Ma et al. *Journal of Inequalities and Applications* 2014, 2014:287
http://www.journalofinequalitiesandapplications.com/content/2014/1/287

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A regularity criterion for the Cahn-Hilliard-Boussinesq system with zero viscosity

Caochuan Ma[1], Faris Saeed Alzahrani[2], Tasawar Hayat[3,2] and Yong Zhou[4,2*]

*Correspondence: yzhou@mail.shufe.edu.cn
[2]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia
[4]School of Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, P.R. China
Full list of author information is available at the end of the article

Abstract
This paper studies a coupled Cahn-Hilliard-Boussinesq system with zero viscosity. We prove a regularity criterion in terms of vorticity in the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$.

MSC: 35Q30; 76D03; 76D05; 76D07

Keywords: regularity criterion; Cahn-Hilliard-Boussinesq; zero viscosity

1 Introduction

In this paper, we study the following Cahn-Hilliard-Boussinesq system with zero viscosity [1]:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\phi + \theta e_3, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t\theta + u \cdot \nabla\theta = \Delta\theta, \tag{1.3}$$

$$\partial_t\phi + u \cdot \nabla\phi = \Delta\mu, \tag{1.4}$$

$$-\Delta\phi + f'(\phi) = \mu, \tag{1.5}$$

$$(u,\theta,\phi)(x,0) = (u_0,\theta_0,\phi_0)(x), \quad x \in \mathbb{R}^3, \tag{1.6}$$

with u the fluid velocity field, θ the temperature, ϕ the order parameter, and π the pressure, are the unknowns. $e_3 := (0,0,1)^t$. μ is the chemical potential. $f(\phi) := \frac{1}{4}(\phi^2-1)^2$ is the double well potential.

When $\theta = \phi \equiv 0$, (1.1) and (1.2) are the well-known Euler system; Kozono, Ogawa and Taniuchi [2] proved the following regularity criterion:

$$\omega := \operatorname{curl} u \in L^1(0,T;\dot{B}^0_{\infty,\infty}). \tag{1.7}$$

Here $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space.

When $\phi = 0$, (1.1), (1.2), and (1.3) are the well-known Boussinesq system with zero viscosity; Fan and Zhou [3] also showed the regularity criterion (1.7).

When $u = 0$, (1.4) and (1.5) are the well-known Cahn-Hilliard system.

It is easy to show that the problem (1.1)-(1.6) has a unique local smooth solution, thus we omit the details here. However, the global regularity is still open. The aim of this paper is to study the blow up criterion. We will prove the following.

Theorem 1.1 *Let $T > 0$ and $u_0 \in H^3$, $\theta_0 \in H^2$, $\phi_0 \in H^3$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. Suppose that (u, θ, ϕ) is a local smooth solution to the problem* (1.1)-(1.6). *Then (u, θ, ϕ) is smooth up to time T provided that* (1.7) *is satisfied.*

We will use the following logarithmic Sobolev inequality [2]:

$$\|\nabla u\|_{L^\infty} \leq C\bigl(1 + \|\operatorname{curl} u\|_{\dot{B}^0_{\infty,\infty}} \log\bigl(e + \|\Lambda^3 u\|_{L^2}\bigr)\bigr), \tag{1.8}$$

and the bilinear product and commutator estimates due to Kato and Ponce [4]:

$$\|\Lambda^s(fg)\|_{L^p} \leq C\bigl(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}\bigr), \tag{1.9}$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\bigl(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\bigr), \tag{1.10}$$

with $s > 0$, $\Lambda := (-\Delta)^{1/2}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

2 Proof of Theorem 1.1

First, it follows from (1.2), (1.3), and the maximum principle that

$$\|\theta\|_{L^\infty(0,T;L^\infty)} \leq C. \tag{2.1}$$

Testing (1.3) by θ, using (1.2), we see that

$$\frac{1}{2}\frac{d}{dt}\int \theta^2\,dx + \int |\nabla\theta|^2\,dx = 0,$$

whence

$$\|\theta\|_{L^\infty(0,T;L^2)} + \|\theta\|_{L^2(0,T;H^1)} \leq C. \tag{2.2}$$

Testing (1.4) by ϕ, using (1.2) and (1.5), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \phi^2\,dx + \int |\Delta\phi|^2\,dx \\
&\quad = \int f'(\phi)\Delta\phi\,dx = \int \bigl(\phi^3 - \phi\bigr)\Delta\phi\,dx \\
&\quad = -\int 3\phi^2|\nabla\phi|^2\,dx - \int \phi\Delta\phi\,dx \\
&\quad \leq -\int \phi\Delta\phi\,dx \leq \|\phi\|_{L^2}\|\Delta\phi\|_{L^2} \\
&\quad \leq \frac{1}{2}\|\Delta\phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\phi\|_{L^\infty(0,T;L^2)} + \|\phi\|_{L^2(0,T;H^2)} \leq C. \tag{2.3}$$

