Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Nonhomogeneous boundary value problem for (I, J) similar solutions of incompressible two-dimensional Euler equations

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This paper is dedicated to Professor Shisheng Zhang for his 80th birthday

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Abstract

In this paper we introduce the (I, J) similar method for incompressible two-dimensional Euler equations, and obtain a series of explicit (I, J) similar solutions to the incompressible two-dimensional Euler equations. These solutions include all of the twin wave solutions, some new singularity solutions, and some global smooth solutions with a finite energy. We also reveal that the twin wave solution and an affine solution to the two-dimensional incompressible Euler equations are, respectively, a plane wave and constant vector. We prove that the initial boundary value problem of the incompressible two-dimensional Euler equations admits a unique solution and discuss the stability of the solution. Finally, we supply some explicit piecewise smooth solutions to the incompressible three-dimensional Euler case and an example of the incompressible three-dimensional Navier-Stokes equations which indicates that the viscosity limit of a solution to the Navier-Stokes equations does not need to be a solution to the Euler equations.

MSC: 35Q30; 76D05; 76D10

Keywords: Euler equation; (I, J) similar method; twin wave solution; affine solution; explicit smooth solution; uniqueness; stability

1 Introduction

In this paper we consider the Euler equations ($\sigma = 0$) or the Navier-Stokes equations below:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \sigma \triangle u, & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0, & \text{in } \Omega \times [0, \infty), n = 2, 3, \end{cases} \tag{1.1}$$

where $\Omega \subset R_+^n = \{x \in R^n | x_n > 0\}$; $u = u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ and $p = p(x_1, x_2, t)$ denote the velocity and pressure, respectively. Though there is a large amount of physics and mathematics literature on the Euler and Navier-Stokes equations, many basic questions remain open.

There are various open problems in fluid physics. The Navier-Stokes equation has been recognized as the basic equation and the very starting point of all problems in fluid physics (see [1]). One of the most significant developments related to the above problem may be the discovery of Lax pairs of the two-dimensional and three-dimensional Euler equations (see [2])

$$\omega_t + [\psi,\omega] = 0, \qquad \omega = \psi_{x_1x_1} + \psi_{x_2x_2}, \tag{1.2}$$

where the velocity $u = (u_1, u_2)$ is determined by the stream function ψ through

$$u_1 = -\psi_{x_1}, \qquad u_2 = \psi_{x_2}. \tag{1.3}$$

Is an exact solution of the Euler equations explicitly given via solving the vortex equations (the weak Lax pair) to the Euler equations? Since the Lax pair has still only weak meaning, one cannot get the solutions to the Euler equations from those solutions of vortex equations by the Biot-Savart law. Thus whether the integrable two-dimensional Euler equations in some stronger sense are similar to those of the three-dimensional Euler equations is still an open question. In this paper we find a so-called (I,J) similar method which can give some explicit smooth solutions to two-dimensional incompressible Euler equations (see Section 2). As applications of the (I,J) similar method, a large amount of explicit twin wave solutions are constructed in Section 3.

There are various open problems in mathematics, such as: how to establish the global existence of smooth solutions, and how to establish the blow-up solution at least when the space dimension equals three (see [3]), and so on. The study of the incompressible Navier-Stokes equations has a long history. A deeper result on the weak solution was obtained by Caffarelli *et al.* in [4]. On the blow-up problem of the incompressible Navier-Stokes equations, Tsai in [5] proved that the Leray self-similar solutions to (1.1) must be zero if they satisfy local energy estimates. So in Section 4 we discuss the method of determining the nonexistence of a non-constant affine solution to the two-dimensional Euler equations, which we can correctly obtain due to the (I,J) similar method.

The blow-up problem of the compressible Navier-Stokes equation has been established by Xin (see [6]). He proved that any smooth solution to the multidimensional Navier-Stokes equation for polytropic fluids in the absence of heat conduction will blow up in finite time if the initial density is compactly supported (see [6]).

