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Some properties of relative efficiency of estimators in a two linear regression equations system with identical parameter vectors

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Abstract
Two normal linear models with some of the parameters identical are discussed in this article. We introduce four relative efficiencies to define the efficiency of estimator in two linear regression equations system with identical parameter vectors, also we give the lower and upper bounds of the four relative efficiencies.

Keywords: best linear unbiased estimator; common parameter; relative efficiency

1 Introduction

Consider a system (H) formed by two linear models:

$$y_1 = X_1\beta + Z_1\beta_1 + \varepsilon_1, \tag{1}$$

$$y_2 = X_2\beta + Z_2\beta_2 + \varepsilon_2, \tag{2}$$

where for $i = 1, 2$, y_i is $n_i \times 1$ vector of observations, X_i and Z_i are $n_i \times p$ and $n_i \times t_i$ full rank matrices satisfying $\operatorname{rank}(X_i, Z_i) = \operatorname{rank}(X_i) + \operatorname{rank}(Z_i)$ with $\operatorname{rank}(\cdot)$ denoting the rank of a matrix, β and β_i are $p \times 1$ and $t_i \times 1$ unknown parameters, ε_i is $n_i \times 1$ random vector supposed to follow a multivariate normal distribution mean 0 and variance covariance matrix $\sigma_i I$, σ_i being a known parameter, ε_1 and ε_2 are independent.

Define $Q_i = I - Z_i(Z_i'Z_i)^{-1}Z_i'$, $T_i = (Z_i'Z_i)^{-1}Z_i'X_i$ and $r = \frac{\sigma_1}{\sigma_2}$. Then by Liu [1] we have the following:

(1) In the single equation (1), the best linear unbiased estimators (BLUE) of β and β_1 are given respectively by

$$\hat{\beta} = \left(X_1'Q_1X_1\right)^{-1}X_1'Q_1y_1, \tag{3}$$

$$\hat{\beta}_1 = \left(Z_1'Z_1\right)^{-1}Z_1'y_1 - T_1\hat{\beta}. \tag{4}$$

(2) In the single equation (2), the best linear unbiased estimators (BLUE) of β and β_2 are given respectively by

$$\tilde{\beta} = \left(X_2'Q_2X_2\right)^{-1}X_2'Q_2y_2, \tag{5}$$

$$\tilde{\beta}_2 = \left(Z_2'Z_2\right)^{-1}Z_2'y_2 - T_2\tilde{\beta}. \tag{6}$$

(3) For the system (H), the BLUE of β, β_1 and β_2 are given respectively by

$$\beta^*(r) = \left(X_1'Q_1X_1 + rX_2'Q_2X_2\right)^{-1}\left(X_1'Q_1y_1 + rX_2'Q_2y_2\right), \tag{7}$$

$$\beta_1^* = \left(Z_1'Z_1\right)^{-1}Z_1'y_1 - T_1\beta^*(r), \tag{8}$$

$$\beta_2^* = \left(Z_2'Z_2\right)^{-1}Z_2'y_2 - T_2\beta^*(r). \tag{9}$$

In this article, we only discuss the estimation of the parameter β. Liu [1] gave the comparison between the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the mean squared error criterion when σ_i are known. He also gave an estimator when σ_i are unknown and discussed the statistical properties of the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$. Ma and Wang [2] also studied the estimators $\hat{\beta}$, $\tilde{\beta}$ and $\beta^*(r)$ in the mean squared error criterion.

It is easy to compute that

$$\operatorname{Cov}(\hat{\beta}) = \sigma_1\left(X_1'Q_1X_1\right)^{-1}, \tag{10}$$

$$\operatorname{Cov}(\tilde{\beta}) = \sigma_2\left(X_2'Q_2X_2\right)^{-1}, \tag{11}$$

$$\operatorname{Cov}\left(\beta^*(r)\right) = \sigma_1\left(X_1'Q_1X_1 + rX_2'Q_2X_2\right)^{-1}. \tag{12}$$

From Equations (10)-(12), we can see that

$$\operatorname{Cov}\left(\beta^*(r)\right) \le \operatorname{Cov}(\hat{\beta}), \qquad \operatorname{Cov}\left(\beta^*(r)\right) \le \operatorname{Cov}(\tilde{\beta}). \tag{13}$$

In practice, σ_i may be unknown, in this case we can use $\hat{\beta}$ or $\tilde{\beta}$ to replace $\beta^*(r)$. However, this will lead to loss, we introduce the relative efficiency to define the loss. Relative efficiency has been studied by many researchers such as Yang [3], Wang and Ip [4], Liu *et al.* [5, 6], Yang and Wang [7], Wang and Yang [8, 9] and Yang and Wu [10].