Testing (1.1) and (1.4) by u and μ, respectively, using (1.2), (1.5), and (2.2), summing up the result, we deduce that

$$\begin{aligned}&\frac{d}{dt}\int \frac{1}{2}|\nabla\phi|^2 + f(\phi) + \frac{1}{2}u^2\,dx + \int |\nabla\mu|^2\,dx \\ &\quad = \int \theta e_3 u\,dx \le \|\theta\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2},\end{aligned}$$

which gives

$$\|\phi\|_{L^\infty(0,T;H^1)} \le C, \tag{2.4}$$

$$\|u\|_{L^\infty(0,T;L^2)} \le C, \tag{2.5}$$

$$\|\nabla\mu\|_{L^2(0,T;L^2)} \le C. \tag{2.6}$$

In the following calculations, we will use the following Gagliardo-Nirenberg inequality:

$$\|\phi\|_{L^\infty}^2 \le C\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2}. \tag{2.7}$$

It follows from (2.6), (1.5), (2.3), (2.4), and (2.7) that

$$\begin{aligned}\int_0^T\int |\nabla\Delta\phi|^2\,dx\,dt &= \int_0^T\int \big|\nabla\big(f'(\phi)-\mu\big)\big|^2\,dx\,dt \\ &\le C\int_0^T\int |\nabla\mu|^2\,dx\,dt + C\int_0^T\int \big|\nabla f'(\phi)\big|^2\,dx\,dt \\ &\le C + C\int_0^T\int \big|\nabla\big(\phi^3-\phi\big)\big|^2\,dx\,dt \\ &\le C + C\int_0^T\int \phi^4|\nabla\phi|^2\,dx\,dt \\ &\le C + C\|\nabla\phi\|_{L^\infty(0,T;L^2)}^2\int_0^T \|\phi\|_{L^\infty}^4\,dt \\ &\le C + C\int_0^T \|\phi\|_{L^\infty}^4\,dt \\ &\le C + C\int_0^T \|\nabla\phi\|_{L^2}^2\|\Delta\phi\|_{L^2}^2\,dt \\ &\le C + C\sup_t\big\|\nabla\phi(t)\big\|_{L^2}^2\int_0^T \|\Delta\phi\|_{L^2}^2\,dt \le C,\end{aligned} \tag{2.8}$$

which yields

$$\|\nabla\phi\|_{L^2(0,T;L^\infty)} \le C. \tag{2.9}$$

Applying Λ^3 to (1.1), testing by $\Lambda^3 u$, using (1.2), (1.9), (1.10), and noting that

$$\Delta\phi\nabla\phi = \sum_j \partial_j(\partial_j\phi\nabla\phi) - \frac{1}{2}\nabla|\nabla\phi|^2,$$

we derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\left|\Lambda^3 u\right|^2 dx &= -\int\left(\Lambda^3(u\cdot\nabla u)-u\nabla\Lambda^3 u\right)\Lambda^3 u\,dx \\
&\quad -\int\sum_j \Lambda^3\partial_j(\partial_j\phi\nabla\phi)\cdot\Lambda^3 u\,dx+\int\Lambda^3\theta e_3\cdot\Lambda^3 u\,dx \\
&\leq C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 u\right\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}\left\|\Lambda^5\phi\right\|_{L^2}\left\|\Lambda^3 u\right\|_{L^2} \\
&\quad +C\left\|\Lambda^3\theta\right\|_{L^2}\left\|\Lambda^3 u\right\|_{L^2} \\
&\leq C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 u\right\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}^2\left\|\Lambda^3 u\right\|_{L^2}^2+C\left\|\Lambda^3 u\right\|_{L^2}^2 \\
&\quad +\epsilon\left\|\Lambda^5\phi\right\|_{L^2}^2+\epsilon\left\|\Lambda^3\theta\right\|_{L^2}^2
\end{aligned} \tag{2.10}$$

for any $0<\epsilon<1$.

Taking Δ to (1.3), testing by $\Delta\theta$, using (1.2), (1.10), and (2.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\Delta\theta|^2 dx+\int|\nabla\Delta\theta|^2 dx \\
&\quad =-\int\left(\Delta(u\cdot\nabla\theta)-u\nabla\Delta\theta\right)\Delta\theta\,dx \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^4}\|\Delta u\|_{L^4}\|\Delta\theta\|_{L^2} \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^{1/2}\|\Delta\theta\|_{L^2}^{1/2}\cdot\|\nabla u\|_{L^\infty}^{1/2}\|\nabla\Delta u\|_{L^2}^{1/2}\cdot\|\Delta\theta\|_{L^2} \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}^2+C\left(\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}+\|\nabla\Delta u\|_{L^2}\right)\|\Delta\theta\|_{L^2} \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}^2+C\|\Delta\theta\|_{L^2}^2+C\|\nabla\Delta u\|_{L^2}^2.
\end{aligned} \tag{2.11}$$

Here we have used the Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
&\|\nabla\theta\|_{L^4}^2\leq C\|\theta\|_{L^\infty}\|\Delta\theta\|_{L^2}, \\
&\|\Delta u\|_{L^4}^2\leq C\|\nabla u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}.
\end{aligned}$$