In Section 5, we prove that incompressible two-dimensional Euler equations under a class of initial boundary values has a unique solution $u(x,t) \in C^\infty([0,\infty); L^2(\Omega))$ for every bounded domain $\bar{\Omega} \subset R^n_+$, and we discuss the stability of solutions in Section 6.

Since it is very hard to solve the Navier-Stokes equations in a three-dimensional space, we consider the two equations in the half space case. In Section 7, we construct some explicit smooth solutions to the incompressible three-dimensional Euler and Navier-Stokes equations and an example of the three-dimensional Navier-Stokes equations which indicates that a solution to the Navier-Stokes equations does not need to tend to a solution to the Euler equations in the continuous function space on the half space.

For open problems of the Euler equations and the Navier-Stokes equations to the incompressible cases, we refer to [7, 8] for more information.

2 (I,J) similar method in solving the Euler equations

We first have the following definition.

Definition 2.1 A piecewise smooth solution $u(x,t)$ to (1.1) is called a (I,J) similar solution, if

$$u(x,t) = \sum_{i=1}^{I} \alpha_i(t) v_i\left(\sum_{j=1}^{J} \beta_j(t) M_j(x)\right), \tag{2.1}$$

where $\alpha_i(t)$ and $\beta_j(t)$ are smooth functions on $[0,\infty)$, $M_j(x) = (M_{j1}(x), M_{j2}(x), \dots, M_{jn}(x))$ is a n-dimensional smooth vector function independent of t, and $v_i(y_1, y_2, \dots, y_n)$ is a piecewise smooth vector function from R^n to R^n.

Here a vector valued function $f(t)$ is called piecewise smooth on $[0,\infty)$, if there exist $0 < t_1 < t_2 < \cdots < t_k < +\infty$ such that $f(t)$ is a smooth function on $(0,t_1)$, (t_i, t_{i+1}), $i = 1,2,\dots,k-1$ and (t_k,∞), respectively. Similarly we call a vector valued function $u(x,t)$ piecewise smooth on R^n if there exist $0 < r_1 < r_2 < \cdots < r_k < +\infty$ such that $u(x,t)$ is a smooth function on $\{x \mid 0 < |x| < r_1\}$, $\{x \mid r_i < |x| < r_{i+1}\}$, $i = 1,2,\dots,k-1$, and $\{x \mid r_k < |x| < \infty\}$, respectively. We rewrite

$$\begin{aligned}\left(\sum_{j=1}^{J} \beta_j(t) M_j(x)\right) &= \left(\sum_{j=1}^{J} \beta_j(t) M_{j1}(x), \sum_{j=1}^{J} \beta_j(t) M_{j2}(x), \dots, \sum_{j=1}^{J} \beta_j(t) M_{jn}(x)\right) \\ &=: (y_1, y_2, \dots, y_n).\end{aligned}$$

By inserting (2.1) into (1.1), we have

$$\begin{aligned}&\sum_{i=1}^{I} \alpha_{it} v_i + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} \beta_{jt} M_{jk} v_{iy_k} + \sum_{i_1,i_2=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n} \alpha_{i_1}\alpha_{i_2}\beta_j M_{jkx_k} v_{i_2k} v_{i_1y_k} + \nabla p \\ &\quad = \sum_{i=1}^{I}\sum_{j_1,j_2=1}^{J}\sum_{k,l,m,s=1}^{n} \alpha_i \beta_{j_1}\beta_{j_2} M_{j_1mx_k} M_{j_2sx_k} v_{iy_ky_l} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k,l=1}^{n} \alpha_i\beta_j v_{iy_l} M_{jlx_kx_k}, \\ &\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k,l=1}^{n} \alpha_i \beta_j v_{iy_l} M_{jlx_k} = 0.\end{aligned} \tag{2.2}$$

For the incompressible Euler equations, we take $n = 2$, $I = 1$, $J = 2$, $\alpha_1 = c(t)$, $\beta_1 = \beta_2 = 1$, $v_1(y_1,y_2) = (y_1,y_2)$, and we also set

$$M_1(x) = \left(\frac{x_2}{r^2}, -\frac{x_1}{r^2}\right), \qquad M_2(x) = \big(h(r)x_2, -h(r)x_1\big), \tag{2.3}$$

where $r = \sqrt{x_1^2 + x_2^2}$. Then u and p must satisfy the following equation:

$$\frac{c'(t)}{r^2}(x_2, -x_1) - \left(\frac{c(t)}{r^2} + h(r)\right)^2 (x_1, x_2) + \nabla p = 0. \tag{2.4}$$