In this article, we introduce four relative efficiencies in system (H), and we also give the lower and upper bounds of the four relative efficiencies.

The rest of the article is organized as follows. In Section 2, we propose the new relative efficiency. Sections 3 and 4 give the lower and upper bounds of the relative efficiencies proposed in Section 2. Some concluding remarks are given in Section 5.

2 New relative efficiency

In order to define the loss when we use $\hat{\beta}$ or $\tilde{\beta}$ to replace $\beta^*(r)$, we introduce four relative efficiencies as follows:

$$e_1\left(\beta^*(r)|\hat{\beta}\right) = \frac{|\operatorname{Cov}(\beta^*(r))|}{|\operatorname{Cov}(\hat{\beta})|}, \tag{14}$$

$$e_2\left(\beta^*(r)|\tilde{\beta}\right) = \frac{|\operatorname{Cov}(\beta^*(r))|}{|\operatorname{Cov}(\tilde{\beta})|}, \tag{15}$$

$$e_3\left(\beta^*(r)|\hat{\beta}\right) = \frac{\operatorname{tr}(\operatorname{Cov}(\beta^*(r)))}{\operatorname{tr}(\operatorname{Cov}(\hat{\beta}))}, \tag{16}$$

$$e_4\left(\beta^*(r)|\tilde{\beta}\right) = \frac{\operatorname{tr}(\operatorname{Cov}(\beta^*(r)))}{\operatorname{tr}(\operatorname{Cov}(\tilde{\beta}))}, \tag{17}$$

where $|A|$ and $\operatorname{tr}(A)$ denote the determinant and trace of matrix A, respectively. By Equation (13), we have $0 < e_i(\cdot|\cdot) \leq 1$, $i = 1, 2, 3, 4$. In the next section we will give the lower and upper bounds of $e_1(\beta^*(r)|\hat{\beta})$ and $e_2(\beta^*(r)|\tilde{\beta})$.

3 The lower and upper bounds of $e_1(\beta^*(r)|\hat{\beta})$ and $e_2(\beta^*(r)|\tilde{\beta})$

In this section we give the lower and upper bounds of $e_1(\beta^*(r)|\hat{\beta})$ and $e_2(\beta^*(r)|\tilde{\beta})$. Firstly, we give some lemmas and notations which are needed in the following discussion. Let A be an $n \times n$ nonnegative definite matrix, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ stands for the ordered eigenvalues of matrix A.

Lemma 3.1 [11] *Let A be an $n \times n$ nonnegative definite matrix, and let B be an $n \times n$ nonnegative definite matrix, then we have*

$$\lambda_n(A)\lambda_i(B) \leq \lambda_i(AB) \leq \lambda_1(A)\lambda_i(B), \quad i = 1, 2, \ldots, n. \tag{18}$$

Lemma 3.2 [12] *Let $\Delta_1 = \operatorname{diag}(\tau_1, \tau_2, \ldots, \tau_p)$, $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_p > 0$, $\Delta_2 = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_p)$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 0$ and A be an $p \times p$ orthogonal matrix, then we have*

$$\sum_{i=1}^{p} \tau_i \mu_{p+1-i} \leq \operatorname{tr}(\Delta_1 A' \Delta_2 A) \leq \sum_{i=1}^{p} \tau_i \mu_i. \tag{19}$$

Now we will give the lower and upper bounds of $e_1(\beta^*(r)|\hat{\beta})$.

