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Weak and strong convergence theorems for common zeros of accretive operators

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Abstract
In this paper, we propose two proximal point algorithms for investigating common zeros of a family of accretive operators. Weak and strong convergence of the two algorithms are obtained in a Banach space.
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1 Introduction and preliminaries

Let E be a real Banach space, and let E^* be the dual space of E. Let R^+ be a positive real number set. Let $\varphi : [0,\infty] := R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a gauge function. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality mapping.

Following Browder [1], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to J_φ). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$.

Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be smooth or is said to be have a Gâteaux differentiable norm if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. E is said to be uniformly smooth or is said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U_E$.

It is well known that Fréchet differentiability of the norm of E implies Gâteaux differentiability of the norm of E. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single-valued and uniformly norm-to-weak* continuous on each bounded subset of E.

Let D be a nonempty subset of a set C. Let $\mathrm{Proj}_D : C \to D$. Q is said to be

(1) a contraction if $\mathrm{Proj}_D^2 = \mathrm{Proj}_D$;

(2) sunny if for each $x \in C$ and $t \in (0,1)$, we have $\operatorname{Proj}_D(tx + (1-t)\operatorname{Proj}_D x) = \operatorname{Proj}_D x$;

(3) a sunny nonexpansive retraction if Proj_D is sunny, nonexpansive and a contraction.

D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. The following result, which was established in [2–4], describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Let E be a smooth Banach space, and let C be a nonempty subset of E. Let $\operatorname{Proj}_C : E \to C$ be a retraction and J_φ be the duality mapping on E. Then the following are equivalent:

(1) Proj_C is sunny and nonexpansive;

(2) $\|\operatorname{Proj}_C x - \operatorname{Proj}_C y\|^2 \leq \langle x - y, J_\varphi(\operatorname{Proj}_C x - \operatorname{Proj}_C y)\rangle$, $\forall x, y \in E$;

(3) $\langle x - \operatorname{Proj}_C x, J_\varphi(y - \operatorname{Proj}_C x)\rangle \leq 0$, $\forall x \in E$, $y \in C$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Proj_C is coincident with the metric projection from E onto C. Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$ and let $x_0 \in C$. Then we have from the above that $x_0 = \operatorname{Proj}_C x$ if and only if $\langle x - x_0, J_\varphi(y - x_0)\rangle \leq 0$ for all $y \in C$, where Proj_C is a sunny nonexpansive retraction from E onto C.

A Banach space E is said to be strictly convex if and only if

$$\|x\| = \|y\| = \big\|(1-\lambda)x + \lambda y\big\|$$

for $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$.

E is said to be uniformly convex if for any $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Let C be a nonempty closed convex subset of E. Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to denote the set of fixed points of S. Recall that S is said to be nonexpansive iff $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. For the existence of fixed points of a nonexpansive mapping, we refer the readers to [5]. Let x be a fixed element in C, and let S be a nonexpansive mapping with a nonempty fixed point set. For each $t \in (0,1)$, let x_t be the unique solution of the equation $y = tx + (1-t)Sy$. In the framework of uniformly smooth Banach spaces, Reich [6] proved that $\{x_t\}$ converges strongly to a fixed point $\operatorname{Proj}_{F(S)} x$, where $\operatorname{Proj}_{F(S)}$ is the unique sunny nonexpansive retraction from C onto $F(S)$, of S as $t \to 0$. Xu [7] further extended the results to the framework of reflexive Banach spaces; for more details, see [7] and [8] and the references therein.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to be M-accretive if $R(I + rA) = E$ for all $r > 0$. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi\bigl(\|x\|\bigr), \quad \forall x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis.

Zero problems of accretive operators recently have been extensively studied (see [6–21] and the references therein) because of their important applications in real world. Proximal point algorithm, which was proposed by Martinet [22, 23] and generalized by Rockafellar [24, 25], is a classical method for investigating zeros of monotone operators. In this paper, we propose two proximal point algorithms for investigating common zeros of a family of m-accretive operators. Weak and strong convergence theorems are established in Banach spaces.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [26].