So the incompressible Euler equations (2.6) below in Theorem 2.2 have a family of (I,J) similar solutions,

$$\begin{aligned} &u = \left(\left(\frac{c(t)}{r^2} + h(r)\right)x_2, -\left(\frac{c(t)}{r^2} + h(r)\right)x_1\right), \\ &p = -c'(t)\arctan\frac{x_1}{x_2} + F(r,t), \quad x_2 \neq 0, \end{aligned} \tag{2.5}$$

where $F(r,t) = \int r(\frac{c(t)}{r^2} + h(r))^2\,dr$, c is an arbitrary smooth function of t, h is an arbitrary smooth function of r.

Thus we have the following result.

Theorem 2.2 *Equation* (2.5) *is a family of* (I,J) *similar solutions to the incompressible Euler equations*,

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = 0, & \text{in } R^2 \times (0,\infty), \\ \operatorname{div} u = 0, & \text{in } R^2 \times (0,\infty). \end{cases} \tag{2.6}$$

Remark 2.3 To the best of our knowledge, there is little known of exact solutions to vortex equations, but they are not solutions to the two-dimensional Euler equations (2.6) except for the zero solution and they did not bring about any solution to the two-dimensional Euler equations (2.6) by the Biot-Savart law as they have a singularity (see [9, 10]), as seen by using VIM (see [11]), and by a Bäcklund transformation (see [12]) method. Notice that (2.5) correctly is a family of exact solutions to the two-dimensional Euler equations (2.6).

Remark 2.4 It is interesting to get many properties by choosing $c(t)$, $h(r)$.

Example 2.5 According to [10], we see that

$$\begin{aligned} &u = \left(\left(\frac{c(t)}{|x-Ct|^2} + h\big(|x-Ct|\big)\right)x_2 + c_1, -\left(\frac{c(t)}{|x-Ct|^2} + h\big(|x-Ct|\big)\right)x_1 + c_2\right), \\ &p = -c'(t)\arctan\frac{x_1 - c_1 t}{x_2 - c_2 t} + F\big(|x-Ct|, t\big), \\ &F(r,t) = \int r\left(\frac{c(t)}{r^2} + h(r)\right)^2 dr \end{aligned} \tag{2.7}$$

is also a solution pair for any constant vectors $C \in R^2$. $w = Q^T u(Qx,t)$, $\bar{p} = p(Qx,t)$ is also a solution pair for any rotation matrices Q.

Example 2.6 There are some u with finite energy only at some points, such as $t = 1$. Taking $c(t) = t$, $h(r) = -\frac{1}{r^2}$

$$\begin{aligned} &u = \left(\left(\frac{t}{r^2} - \frac{1}{r^2}\right)x_2, -\left(\frac{t}{r^2} - \frac{1}{r^2}\right)x_1\right), \\ &p = -\arctan\frac{x_1}{x_2} - \frac{1}{2r^2}(t-1)^2. \end{aligned} \tag{2.8}$$

Then $u(x,0) \in L^{2+\epsilon}(R^2\backslash B_\delta)$, however, for every $\epsilon > 0$ and $\delta > 0$, we have $u(x,t) \in L^{2+\epsilon}(R^2\backslash B_\delta)$ but $u(x,t) \notin L^2(R^2\backslash B_\delta)$, where $B_\delta = \{x \in R^2 \,|\, |x| < \delta\}$.

Example 2.7 There are some explicit solutions u with singularities only at some points. Taking $c(t) = \frac{1}{T-t}$, $h(r) = -\frac{1}{r^2}$,

$$\begin{aligned} u &= \left(\left(\frac{1}{r^2(T-t)} - \frac{1}{r^2}\right)x_2, -\left(\frac{1}{r^2(T-t)} - \frac{1}{r^2}\right)x_1\right), \\ p &= -\frac{1}{(T-t)^2}\arctan\frac{x_1}{x_2} - \frac{1}{2r^2}\left(\frac{1}{T-t} - 1\right)^2. \end{aligned} \tag{2.9}$$

Then u is singular at $r = 0$ and blows up at $t = T$.

