_{\dot{K}_p^{\alpha,p}}.
\end{aligned}$$

But

$$\begin{aligned}
\bigl\|\mathcal{X}_{A_j}f(x)\bigr\|_{\dot{K}_p^{\alpha,p}}^q &\le \Biggl(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\mathcal{X}_{B_k}\mathcal{X}_{A_j}f\|_p^p\Biggr)^{\frac{q}{p}} \\
&\le c\Biggl(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{z\in\mathbf{Z}^n}\|\mathcal{X}_{B_k}\mathcal{X}_{Q_z}\mathcal{X}_{A_j}f\|_p^p\Biggr)^{\frac{q}{p}}
\end{aligned}$$

$$\begin{aligned}
&\le c\Biggl(\sum_{z\in\mathbf{Z}^n}\Biggl(\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|\mathcal{X}_{B_k}\mathcal{X}_{Q_z}\mathcal{X}_{A_j}f\|_p^p\Biggr)\Biggr)^{\frac{q}{p}}\\
&\le c\Biggl(\sum_{|z|\backsim 2^{jn}}\|\mathcal{X}_{Q_z}\mathcal{X}_{A_j}f\|_{\dot K_p^{\alpha,p}}^p\Biggr)^{\frac{q}{p}}\\
&\le c2^{jn(\frac{q}{p}-1)_+}\sum_{|z|\backsim 2^{jn}}\|\mathcal{X}_{Q_z}\mathcal{X}_{A_j}f\|_{\dot K_p^{\alpha,p}}^q,
\end{aligned}$$

where we use the inequality $(\sum_{j=1}^{L}|a_j|)^{\frac{q}{p}}\le L^{(\frac{q}{p}-1)_+}\sum_{j=1}^{L}|a_j|^{\frac{q}{p}}$ (see [5]).

If we put these estimates together, we have

$$\begin{aligned}
\bigl\|\bigl\{\bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\phi_j(D)\mathcal{F}f\bigr\|_{\dot K_2^{-\alpha,p'}}\bigr\}\bigr\|_{l^q}&\le c\Biggl\|\Biggl\{\Biggl(\sum_z\|\mathcal{X}_{A_j}\mathcal{X}_{Q_z}f\|_{\dot K_p^{\alpha,p}}^q\Biggr)^{\frac{1}{q}}\Biggr\}\Biggr\|_{l^q}\\
&\le c\Biggl(\sum_z\sum_j\|\mathcal{X}_{A_j}\mathcal{X}_{Q_z}f\|_{\dot K_p^{\alpha,p}}^q\Biggr)^{\frac{1}{q}}.
\end{aligned}$$

Given $z\in\mathbf{Z}^n$, from the definition of the ϕ_j, there are at most three j such that $A_j\cap Q_z\neq\emptyset$. So we have

$$\begin{aligned}
\bigl\|\bigl\{\bigl\|2^{-jn(\frac{1}{p}-\frac{1}{q})_+}\phi_j(D)\mathcal{F}f\bigr\|_{\dot K_2^{-\alpha,p'}}\bigr\}\bigr\|_{l^q}&\le c\Biggl(\sum_z\sum_j\|\mathcal{X}_{A_j}\mathcal{X}_{Q_z}f\|_{\dot K_p^{\alpha,p}}^q\Biggr)^{\frac{1}{q}}\\
&\le c\Biggl(\sum_z\|\mathcal{X}_{Q_z}f\|_{\dot K_p^{\alpha,p}}^q\Biggr)^{\frac{1}{q}}\\
&\le c\|f\|_{(\dot K_p^{\alpha,p},l^q(\langle z\rangle^s))}.
\end{aligned}$$

This is the desired result. □

Theorem 3.2 *Let $1<p\le 2$, $0<q\le\infty$, $0\le\alpha<\frac{1}{p'}$ and $s\in\mathbf{R}$. Then the Fourier transform is bounded from the Herz type Besov space $\dot K_p^{\alpha,p}B_q^s$ to the Herz type amalgams space $(\dot K_2^{-\alpha,p'},l^q(\langle z\rangle^{s-n(\frac{1}{q}-\frac{1}{p'})_+}))$.*