Lemma 1.1 *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds*:

$$\Phi\bigl(\|x+y\|\bigr) \le \Phi\bigl(\|x\|\bigr) + \bigl\langle y, J_\varphi(x+y)\bigr\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \le \|x\|^2 + 2\bigl\langle y, J(x+y)\bigr\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*
Then the following identity holds:

$$\limsup_{n\to\infty} \Phi\bigl(\|x_n - y\|\bigr) = \limsup_{n\to\infty} \Phi\bigl(\|x_n - x\|\bigr) + \Phi\bigl(\|y-x\|\bigr), \quad \forall x, y \in E.$$

Lemma 1.2 [7] *Let E be a reflexive Banach space and have a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be a closed convex subset of E, and let $S : C \to C$ be a nonexpansive mapping. Fix $x \in C$ and $t \in (0,1)$. Let $x_t \in C$ be the unique fixed point of the mapping $tx + (1-t)S$. Then S has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of S. Define a mapping $\mathrm{Proj}_{F(S)} : C \to F(S)$ by $\mathrm{Proj}_{F(S)} x := \lim_{t\to 0} x_t$. Then $\mathrm{Proj}_{F(S)}$ is the sunny nonexpansive retraction from C onto $F(S)$.*

Lemma 1.3 [27] *Let C be a closed convex subset of a strictly convex Banach space E. Let $N \ge 1$ be some positive integer, and let $S_m : C \to C$ be a nonexpansive mapping. Suppose that $\bigcap_{m=1}^N F(S_m)$ is nonempty. Then the mapping $\sum_{m=1}^N \beta_m S_m$, where $\{\beta_m\}$ is a real number sequence in $(0,1)$ such that $\sum_{m=1}^N \beta_m = 1$, is nonexpansive with $F(\sum_{m=1}^N \beta_m S_m) = \bigcap_{m=1}^N F(S_m)$.*

Lemma 1.4 [28] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying $b_{n+1} \le (1-a_n)b_n + a_n c_n$, $\forall n \ge n_0$, where n_0 is some positive integer, $\{a_n\}$ is a number sequence in $(0,1)$ such that $\sum_{n=n_0}^\infty a_n = \infty$, $\{c_n\}$ is a number sequence such that $\limsup_{n\to\infty} c_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.5 [29] *Let E be a uniformly convex Banach space, $s > 0$ be a positive number, and $B_s(0)$ be a closed ball of E. There exits a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{m=1}^{N} \beta_m x_m\right\|^2 \le \sum_{m=1}^{N} \beta_m \|x_m\|^2 - \beta_1\beta_2 g\big(\|x_1 - x_2\|\big)$$

for all $x_1, x_2, \dots, x_N \in B_s(0) = \{x \in E : \|x\| < s\}$ and $\beta_1, \beta_2, \dots, \beta_N \in (0,1)$ such that $\sum_{m=1}^{N} \beta_m = 1$.

Lemma 1.6 [30] *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E, and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

2 Main results

Theorem 2.1 *Let E be a uniformly convex Banach space which has the Opial condition. Let $N \ge 1$ be some positive integer, and let A_m be an M-accretive operator in E for each $1 \le m \le N$. Assume that $\bigcap_{m=1}^{N} \overline{D(A_i)}$ is convex. Let $\{\alpha_n\}$ and $\{\beta_{n,m}\}$ be real number sequences in $(0,1)$, and let $\{r_m\}$ be a positive real number sequence. Assume that $\bigcap_{m=1}^{N} A_m^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in \bigcap_{m=1}^{N} \overline{D(A_i)}$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{N} \beta_{n,m} J_{r_m} x_n, \quad \forall n \ge 1,$$

where $J_{r_m} = (I + r_m A_m)^{-1}$. Assume that the following restrictions are satisfied:
(a) $0 < a \le \alpha_n \le b < 1$;
(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$ *and* $0 < c \le \beta_{n,m} < 1$,
where a, b and c are real numbers. Then the sequence $\{x_n\}$ converges weakly to $x^* \in \bigcap_{m=1}^{N} A_m^{-1}(0)$.