3 Twin wave solutions

In this section we give more explicit nonzero solutions by considering explicit twin wave solutions to the two-dimensional Euler equations. Here a twin wave solution has the form of $u = u(x_1 - c_1 t, x_2 - c_2 t)$. The twin wave solution is a (I, J) similar solution. In fact, if we take $I = 1, J = 3, \beta_1 = 1, \beta_2 = -c_1 t, \beta_3 = -c_2 t, M_1 = (x_1, x_2), M_2 = (1, 0), M_3 = (0, 1), v_1(y_1, y_2) = u(y_1, y_2)$, then $u = u(x_1 - c_1 t, x_2 - c_2 t)$. Inserting them into (2.6), we have the following theorem.

Theorem 3.1 *If the pressure is independent of x, all twin wave solutions to the two-dimensional Euler equations $u = u(x_1 - c_1 t, x_2 - c_2 t)$ will be given by $u(x, t) = (v(c_3 x_1 - x_2 - (c_3 c_1 - c_2)t) + c_1, c_3 v(c_3 x_1 - x_2 - (c_3 c_1 - c_2)t) + c_2)$, where v is any function of $c_3 x_1 - x_2 - (c_3 c_1 - c_2)t$, and c_1, c_2, c_3 are arbitrary constants.*

Proof Inserting $u = u(x_1 - c_1 t, x_2 - c_2 t)$ and $p(x, t) = p(x_1, x_2, t)$ into (2.6),

$$\begin{cases} (-c_1 + u_1)u_{1x_1} + (-c_2 + u_2)u_{1x_2} + p_{x_1} = 0, \\ (-c_1 + u_1)u_{2x_1} + (-c_2 + u_2)u_{2x_2} + p_{x_2} = 0, \\ u_{1x_1} + u_{2x_2} = 0. \end{cases} \tag{3.1}$$

Then (3.1) is rewritten as

$$\begin{cases} (u_1 - c_1)(u_1 - c_1)_{x_1} + (u_2 - c_2)(u_1 - c_1)_{x_2} + p_{x_1} = 0, \\ (u_1 - c_1)(u_2 - c_2)_{x_1} + (u_2 - c_2)(u_2 - c_2)_{x_2} + p_{x_2} = 0, \\ (u_1 - c_1)_{x_1} + (u_2 - c_2)_{x_2} = 0. \end{cases} \tag{3.2}$$

Let us take $y_1 = x_1 - c_1 t$, $y_2 = x_2 - c_2 t$, $v_1(y_1, y_2) = u_1 - c_1$, $v_2(y_1, y_2) = u_2 - c_2$. By (3.2), we have

$$\begin{cases} v_1 v_{1y_1} + v_2 v_{1y_2} + p_{x_1} = 0, \\ v_1 v_{2y_1} + v_2 v_{2y_2} + p_{x_2} = 0, \\ v_{1y_1} + v_{2y_2} = 0. \end{cases} \tag{3.3}$$

If $p \equiv p(t)$, this may be interpreted as the equations in $v_1(y_1, y_2)$, $v_2(y_1, y_2)$.

We now consider the following equations:

$$\begin{cases} v_1 v_{2y_2} = v_2 v_{1y_2}, \\ v_1 v_{2y_1} = v_2 v_{1y_1}, \\ v_{1y_1} + v_{2y_2} = 0. \end{cases} \tag{3.4}$$

So

$$\begin{cases} v_2 = f(y_1) v_1, \\ v_2 = g(y_2) v_1, \\ v_{1y_1} + v_{2y_2} = 0, \end{cases} \tag{3.5}$$

where $f(y_1)$, $g(y_2)$ are determinate functions. Hence, we have

$$f(y_1) = g(y_2) = c_3.$$

Further let

$$\begin{cases} v_1 = v(c_3 y_1 - y_2), \\ v_2 = c_3 v(c_3 y_1 - y_2). \end{cases} \tag{3.6}$$