Taking $\nabla\Delta$ to (1.4), testing by $\nabla\Delta\phi$, using (1.5), (1.2), and (1.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\Delta\phi|^2 dx+\int\left|\Lambda^5\phi\right|^2 dx \\
&\quad =-\sum_i\int\left(\nabla\Delta(u_i\partial_i\phi)-u_i\nabla\Delta\partial_i\phi\right)\nabla\Delta\phi\,dx+\int\nabla\Delta\cdot\Delta f'(\phi)\cdot\nabla\Delta\phi\,dx \\
&\quad \leq C\|\nabla u\|_{L^\infty}\|\nabla\Delta\phi\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}\left\|\Lambda^3 u\right\|_{L^2}\|\nabla\Delta\phi\|_{L^2} \\
&\qquad +\int\nabla\Delta f'(\phi)\cdot\nabla\Delta^2\phi\,dx.
\end{aligned} \tag{2.12}$$

Using (2.4), we get

$$\begin{aligned}
&\int\nabla\Delta f'(\phi)\cdot\nabla\Delta^2\phi\,dx \\
&\quad \leq C\int\left(|\nabla\Delta\phi|+\left|\phi^2\right||\nabla\Delta\phi|+|\phi||\nabla\phi|\left|\nabla^2\phi\right|+|\nabla\phi|^3\right)\left|\nabla\Delta^2\phi\right|dx
\end{aligned}$$

$$\begin{aligned}
&\leq C\big(\|\nabla\Delta\phi\|_{L^2} + \|\phi\|_{L^\infty}^2\|\nabla\Delta\phi\|_{L^2} \\
&\quad + \|\phi\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\nabla^2\phi\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}^2\|\nabla\phi\|_{L^2}\big)\|\Lambda^5\phi\|_{L^2} \\
&\leq C\|\nabla\Delta\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}^4\|\nabla\Delta\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}\|\nabla^2\phi\|_{L^\infty}\|\Lambda^5\phi\|_{L^2} \\
&\quad + C\|\nabla\phi\|_{L^\infty}^2\|\Lambda^5\phi\|_{L^2} + \frac{1}{4}\|\Lambda^5\phi\|_{L^2}^2. \qquad (2.13)
\end{aligned}$$

Now we use the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
&\|\nabla^2\phi\|_{L^\infty} \leq C\|\nabla\Delta\phi\|_{L^2}^{3/4}\|\Lambda^5\phi\|_{L^2}^{1/4}, \\
&\|\nabla\phi\|_{L^\infty}^2 \leq C\|\Delta\phi\|_{L^2}\|\Lambda^3\phi\|_{L^2}.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\int \nabla\Delta f'(\phi)\cdot\nabla\Delta^2\phi\,dx \\
&\quad\leq C\|\nabla\Delta\phi\|_{L^2}^2 + C\|\phi\|_{L^\infty}^4\|\nabla\Delta\phi\|_{L^2}^2 \\
&\qquad + C\|\phi\|_{L^\infty}^{8/3}\|\nabla\Delta\phi\|_{L^2}^2 + C\|\Delta\phi\|_{L^2}^2\|\Lambda^3\phi\|_{L^2}^2 + \frac{1}{2}\|\Lambda^5\phi\|_{L^2}. \qquad (2.14)
\end{aligned}$$

Combining (2.10), (2.11), (2.12), and (2.14), taking ϵ small enough, using (1.8), (2.5), Gronwall's inequality and

$$\|\phi\|_{L^4(0,T;L^\infty)} \leq C,$$

we conclude that

$$\begin{aligned}
&\|u\|_{L^\infty(0,T;H^3)} \leq C, \\
&\|\theta\|_{L^\infty(0,T;H^2)} + \|\theta\|_{L^2(0,T;H^3)} \leq C, \\
&\|\phi\|_{L^\infty(0,T;H^3)} + \|\phi\|_{L^2(0,T;H^5)} \leq C.
\end{aligned}$$

This completes the proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Tianshui Normal University, Tianshui, Gansu 741001, P.R. China. [2]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [3]Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad, 44000, Pakistan. [4]School of Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, P.R. China.

Acknowledgements
This project is funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant No. 27-130-35-HiCi. The authors, therefore, acknowledge technical and financial support of KAU.

Received: 20 May 2014 Accepted: 15 July 2014 Published: 18 Aug 2014

References

1. Boyer, F: Mathematical study of multi-phase flow under shear through order parameter formulation. Asymptot. Anal. **20**, 175-212 (1999)
2. Kozono, H, Ogawa, T, Taniuchi, Y: The critical Sobolev inequalities in Besov space and regularity criterion to some semilinear evolution equations. Math. Z. **242**, 251-278 (2002)
3. Fan, J, Zhou, Y: A note on regularity criterion for the 3D Boussinesq system with partial viscosity. Appl. Math. Lett. **22**(5), 802-805 (2009)
4. Kato, T, Ponce, G: Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. **41**, 891-907 (1988)

10.1186/1029-242X-2014-287
Cite this article as: Ma et al.: A regularity criterion for the Cahn-Hilliard-Boussinesq system with zero viscosity. *Journal of Inequalities and Applications* 2014, 2014:287