Theorem 3.1 *Let $\beta^*(r)$ and $\hat{\beta}$ be given in Equations* (7) *and* (3), *let $e_1(\beta^*(r)|\hat{\beta})$ be defined in Equation* (14), *then we have*

$$\frac{1}{\prod_{i=1}^{p}(1 + r\theta_p^{-1}\eta_i)} \leq e_1(\beta^*(r)|\hat{\beta}) \leq \frac{1}{\prod_{i=1}^{p}(1 + r\theta_1^{-1}\eta_i)}, \tag{20}$$

where $\theta_1 \geq \cdots \geq \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \geq \cdots \geq \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

Proof By the definition of $e_1(\beta^*(r)|\hat{\beta})$, we have

$$\begin{aligned} e_1(\beta^*(r)|\hat{\beta}) &= \frac{|\operatorname{Cov}(\beta^*(r))|}{|\operatorname{Cov}(\hat{\beta})|} \\ &= \frac{|\sigma_1(X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}|}{|\sigma_1(X_1'Q_1X_1)^{-1}|} \\ &= \frac{|X_1'Q_1X_1|}{|X_1'Q_1X_1 + rX_2'Q_2X_2|}. \end{aligned} \tag{21}$$

It is easy to see that $X_1'Q_1X_1 > 0$ and $X_2'Q_2X_2 > 0$. Define

$$A = (X_1'Q_1X_1)^{-1/2}(X_2'Q_2X_2)(X_1'Q_1X_1)^{-1/2},$$

then $A > 0$, there exists an orthogonal matrix N such that

$$NAN' = \operatorname{diag}(\zeta_1, \ldots, \zeta_p) \triangleq \Delta, \tag{22}$$

where $\zeta_1 \geq \cdots \geq \zeta_p$ is the eigenvalues of A. Now we define $M = N(X_1'Q_1X_1)^{-1/2}$, then we have

$$M\left(X_1'Q_1X_1\right)M' = NN' = I_p, \tag{23}$$

$$\begin{aligned} M\left(X_2'Q_2X_2\right)M' &= N\left(X_1'Q_1X_1\right)^{-1/2}\left(X_2'Q_2X_2\right)\left(X_1'Q_1X_1\right)^{-1/2}N' \\ &= NAN' = \Delta. \end{aligned} \tag{24}$$

Thus

$$X_1'Q_1X_1 = M^{-1}M'^{-1}, \tag{25}$$

$$X_2'Q_2X_2 = M^{-1}\Delta M'^{-1}. \tag{26}$$

Then we put Equations (25) and (26) into Equation (21), and we have

$$\begin{aligned} e_1\left(\beta^*(r)|\hat{\beta}\right) &= \frac{|X_1'Q_1X_1|}{|X_1'Q_1X_1 + rX_2'Q_2X_2|} \\ &= \frac{|M^{-1}M'^{-1}|}{|M^{-1}M'^{-1} + rM^{-1}\Delta M'^{-1}|} \\ &= \frac{|M^{-1}||M'^{-1}|}{|M^{-1}||I_p + r\Delta||M'^{-1}|} = \frac{1}{|I_p + r\Delta|}. \end{aligned} \tag{27}$$

Since $A = (X_1'Q_1X_1)^{-1/2}(X_2'Q_2X_2)(X_1'Q_1X_1)^{-1/2}$ has the same eigenvalues of $(X_2'Q_2X_2) \times (X_1'Q_1X_1)^{-1}$, we have $\lambda_i(A) = \lambda_i((X_2'Q_2X_2)(X_1'Q_1X_1)^{-1})$, $i = 1, 2, \ldots, p$. Then by Lemma 3.1 we have

$$\begin{aligned} \lambda_p\left(\left(X_1'Q_1X_1\right)^{-1}\right)\lambda_i\left(X_2'Q_2X_2\right) &\leq \lambda_i\left(\left(X_2'Q_2X_2\right)\left(X_1'Q_1X_1\right)^{-1}\right) \\ &\leq \lambda_1\left(\left(X_1'Q_1X_1\right)^{-1}\right)\lambda_i\left(X_2'Q_2X_2\right). \end{aligned} \tag{28}$$