Therefore,

$$\begin{cases} u_1 = v(c_3 x_1 - x_2 - (c_3 c_1 - c_2)t) + c_1, \\ u_2 = c_3 v(c_3 x_1 - x_2 - (c_3 c_1 - c_2)t) + c_2, \end{cases} \tag{3.7}$$

where v is any function of $c_3 x_1 - x_2 - (c_3 c_1 - c_2)t$, and c_1, c_2, c_3 are arbitrary constants. □

Example 3.2 By taking $c(t) = 1$, $h(r) = -\frac{1}{r^2} + \frac{1}{(1+r^2)^2}$ in Example 2.5, we have

$$\begin{aligned} u &= \left(\frac{1}{(1 + |x - Ct|^2)^2} x_2 + c_1, -\frac{1}{(1 + |x - Ct|^2)^2} x_1 + c_2 \right), \\ p &= -\frac{1}{6(1 + |x - Ct|^2)^3}. \end{aligned} \tag{3.8}$$

These are a global smooth twin wave solutions pair for any constant vectors $C \in R^2$. $w = Q^T u(Qx, t)$, $\bar{p} = p(Qx, t)$ are also a twin wave solutions pair for any rotation matrices Q.

Remark 3.3 These solutions in (3.8) are symmetric only in some domains. In particular, if $c_1 = c_2$, they are symmetry solutions for all $t \geq 0$, and if $c_1 \neq c_2$ they are not symmetric for all $t \neq 0$ and symmetric only at $t = 0$. These examples show that the difference between the velocity of flow and its wave speed $u - C$ has a finite energy over R^2, *i.e.* $u - C \in L^2(R^2)$.

Example 3.4 There are some forms of symmetric solutions u only in some domains; for example, taking $v(\xi) = \frac{1}{|\xi + T(c_1-c_2)|^2}$, $c_3 = 1$,

$$\begin{cases} u_1 = \frac{1}{|x_1-x_2+(c_1-c_2)(T-t)|^2} + c_1, \\ u_2 = \frac{1}{|x_1-x_2+(c_1-c_2)(T-t)|^2} + c_2, \\ p = p(t) \end{cases} \tag{3.9}$$

are some twin wave solutions to (2.6), and they form a symmetry only at $t = T$, or they are static.

Example 3.5 Take $v(\xi) = \frac{1}{(1+|\xi|^2)^2} - c_1$, $c_3 = 1$. Then we find that

$$\begin{cases} u_1 = \frac{1}{(1+|x_1-x_2-(c_1-c_2)t|^2)^2}, \\ u_2 = \frac{1}{(1+|x_1-x_2-(c_1-c_2)t|^2)^2} + c_2 - c_1, \\ p = p(t) \end{cases} \tag{3.10}$$

are global smooth twin wave solutions to the Euler equations (2.6) with finite energy in any bounded domain, but $u - C$ with infinite energy over R^2 except for the static case. If the components of the wave speed are equal, then the system is static.

Example 3.6 If we take $v(\xi) = \frac{1}{|\xi|^2}$, $c_3 = 1$, then

$$\begin{cases} u_1 = \frac{1}{|x_1-x_2-(c_1-c_2)t|^2} + c_1, \\ u_2 = \frac{1}{|x_1-x_2-(c_1-c_2)t|^2} + c_2, \\ p = p(t) \end{cases} \tag{3.11}$$

are some twin wave solutions with singularity to the Euler equations (2.6).

Remark 3.7 These solutions in (3.11) have a singularity on the line $\{(x_1,x_2)|x_1 - x_2 = (c_1 - c_2)t\}$ for every $t \geq 0$. In particular, we have the following result:

For every given time $t \geq 0$ and arbitrary line $\{(x_1,x_2)|Ax_1 - Bx_2 = C, A^2 + B^2 \neq 0\}$, there exist some solutions with singularity over the line $\{(x_1,x_2)|Ax_1 - Bx_2 = C, A^2 + B^2 \neq 0\}$.