Proof We start the proof with the boundedness of the sequence $\{x_n\}$. Fixing $p \in \bigcap_{m=1}^{N} A_m^{-1}(0)$, we find that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\left\|\sum_{m=1}^{N} \beta_{n,m} J_{r_m} x_n - p\right\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\sum_{m=1}^{N} \beta_{n,m} \|J_{r_m} x_n - p\| \le \|x_n - p\|.\end{aligned}$$

This shows that the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded. Using Lemma 1.5, we find that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|\sum_{m=1}^{N} \beta_{n,m} J_{r_m} x_n - p\right\|^2 \\ &\quad - \alpha_n(1-\alpha_n) g\left(\left\|x_n - \sum_{m=1}^{N} \beta_{n,m} J_{r_m} x_n\right\|\right) \\ &\le \|x_n - p\|^2 - \alpha_n(1-\alpha_n) g\left(\sum_{m=1}^{N} \beta_{n,m} \|x_n - J_{r_m} x_n\|\right).\end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)g\left(\sum_{m=1}^{N}\beta_{n,m}\|x_n - J_{r_m}x_n\|\right) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

In view of restriction (a), we find that $\lim_{n\to\infty} g(\sum_{m=1}^{N}\beta_{n,m}\|x_n - J_{r_m}x_n\|) = 0$. It follows that $\lim_{n\to\infty}\sum_{m=1}^{N}\beta_{n,m}\|x_n - J_{r_m}x_n\| = 0$. Using restriction (b), we arrive at $\lim_{n\to\infty}\|x_n - J_{r_m}x_n\| = 0$ for each $m \in \{1,2,\ldots,N\}$. Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to x^*. Using Lemma 1.6, we obtain that $x^* \in F(J_{r_m})$. This proves that $x^* \in \bigcap_{m=1}^{N} A_m^{-1}(0)$.

Next we show that $\{x_n\}$ converges weakly to x^*. Supposing the contrary, we see that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\hat{x} \in C$, where $\hat{x} \neq x^*$. Similarly, we can show $\hat{x} \in \bigcap_{m=1}^{N} A_m^{-1}(0)$. Note that we have proved that $\lim_{n\to\infty}\|x_n - p\|$ exists for every $p \in \bigcap_{m=1}^{N} A_m^{-1}(0)$. Assume that $\lim_{n\to\infty}\|x_n - x^*\| = d$, where d is a nonnegative number. Since the space has the Opial condition [31], we see that

$$d = \liminf_{i\to\infty}\|x_{n_i} - x^*\| < \liminf_{i\to\infty}\|x_{n_i} - \hat{x}\| = \liminf_{j\to\infty}\|x_{n_j} - \hat{x}\| < \liminf_{j\to\infty}\|x_{n_j} - x^*\| = d.$$

This is a contradiction. Hence $x^* = \hat{x}$. This completes the proof. □

Theorem 2.2 *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let $N \ge 1$ be some positive integer, and let A_m be an M-accretive operator in E for each $1 \le m \le N$. Assume that $\bigcap_{m=1}^{N}\overline{D(A_i)}$ is convex. Let $\{\alpha_n\}$ and $\{\beta_{n,m}\}$ be real number sequences in $(0,1)$, and let $\{r_m\}$ be a positive real number sequence for each $1 \le m \le N$. Assume that $\bigcap_{m=1}^{N} A_m^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in \bigcap_{m=1}^{N}\overline{D(A_i)}$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\sum_{m=1}^{N}\beta_{n,m}J_{r_m}x_n, \quad \forall n \ge 1,$$

where x is a fixed element in $\bigcap_{m=1}^{N}\overline{D(A_m)}$ and $J_{r_m} = (I + r_m A_m)^{-1}$. Assume that the following restrictions are satisfied:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;

(b) $\sum_{m=1}^{N}\beta_{n,m} = 1$, $\lim_{n\to\infty}\beta_{n,m} = \beta_m$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1,m} - \beta_{n,m}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = \mathrm{Proj}_{\bigcap_{m=1}^{N} A_m^{-1}(0)}x$, where $\mathrm{Proj}_{\bigcap_{m=1}^{N} A_m^{-1}(0)}$ is the unique sunny nonexpansive retract from $\bigcap_{m=1}^{N}\overline{D(A_m)}$ onto $\bigcap_{m=1}^{N} A_m^{-1}(0)$.