On the other hand,

$$\lambda_p\left(\left(X_1'Q_1X_1\right)^{-1}\right) = \lambda_1^{-1}\left(X_1'Q_1X_1\right) = \theta_1^{-1}, \tag{29}$$

$$\lambda_1\left(\left(X_1'Q_1X_1\right)^{-1}\right) = \lambda_p^{-1}\left(X_1'Q_1X_1\right) = \theta_p^{-1}, \tag{30}$$

where $\theta_1 \geq \cdots \geq \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$. By Equations (28)-(30), we obtain

$$\theta_1^{-1}\eta_i \leq \lambda_i\left(\left(X_2'Q_2X_2\right)\left(X_1'Q_1X_1\right)^{-1}\right) \leq \theta_p^{-1}\eta_i, \quad i = 1, \ldots, p, \tag{31}$$

where $\eta_1 \geq \cdots \geq \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$. Thus by Equations (27) and (31), we have

$$\frac{1}{\prod_{i=1}^{p}(1 + r\theta_p^{-1}\eta_i)} \leq e_1\left(\beta^*(r)|\hat{\beta}\right) \leq \frac{1}{\prod_{i=1}^{p}(1 + r\theta_1^{-1}\eta_i)}. \tag{32}$$

□

Corollary 3.1 *Let $\beta^*(r)$ and $\hat{\beta}$ be given in Equations* (7) *and* (3), *let $e_1(\beta^*(r)|\hat{\beta})$ be defined in Equation* (14), *$X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, then we have*

$$\frac{\theta_p^p}{(\theta_1 + r\eta_1)^p} \le e_1\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\theta_1^p}{(\theta_p + r\eta_p)^p}, \tag{33}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \ge \cdots \ge \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

Proof Since $X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, there exists an orthogonal matrix G such that

$$G'X_1'Q_1X_1G = \operatorname{diag}(\theta_1, \dots, \theta_p) \triangleq \Sigma, \tag{34}$$

$$G'X_2'Q_2X_2G = \operatorname{diag}(\eta_1, \dots, \eta_p) \triangleq \Omega, \tag{35}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \ge \cdots \ge \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

By the definition of $e_1(\beta^*(r)|\hat{\beta})$, we have

$$\begin{aligned} e_1\bigl(\beta^*(r)|\hat{\beta}\bigr) &= \frac{|\operatorname{Cov}(\beta^*(r))|}{|\operatorname{Cov}(\hat{\beta})|} \\ &= \frac{|X_1'Q_1X_1|}{|X_1'Q_1X_1 + rX_2'Q_2X_2|} \\ &= \frac{|G\Sigma G'|}{|G\Sigma G' + rG\Omega G'|} \\ &= \frac{\prod_{i=1}^p \theta_i}{\prod_{i=1}^p (\theta_i + r\eta_i)}. \end{aligned} \tag{36}$$

Thus we have

$$\frac{\theta_p^p}{(\theta_1 + r\eta_1)^p} \le e_1\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\theta_1^p}{(\theta_p + r\eta_p)^p}. \tag{37}$$

□

Using the same way, we can give the lower and upper bounds of $e_2(\beta^*(r)|\tilde{\beta})$.

Theorem 3.2 *Let $\beta^*(r)$ and $\tilde{\beta}$ be given in Equations* (7) *and* (5), *let $e_2(\beta^*(r)|\tilde{\beta})$ be defined in Equation* (15), *then we have*

$$\frac{1}{\prod_{i=1}^p (r + \eta_p^{-1}\theta_i)} \le e_2\bigl(\beta^*(r)|\tilde{\beta}\bigr) \le \frac{1}{\prod_{i=1}^p (r + \eta_1^{-1}\theta_i)}, \tag{38}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \ge \cdots \ge \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

Corollary 3.2 *Let $\beta^*(r)$ and $\tilde{\beta}$ be given in Equations* (7) *and* (5), *let $e_2(\beta^*(r)|\tilde{\beta})$ be defined in Equation* (15), *$X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, then we have*

$$\frac{\eta_p^p}{(\theta_1 + r\eta_1)^p} \le e_2\bigl(\beta^*(r)|\tilde{\beta}\bigr) \le \frac{\eta_1^p}{(\theta_p + r\eta_p)^p}. \tag{39}$$

4 The lower and upper bounds of $e_3(\beta^*(r)|\hat{\beta})$ and $e_4(\beta^*(r)|\tilde{\beta})$

In this section we give the lower and upper bounds of $e_3(\beta^*(r)|\hat{\beta})$ and $e_4(\beta^*(r)|\tilde{\beta})$. Firstly we give the lower and upper bounds of $e_3(\beta^*(r)|\hat{\beta})$.