Example 3.8 According to [10], $w = u(x - Ct, t) + C$, $\bar{p} = p(x - Ct, t)$ is also a solution pair for any constant vectors $C \in R^2$. $w = Q^T u(Qx, t)$, $\bar{p} = p(Qx, t)$, is also a solution pair for any rotation matrices Q. $w = \frac{\lambda}{\tau}(\frac{x}{\lambda}, \frac{t}{\tau})$, $\bar{p} = \frac{\lambda^2}{\tau^2}(\frac{x}{\lambda}, \frac{t}{\tau})$ is also a solution pair.

4 Nonexistence

In this section we consider the explicit affine solution to the two-dimensional Euler equations. Here a solution $u(x,t)$ is called an affine solution, if the $u(x,t)$ is denoted by $u(x,t) = (v_1(\frac{x_1-c_1t}{x_2-c_2t}), v_2(\frac{x_1-c_1t}{x_2-c_2t}))$, $c_2 \neq 0$. The affine solution indeed is a (I,J) similar solution. In fact, this is the case: $\beta_1 = 1$, $\beta_2 = -c_1t$, $\beta_3 = 1$, $\beta_4 = -c_2t$, $M_1 = (x_1, 0)$, $M_2 = (1, 0)$, $M_3 = (0, x_2)$, $M_4 = (0, 1)$, $z_1 = \beta_1 M_1 + \beta_2 M_2$, $z_2 = \beta_3 M_3 + \beta_4 M_4$, $w(z_1, z_2) = v(\frac{z_1}{z_2})$. We have the following result.

Theorem 4.1 *All affine solutions must be twin wave solutions. Affine solutions to the two-dimensional Euler equations are constant vectors. That is to say there does not exist a non-constant affine solution to the two-dimensional Euler equations.*

Proof Since we are concerned with an affine solution here, let

$$u(x,t) = \left(v_1\left(\frac{x_1 - c_1 t}{x_2 - c_2 t}\right), v_2\left(\frac{x_1 - c_1 t}{x_2 - c_2 t}\right)\right), \quad c_2 \neq 0,$$

by using (2.6), and letting $\xi = \frac{x_1 - c_1 t}{x_2 - c_2 t}$, a straightforward calculation shows that

$$\begin{cases} \frac{1}{x_2 - c_2 t} v_{1\xi} - \frac{x_1 - c_1 t}{(x_2 - c_2 t)^2} v_{2\xi} = 0, \\ v_\xi\left(\frac{-c_1}{x_2 - c_2 t} + \frac{c_2(x_1 - c_1 t)}{(x_2 - c_2)^2}\right) + \frac{1}{x_2 - c_2 t} v_1 v_\xi - \frac{x_1 - c_1 t}{(x_2 - c_2 t)^2} v_2 v_\xi + \nabla p = 0, \end{cases} \tag{4.1}$$

$$\begin{cases} v_\xi(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) + (x_1 - c_1 t)\nabla p = 0, \\ v_{1\xi} = \xi v_{2\xi}, \end{cases} \tag{4.2}$$

$$\begin{cases} v_{1\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) + (x_1 - c_1 t)p_{x_1} = 0, \\ v_{2\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) + (x_1 - c_1 t)p_{x_2} = 0, \\ v_{1\xi} = \xi v_{2\xi}, \end{cases} \tag{4.3}$$

$$\begin{cases} v_{1\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) + \xi p_\xi = 0, \\ v_{2\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) - \xi^2 p_\xi = 0, \\ v_{1\xi} = \xi v_{2\xi}, \end{cases} \tag{4.4}$$

$$\begin{cases} v_{2\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) + p_\xi = 0, \\ v_{2\xi}(-c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2) - \xi^2 p_\xi = 0, \\ v_{1\xi} = \xi v_{2\xi}. \end{cases} \tag{4.5}$$