Proof of Theorem 3.2 As before, we assume $s=0$. Let $|z|_\infty=\max\{|z_1|,\dots,|z_n|\}$. By the definition of the Herz type amalgam space, we have

$$\|\mathcal{F}f\|^q_{(\dot K_2^{-\alpha,p'},l^q(\langle z\rangle^{-n(\frac{1}{q}-\frac{1}{p'})_+}))}\backsim\Biggl\|\Biggl\{2^{-jnq(\frac{1}{q}-\frac{1}{p'})_+}\sum_{|z|_\infty\backsim 2^{jn}}\|\mathcal{F}f\|^q_{\dot K_2^{-\alpha,p'}}\Biggr\}\Biggr\|_{l^1}.$$

When $|z|_\infty\backsim 2^j$, from the definition of B_k, there are at most three k ($k=j-1,j,j+1$) such that $Q_z\cap B_k\neq\emptyset$. If $|z_1|_\infty,\dots,|z_i|_\infty\backsim 2^j$ and $Q_{z_1}\cap\cdots\cap Q_{z_i}=\emptyset$, we have

$$\begin{aligned}
&\|\mathcal{X}_{Q_{z_1}}\mathcal{F}f\|^{p'}_{\dot K_2^{-\alpha,p'}}+\cdots+\|\mathcal{X}_{Q_{z_i}}\mathcal{F}f\|^{p'}_{\dot K_2^{-\alpha,p'}}\\
&\quad\le c\sum_{k=j-1}^{j+1}2^{-k\alpha p'}\|\mathcal{X}_{Q_{z_1}}\mathcal{X}_{B_k}\mathcal{F}f\|_{p'}^{p'}+\cdots+c\sum_{k=j-1}^{j+1}2^{-k\alpha p'}\|\mathcal{X}_{Q_{z_i}}\mathcal{X}_{B_k}\mathcal{F}f\|_{p'}^{p'}
\end{aligned}$$

$$= c\sum_{k=j-1}^{j+1} 2^{-k\alpha p'} \|\mathcal{X}_{Q_{z_1}\cup\cdots\cup Q_{z_i}}\mathcal{X}_{B_k}\mathcal{F}f\|_{p'}^{p'}$$

$$= c\|\mathcal{X}_{Q_{z_1}\cup\cdots\cup Q_{z_i}}\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}^{p'}.$$

Using the above inequality and $\sum_{i=1}^{L}|a_i|^{\frac{q}{p'}} \le L^{(1-\frac{q}{p'})_+}(\sum_{i=1}^{L}|a_i|)^{\frac{q}{p'}}$ (see [5]) we have

$$2^{-jnq(\frac{1}{q}-\frac{1}{p'})_+}\sum_{|z|_\infty\backsim 2^{jn}}\|\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}^{q} \le 2^{-jnq(\frac{1}{q}-\frac{1}{p'})_+}\sum_{|z|_\infty\backsim 2^{jn}}\|\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}^{p'\frac{q}{p'}}$$

$$\le \Big(\sum_{|z|_\infty\backsim 2^{jn}}\|\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}^{p'}\Big)^{\frac{q}{p'}} \backsim \|\mathcal{X}_{B_j}\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}^{q}.$$

So

$$\begin{aligned}
\|\mathcal{F}f\|_{(\dot{K}_2^{-\alpha,p'},l^q(\langle z\rangle^{-n(\frac{1}{q}-\frac{1}{p'})_+}))} &\le \big\|\big\{\|\mathcal{X}_{B_j}\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}\big\}\big\|_{l^q} \\
&\le c\big\|\big\{\|\phi_j\mathcal{F}f\|_{\dot{K}_2^{-\alpha,p'}}\big\}\big\|_{l^q} \\
&\le c\big\|\big\{\|\phi_j(D)f\|_{\dot{K}_p^{\alpha,p}}\big\}\big\|_{l^q} \\
&\le c\|f\|_{\dot{K}_p^{\alpha,p}B_q^0}.
\end{aligned}$$

This proves Theorem 3.2. □

We note that $\dot{K}_p^{\alpha,p} = L^p(|x|^\alpha)$ (see [2]) and $\dot{K}_2^{\alpha,p} \subset \dot{K}_q^{\alpha,p}$, $q \ge 2$. Then we have the following.