Theorem 4.1 *Let $\beta^*(r)$ and $\hat{\beta}$ be given in Equations* (7) *and* (3), *let $e_3(\beta^*(r)|\hat{\beta})$ be defined in Equation* (16), *then we have*

$$\frac{\sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_i)^{-1}}{\sum_{i=1}^{p}\theta_i^{-1}} \le e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_{p+1-i})^{-1}}{\sum_{i=1}^{p}\theta_i^{-1}}, \tag{40}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\zeta_1 \ge \cdots \ge \zeta_p$ is the ordered eigenvalues of $(X_1'Q_1X_1)^{-1/2}(X_2'Q_2X_2)(X_1'Q_1X_1)^{-1/2}$.

Proof Since $X_1'Q_1X_1 > 0$, there exists an orthogonal matrix K_1 such that

$$X_1'Q_1X_1 = K_1'\Sigma K_1, \quad \Sigma = \operatorname{diag}(\theta_1,\ldots,\theta_p), \tag{41}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$. Similar to Theorem 3.1, we define

$$A = \bigl(X_1'Q_1X_1\bigr)^{-1/2}\bigl(X_2'Q_2X_2\bigr)\bigl(X_1'Q_1X_1\bigr)^{-1/2}.$$

Since $A > 0$, there exists an orthogonal matrix K_2 such that

$$A = K_2'\Delta K_2, \quad \Delta = \operatorname{diag}(\zeta_1,\ldots,\zeta_p), \tag{42}$$

where $\zeta_1 \ge \cdots \ge \zeta_p$ is the order eigenvalues of A.

We can easily compute that

$$\operatorname{tr}\bigl(\operatorname{Cov}(\hat{\beta})\bigr) = \sigma_1 \operatorname{tr}\bigl(\bigl(X_1'Q_1X_1\bigr)^{-1}\bigr) = \sigma_1\sum_{i=1}^{p}\theta_i^{-1} \tag{43}$$

and

$$\begin{aligned}
\operatorname{tr}\bigl(\operatorname{Cov}\bigl(\beta^*(r)\bigr)\bigr) &= \sigma_1 \operatorname{tr}\bigl(\bigl(X_1'Q_1X_1 + rX_2'Q_2X_2\bigr)^{-1}\bigr) \\
&= \sigma_1 \operatorname{tr}\bigl(\bigl(X_1'Q_1X_1\bigr)^{-1/2}(I_p + rA)^{-1}\bigl(X_1'Q_1X_1\bigr)^{-1/2}\bigr) \\
&= \sigma_1 \operatorname{tr}\bigl((I_p + rA)^{-1}\bigl(X_1'Q_1X_1\bigr)^{-1}\bigr) \\
&= \sigma_1 \operatorname{tr}\bigl((I_p + r\Delta)^{-1}K_2K_1'\Sigma^{-1}K_1K_2'\bigr) \\
&= \sigma_1 \operatorname{tr}\bigl((I_p + r\Delta)^{-1}K'\Sigma^{-1}K\bigr),
\end{aligned} \tag{44}$$

where $K = K_1K_2'$ is an orthogonal matrix. Thus we have

$$\begin{aligned}
e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) &= \frac{\operatorname{tr}(\operatorname{Cov}(\beta^*(r)))}{\operatorname{tr}(\operatorname{Cov}(\hat{\beta}))} \\
&= \frac{\operatorname{tr}((I_p + r\Delta)^{-1}K'\Sigma^{-1}K)}{\sum_{i=1}^{p}\theta_i^{-1}}.
\end{aligned} \tag{45}$$

Using Lemma 3.2, we have

$$\begin{aligned}\sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_i)^{-1} &\le \operatorname{tr}\bigl((I_p+r\Delta)^{-1}K'\Sigma^{-1}K\bigr)\\ &\le \sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_{p+1-i})^{-1}.\end{aligned}\tag{46}$$