Proof We start the proof with the boundedness of the sequence $\{x_n\}$. Fixing $p \in \bigcap_{m=1}^{N} A_m^{-1}(0)$, we find that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|x - p\| + (1-\alpha_n)\left\|\sum_{m=1}^{N}\beta_{n,m}J_{r_m}x_n - p\right\| \\ &\le \alpha_n\|x - p\| + (1-\alpha_n)\sum_{m=1}^{N}\beta_{n,m}\|J_{r_m}x_n - p\| \\ &\le \alpha_n\|x - p\| + (1-\alpha_n)\|x_n - p\|.\end{aligned}$$

This implies that $\|x_{n+1}-p\| \leq \max\{\|x-p\|, \|x_1-p\|\}$. This shows that $\{x_n\}$ is bounded. Put $y_n = \sum_{m=1}^N \beta_{n,m} J_{r_m} x_n$. It follows that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \left\| \sum_{m=1}^N \beta_{n,m} J_{r_m} x_n - \sum_{m=1}^N \beta_{n,m} J_{r_m} x_{n-1} \right\| \\ &\quad + \left\| \sum_{m=1}^N \beta_{n,m} J_{r_m} x_{n-1} - \sum_{m=1}^N \beta_{n-1,m} J_{r_m} x_{n-1} \right\| \\ &\leq \|x_n - x_{n-1}\| + \sum_{m=1}^N |\beta_{n,m} - \beta_{n-1,m}| \|J_{r_m} x_{n-1}\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x - y_{n-1}\| \\ &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \sum_{m=1}^N |\beta_{n,m} - \beta_{n-1,m}| \|J_{r_m} x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|x - y_{n-1}\|.\end{aligned}$$

In light of restrictions (a) and (b), we find that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.1}$$

Set $S = \sum_{m=1}^N \beta_m J_{r_m}$. It follows from Lemma 1.3 that S is nonexpansive with $F(S) = \bigcap_{m=1}^N F(J_{r_m}) = \bigcap_{m=1}^N A_m^{-1}(0)$. Note that

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sx_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|x - Sx_n\| + \beta_n \left\| \sum_{m=1}^N \beta_{n,m} J_{r_m} x_n - Sx_n \right\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|x - Sx_n\| + \sum_{m=1}^N |\beta_{n,m} - \beta_m| \|J_{r_m} x_n\|.\end{aligned}$$

In view of (2.1), we find from the restrictions (a) and (b) that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{2.2}$$

Now, we are in a position to prove

$$\limsup_{n\to\infty} \langle x - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x, J_\varphi(x_n - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x)\rangle \leq 0. \tag{2.3}$$

By Lemma 1.2, we have the sunny nonexpansive retraction $\operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} : \bigcap_{m=1}^N \overline{D(A_m)} \to \bigcap_{m=1}^N A_m^{-1}(0)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty} \langle x - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x, J_\varphi(x_n - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)})x\rangle \\ &\quad = \lim_{k\to\infty} \langle x - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x, J_\varphi(x_{n_k} - \operatorname{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)})x\rangle.\end{aligned} \tag{2.4}$$

Since E is reflexive, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in \bigcap_{m=1}^N \overline{D(A_m)}$. Since j_φ is weakly continuous, we have from Lemma 1.1

$$\limsup_{k\to\infty} \Phi(\|x_{n_k} - y\|) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|y - \bar{x}\|), \quad \forall y \in E.$$

Put $f(y) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - y\|)$, $\forall y \in E$. It follows that

$$f(y) = f(\bar{x}) + \Phi(\|y - \bar{x}\|), \quad \forall y \in E. \tag{2.5}$$

From (2.2), we have

$$\begin{aligned} f(S\bar{x}) &= \limsup_{k\to\infty} \Phi(\|x_{n_k} - S\bar{x}\|) = \limsup_{k\to\infty} \Phi(\|Sx_{n_k} - S\bar{x}\|) \\ &\leq \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) = f(\bar{x}). \end{aligned} \tag{2.6}$$