Corollary 3.3 *Let $1 < p \le 2$, $0 < q \le \infty$, $0 \le \alpha < \frac{1}{p'}$, and $s \in \mathbf{R}$. Then the Fourier transform is bounded from the weighted amalgam space $(L^p(|x|^\alpha), l^q(\langle z\rangle^s))$ to the weighted Besov space $B_{p',q}^{s-n(\frac{1}{p}-\frac{1}{q})_+}(|x|^{-\alpha})$.*

Corollary 3.4 *Let $1 < p \le 2$, $0 < q \le \infty$, $0 \le \alpha < \frac{1}{p'}$, and $s \in \mathbf{R}$. Then the Fourier transform is bounded from the weighted Besov space $B_{p,q}^s(|x|^\alpha)$ to the weighted amalgam space $(L^{p'}(|x|^{-\alpha}), l^q(\langle z\rangle^{s-n(\frac{1}{q}-\frac{1}{p'})_+}))$.*

If $q = p$ and $s = 0$ in Theorem 3.1, we have the following.

Corollary 3.5 *Let $1 < p \le 2$ and $0 \le \alpha < \frac{1}{p'}$. Then the Fourier transform is bounded from the Herz space $\dot{K}_p^{\alpha,p}$ to the Herz type Besov space $\dot{K}_2^{-\alpha,p'}B_p^0$.*

Next we deduce some necessary condition as regards the boundedness of the Fourier transform.

Proposition 3.6 *Let $1 \le p_1 \le \infty$, $0 < p_2, q_1, q_2, u_1, u_2 \le \infty$, $\alpha_1 \ge 0$, and $\alpha_2, s_1, s_2 \in \mathbf{R}$. If*

$$\|\mathcal{F}f\|_{\dot{K}_{u_2}^{\alpha_2,p_2}B_{q_2}^{s_2}} \le c\|f\|_{(\dot{K}_{u_1}^{\alpha_1,p_1},l^{q_1}(\langle z\rangle^{s_1}))},$$

then $p_1 \ge p_2'$.

Proof of Proposition 3.6 Let $\theta \in \mathbf{S}$ be an even function with $\mathcal{X}_{B(0,\frac{1}{4})} \le \theta \le \mathcal{X}_{B(0,\frac{1}{2})}$. We take $f(x) = |x|^{-\alpha_1}\theta$, $g(x) = |x|^{-\alpha_1}(1-\theta)$. So $\mathcal{F}(f+g)(x) = c|x|^{\alpha_1 - n}$. Since $|x|^{2\nu}\mathcal{F}g(x) = \mathcal{F}((-\Delta)^\nu g)(x) < c$ ($\nu \gg 1$), it follows that $|\mathcal{F}g(x)| \le c|x|^{-2\nu}$. From this we have $|\mathcal{F}f(x)| \backsim |x|^{\alpha_1 - n}$ ($x \to \infty$).

Since $\operatorname{supp} f \subset B(0,\frac{1}{2})$, we have $\|f\|_{(\dot{K}_{u_1}^{\alpha_1,p_1}, l^{q_1}(\langle z\rangle^{s_1}))} \backsim \|f\|_{\dot{K}_{u_1}^{\alpha_1,p_1}}$. But $\|f\mathcal{X}_{B_k}\|_{p_1} \backsim 2^{\frac{k}{p_1} - \alpha_1 k}$. So

$$
\begin{aligned}
\|f\|_{\dot{K}_{u_1}^{\alpha_1,p_1}} &\backsim \left(\sum_{k=-\infty}^{1} 2^{k\alpha_1 u_1}\|f\mathcal{X}_{B_k}\|_{p_1}^{u_1}\right)^{\frac{1}{u_1}} \\
&\backsim \sum_{k=-\infty}^{1} 2^{ku_1\frac{1}{p_1}}.
\end{aligned}
$$