Thus

$$\frac{\sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_i)^{-1}}{\sum_{i=1}^{p}\theta_i^{-1}} \le e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\sum_{i=1}^{p}\theta_{p+1-i}^{-1}(1+r\zeta_{p+1-i})^{-1}}{\sum_{i=1}^{p}\theta_i^{-1}}.\tag{47}$$

□

Corollary 4.1 *Let $\beta^*(r)$ and $\hat{\beta}$ be given in Equations* (7) *and* (3), *let $e_3(\beta^*(r)|\hat{\beta})$ be defined in Equation* (16), *$X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, then we have*

$$\frac{\theta_p}{\theta_1+r\eta_1} \le e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\theta_1}{\theta_p+r\eta_p},\tag{48}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \ge \cdots \ge \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

Proof Since $X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, there exists an orthogonal matrix G such that

$$G'X_1'Q_1X_1G = \operatorname{diag}(\theta_1,\ldots,\theta_p) = \Sigma,\tag{49}$$

$$G'X_2'Q_2X_2G = \operatorname{diag}(\eta_1,\ldots,\eta_p) = \Omega,\tag{50}$$

where $\theta_1 \ge \cdots \ge \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \ge \cdots \ge \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

By the definition of $e_3(\beta^*(r)|\hat{\beta})$, we have

$$\begin{aligned}e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) &= \frac{\operatorname{tr}(\operatorname{Cov}(\beta^*(r)))}{\operatorname{tr}(\operatorname{Cov}(\hat{\beta}))}\\ &= \frac{\sum_{i=1}^{p}(\theta_i+r\eta_i)^{-1}}{\sum_{i=1}^{p}\theta_i^{-1}}.\end{aligned}\tag{51}$$

Thus we have

$$\frac{\theta_p}{\theta_1+r\eta_1} \le e_3\bigl(\beta^*(r)|\hat{\beta}\bigr) \le \frac{\theta_1}{\theta_p+r\eta_p}.\tag{52}$$

□

Then we can give the lower and upper bounds of $e_4(\beta^*(r)|\tilde{\beta})$.

Theorem 4.2 *Let $\beta^*(r)$ and $\tilde{\beta}$ be given in Equations* (7) *and* (5), *let $e_4(\beta^*(r)|\tilde{\beta})$ be defined in Equation* (17), *then we have*

$$\frac{\sum_{i=1}^{p}\eta_{p+1-i}^{-1}(r+\iota_i)^{-1}}{\sum_{i=1}^{p}\eta_i^{-1}} \le e_4\bigl(\beta^*(r)|\tilde{\beta}\bigr) \le \frac{\sum_{i=1}^{p}\eta_{p+1-i}^{-1}(r+\iota_{p+1-i})^{-1}}{\sum_{i=1}^{p}\eta_i^{-1}},\tag{53}$$

where $\eta_1 \geq \cdots \geq \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$, $\iota_1 \geq \cdots \geq \iota_p$ is the ordered eigenvalues of $(X_2'Q_2X_2)^{-1/2}(X_1'Q_1X_1)(X_2'Q_2X_2)^{-1/2}$.

Corollary 4.2 *Let $\beta^*(r)$ and $\tilde{\beta}$ be given in Equations* (7) *and* (5), *let $e_4(\beta^*(r)|\tilde{\beta})$ be defined in Equation* (17), *$X_1'Q_1X_1$ and $X_2'Q_2X_2$ communicate, then we have*

$$\frac{\eta_p}{\theta_1 + r\eta_1} \leq e_4\big(\beta^*(r)|\tilde{\beta}\big) \leq \frac{\eta_1}{\theta_p + r\eta_p}, \tag{54}$$

where $\theta_1 \geq \cdots \geq \theta_p$ is the ordered eigenvalues of $X_1'Q_1X_1$, $\eta_1 \geq \cdots \geq \eta_p$ is the ordered eigenvalues of $X_2'Q_2X_2$.

5 Concluding remarks

In this article, we have introduced four relative efficiencies in two linear regression equations system with identical parameter vectors, and we have also given the lower and upper bounds for the four relative efficiencies.

Competing interests
The author declares that they have no competing interests.

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