Using (2.5), we have

$$f(S\bar{x}) = f(\bar{x}) + \Phi(\|S\bar{x} - \bar{x}\|). \tag{2.7}$$

Combining (2.6) with (2.7), we obtain that

$$\Phi(\|S\bar{x} - \bar{x}\|) \leq 0.$$

Hence $S\bar{x} = \bar{x}$; that is, $\bar{x} \in F(S) = \bigcap_{m=1}^N A_m^{-1}(0)$. It follows from (2.4) that

$$\limsup_{n\to\infty} \langle x - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x, j_\varphi(x_n - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x)\rangle \leq 0.$$

Finally, we prove that $x_n \to \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x$ as $n \to \infty$. Using Lemma 1.1, we find that

$$\begin{aligned} &\Phi(\|x_{n+1} - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x\|) \\ &\quad = \Phi\left(\left\|\alpha_n(x - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x) + (1-\alpha_n)\left(\sum_{m=1}^N \beta_{n,m} J_{r_m} x_n - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x\right)\right\|\right) \\ &\quad \leq (1-\alpha_n)\Phi(\|x_n - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x\|) \\ &\qquad + \alpha_n \langle x - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x, J_\varphi(x_{n+1} - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x)\rangle. \end{aligned}$$

Using Lemma 1.4, we see that $\Phi(\|x_n - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x\|) \to 0$. This implies that

$$\lim_{n\to\infty} \|x_n - \mathrm{Proj}_{\bigcap_{m=1}^N A_m^{-1}(0)} x\| = 0.$$

This completes the proof. □

3 Applications

In this section, we give an application of Theorem 2.1 in the framework of Hilbert spaces.

Let C be a nonempty closed and convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{3.1}$$

To study the equilibrium problem (3.1), we may assume that F satisfies the following restrictions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 3.1 [32] *Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $\mathrm{EP}(F) = A_F^{-1}(0)$, *where* $\mathrm{EP}(F)$ *stands for the solution set of* (3.1), *and*

$$T_r x = (I + rA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_s is defined by

$$T_r x = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}, \quad \forall x \in H.$$

Corollary 3.2 *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $N \geq 1$ be some positive integer, and let $F_m : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $\{\alpha_n\}$ and $\{\beta_{n,m}\}$ be real number sequences in* $(0,1)$, *and let $\{r_m\}$ be a positive real number sequence. Assume that $\bigcap_{m=1}^N \mathrm{EP}(F_m)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner*: $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^N \beta_{n,m} T_{r_m} x_n, \quad \forall n \geq 1,$$

where $T_{r_m} = (I + r_m A_{F_m})^{-1}$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq \alpha_n \leq b < 1$;
(b) $\sum_{m=1}^N \beta_{n,m} = 1$ *and* $0 < c \leq \beta_{n,m} < 1$,

where a, b and c are real numbers. Then the sequence $\{x_n\}$ converges weakly to $x^* \in \bigcap_{m=1}^N \mathrm{EP}(F_m)$.

Corollary 3.3 *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $N \geq 1$ be some positive integer, and let $F_m : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $\{\alpha_n\}$ and $\{\beta_{n,m}\}$ be real number sequences in* $(0,1)$, *and let $\{r_m\}$ be a positive real* *number sequence. Assume that* $\bigcap_{m=1}^{N} \mathrm{EP}(F_m)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner*: $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\sum_{m=1}^{N} \beta_{n,m} T_{r_m} x_n, \quad \forall n \geq 1,$$

where x *is a fixed element in* C *and* $T_{r_m} = (I + r_m A_{F_m})^{-1}$. *Assume that the following restrictions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(b) $\sum_{m=1}^{N} \beta_{n,m} = 1$, $\lim_{n\to\infty} \beta_{n,m} = \beta_m$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = \mathrm{Proj}_{\bigcap_{m=1}^{N} \mathrm{EP}(F_m)} x$, *where* $\mathrm{Proj}_{\bigcap_{m=1}^{N} \mathrm{EP}(F_m)}$ *is the metric projection from* C *onto* $\bigcap_{m=1}^{N} \mathrm{EP}(F_m)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors designed the algorithms and established weak and strong convergence analysis. Both authors read and approved the final manuscript.

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