That is to say, $f \in (\dot{K}_{u_1}^{\alpha_1,p_1}, l^{q_1}(\langle z\rangle^{s_1}))$ if and only if $\alpha_1 p_1 < n$. $\|\mathcal{F}f\|_{\dot{K}_{u_2}^{\alpha_2,p_2}B_{q_2}^{s_2}} < \infty$ forces $p_2(n - a_1) > n$. From this it follows that $\frac{n}{p_1} \le n - \frac{n}{p_2}$, which is equivalent to $p_1 \ge p_2'$. □

Proposition 3.7 *Let* $1 \le p_1 \le \infty$, $0 < p_2, q_1, q_2, u_1, u_2 \le \infty$, $\alpha_1 \ge 0$, *and* $\alpha_2, s_1, s_2 \in \mathbf{R}$. *If*

$$
\|\mathcal{F}f\|_{\dot{K}_{u_2}^{\alpha_2,p_2}B_{q_2}^{s_2}} \le c\|f\|_{(\dot{K}_{u_1}^{\alpha_1,p_1}, l^{q_1}(\langle z\rangle^{s_1}))},
$$

then $s_2 \le \alpha_1 + \alpha_2 + s_1 + \frac{n}{q_1} - \frac{n}{p_2'}$.

Proof of Proposition 3.7 We estimate $\|\phi_j\|_{(\dot{K}_{u_1}^{\alpha_1,p_1}, l^{q_1}(\langle z\rangle^{s_1}))}$ and $\|\mathcal{F}\phi_j\|_{\dot{K}_{u_2}^{\alpha_2,p_2}B_{q_2}^{s_2}}$. We have

$$
\begin{aligned}
\|\phi_j\|_{(\dot{K}_{u_1}^{\alpha_1,p_1}, l^{q_1}(\langle z\rangle^{s_1}))} &= \left(\sum_z \langle z\rangle^{s_1 q_1}\|\phi_j\mathcal{X}_{Q_z}\|_{\dot{K}_{u_1}^{\alpha_1,p_1}}^{q_1}\right)^{\frac{1}{q_1}} \\
&= \left(\sum_{k\le j+1} 2^{ks_1q_1}\sum_{|z|\backsim 2^k}\|\phi_j\mathcal{X}_{Q_z}\|_{\dot{K}_{u_1}^{\alpha_1,p_1}}^{q_1}\right)^{\frac{1}{q_1}} \\
&\backsim \left(\sum_{k\le j+1} 2^{ks_1q_1}\sum_{|z|\backsim 2^k} 2^{k\alpha_1 q_1}\right)^{\frac{1}{q_1}} \\
&\backsim \left(\sum_{k\le j+1} 2^{ks_1q_1}2^{kn}2^{k\alpha_1q_1}\right)^{\frac{1}{q_1}} \backsim 2^{j(s_1+\alpha_1+\frac{n}{q_1})},
\end{aligned}
$$

$$
\begin{aligned}
\|\mathcal{F}\phi_j\|_{\dot{K}_{u_2}^{\alpha_2,p_2}B_{q_2}^{s_2}} &\backsim \left\|2^{js_2}\mathcal{F}\phi_j\right\|_{\dot{K}_{u_2}^{\alpha_2,p_2}} \\
&\sim 2^{js_2}2^{jn}\left\|\mathcal{F}\phi_1(2^{j-1}x)\right\|_{\dot{K}_{u_2}^{\alpha_2,p_2}} \\
&\sim 2^{js_2}2^{jn}2^{-j(\alpha_2+\frac{n}{p_2})}\left\|\mathcal{F}\phi_1(x)\right\|_{\dot{K}_{u_2}^{\alpha_2,p_2}} \\
&\sim 2^{j(s_2-\alpha_2+\frac{n}{p_2'})}.
\end{aligned}
$$

Here we use $\|f(\lambda x)\|_{\dot{K}_q^{\alpha,p}} = \lambda^{-(\alpha+\frac{n}{p})}\|f\|_{\dot{K}_q^{\alpha,p}}$ (see [15]); through $2^{j(s_2-\alpha_2+\frac{n}{p_2'})} \le 2^{j(s_1+\alpha_1+\frac{n}{q_1})}$ we have $s_2 \le \alpha_1 + \alpha_2 + s_1 + \frac{n}{q_1} - \frac{n}{p_2'}$. This completes the proof of Proposition 3.7. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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