Thus

$$p_\xi = -\xi^2 p_\xi$$

and

$$p = p(t). \tag{4.6}$$

Hence

$$\begin{cases} v_{2\xi} = 0, \\ v_{1\xi} = \xi v_{2\xi}, \end{cases} \tag{4.7}$$

or

$$\begin{cases} -c_1\xi + c_2\xi^2 + \xi v_1 - \xi^2 v_2 = 0, \\ v_{1\xi} = \xi v_{2\xi}. \end{cases} \tag{4.8}$$

Therefore, we get

$$v_1 = \text{constant}, \qquad v_2 = \text{constant}, \qquad p = p(t). \qquad \square$$

5 Uniqueness

In this section, we study the uniqueness of the initial boundary value problem for the Euler equations ($\sigma = 0$) below:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \sigma \triangle u, \quad \text{in } R^n \times (0,\infty), \\ &\operatorname{div} u = 0, \quad \text{in } R^n \times [0,\infty), n = 2,3, \end{aligned} \tag{5.1}$$

and we have the following result.

Theorem 5.1 *Assume that $\Omega \subset R^2_+$ is a bounded domain and $\inf_{x\in\Omega}\{x_2\} = a > 0$. If $c \in C^\infty(\overline{R^1_+})$, $h(s)$ is an arbitrary smooth function of s satisfying $|h(r)| \le M_1$, $|h'_r(r)| \le \frac{M_2}{r^2}$ for $r \ge a > 0$, where M_1, M_2, and a are positive constants, and $r = \sqrt{x_1^2 + x_2^2}$. Then the following initial boundary value problem*:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = 0, & \textit{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0, & \textit{in } \Omega \times (0,\infty), \\ u(x,0) = ((\frac{c_0}{r^2} + h(r))x_2, -(\frac{c_0}{r^2} + h(r))x_1), & \textit{in } \Omega, \\ p(x,0) = -c_1 \arctan \frac{x_1}{x_2} + \int_a^r s(\frac{c_0}{s^2} + h(s))^2\, ds, & \textit{in } \Omega, \\ u(x,t) = ((\frac{c(t)}{r^2} + h(r))x_2, -(\frac{c(t)}{r^2} + h(r))x_1), & \textit{on } \partial\Omega \times [0,\infty), \\ p(x,t) = -c'(t) \arctan \frac{x_1}{x_2}, & \textit{on } \partial\Omega \times [0,\infty), \end{cases} \tag{5.2}$$

has a unique smooth solution $u \in (C^\infty([0,\infty) \times \Omega))^2$, $p \in C^\infty([0,\infty) \times \Omega)$.

Proof Notice that $\operatorname{div} u(x,0) = 0$, $\inf_{x\in\Omega}\{x_2\} = a > 0$, $c \in C^\infty[0,\infty)$, and given the assumptions on $h(s)$ satisfying the initial boundary value problems in (2.6), the result about existence is directly derived by Theorem 2.2. To prove the uniqueness we consider two smooth solution pairs, say u, p and v, p_1. Let their difference be $w = u - v$, with initial value w_0, and let $\tilde{p}$ be the difference of the corresponding pressures. Then, subtracting the equations from each other in (5.1), we have

$$\begin{cases} w_t + u \cdot \nabla w + w \cdot \nabla u - w \cdot \nabla w + \nabla \tilde{p} = 0, & \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0, \qquad \operatorname{div} w = 0, & \text{in } \Omega \times [0,\infty), \\ w(x,0) = (0,0), & \text{in } \Omega, \\ \tilde{p}(x,0) = 0, & \text{in } \Omega, \\ w|_{\partial\Omega} = (0,0), \\ \tilde{p}|_{\partial\Omega} = 0. \end{cases} \tag{5.3}$$

Multiplying the first equation of (5.3) by w, integrating over Ω, and using the Gauss formula, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2} = -(w \cdot \nabla u, w), \tag{5.4}$$

since (using the Gauss formula and $\operatorname{div} u = 0$, $\operatorname{div} w = 0$)

$$\begin{aligned}(u\cdot\nabla w, w) &= \sum_{i,j=1}^{2}\int_\Omega u_j\frac{\partial w_i}{\partial x_j}w_i\,dx\\ &= \int_{\partial\Omega}\frac{|w|^2}{2}u\cdot n\,ds\\ &= 0,\\ (\nabla\tilde{p}, w) &= \sum_{i=1}^{2}\int_\Omega\frac{\partial\tilde{p}}{\partial x_i}w_i\,dx\\ &= -\int_\Omega\tilde{p}\operatorname{div} w\,dx\\ &= 0,\end{aligned}$$

where n stands for the outward unit normal to Ω.

Similarly $(w\cdot\nabla w, w) = \int_{\partial\Omega}\frac{|w|^2}{2}w\cdot n\,ds = 0$.

By using $u = ((\frac{c(t)}{r^2} - h(r))x_2, -(\frac{c(t)}{r^2} - h(r))x_1)$ in the term $|-(w\cdot\nabla u, w)|$, we obtain

$$\bigl|-(w\cdot\nabla u, w)\bigr| \le \|w\|_{L^2}^2\|\nabla u\|_{L^\infty}, \tag{5.5}$$

$$\begin{aligned}|\nabla u| &= \sqrt{\left[\left(-2\frac{c(t)}{r^4} + h_r'\right)\right]^2 r^4 + 2\left(\frac{c(t)}{r^2} + h\right)^2}\\ &\le (2+\sqrt{2})\frac{|c(t)|}{r^2} + \sqrt{2}|h| + r^2\bigl|h_r'\bigr|,\end{aligned} \tag{5.6}$$

$$\|\nabla u\|_{L^\infty} \le (2+\sqrt{2})\frac{|c(t)|}{a^2} + M_1 + M_2.$$

Inserting (5.5)-(5.6) into (5.4), it follows that

$$\frac{d}{dt}\|w\|_{L^2}^2 \le 2\left((2+\sqrt{2})\frac{|c(t)|}{a^2} + M_1 + M_2\right)\|w\|_{L^2}^2. \tag{5.7}$$

Thanks to the Gronwall inequality, we have the following:

$$\begin{aligned}\|w\|_{L^2}^2 &\le \exp^{\int_0^t 2((2+\sqrt{2})\frac{|c(t)|}{a^2} + M_1 + M_2)\,d\tau}\bigl\|w(x,0)\bigr\|_{L^2}^2\\ &= 0.\end{aligned} \tag{5.8}$$

Therefore there exists a unique solution in the sense of $L^2(0,T;(L^2(\Omega))^3)$, $\forall T > 0$. The denseness of $C^\infty([0,T]\times\Omega)$ in $L^2(0,T;(L^2(\Omega))^3)$ implies the uniqueness of the solution in the sense of $C^\infty([0,T]\times\Omega)$ from $w = u - v \in (C^\infty([0,\infty)\times\Omega))^2$. We can apply the same argument on the intervals $[T,2T]$, $[2T,3T]$, *etc.*, according to the uniform Gronwall lemma since $c \in C^\infty(\overline{R_+^1})$. We obtain the uniqueness of the solution.

Let us again consider Example 2.6. We have

$$\begin{aligned}
-(w\cdot\nabla u, w) &= -(t-1)\int_\Omega\left[w_1^2\frac{-2x_1x_2}{r^4} + w_1w_2\frac{2(x_1^2-x_2^2)}{r^4} + \frac{2x_1x_2}{r^4}w_2^2\right]dx \\
&= -(t-1)\int_\Omega \left(w_2^2-w_1^2\right)\frac{2x_1x_2}{r^4} + w_1w_2\frac{2(x_1^2-x_2^2)}{r^4}\,dx \\
&= -(t-1)\int_\Omega \frac{1}{r^2}\left[\left(w_2^2-w_1^2\right)\sin(2\theta) + 2w_1w_2\cos(2\theta)\right]dx \\
&\le \frac{|t-1|}{a^2}\|w\|_{L^2}^2.
\end{aligned} \tag{5.9}$$

Using (5.9) for the right-hand side of (5.4), we obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 \le \frac{2|t-1|}{a^2}\|w\|_{L^2}^2. \tag{5.10}$$

Thanks to the Gronwall inequality again, we have the following result:

$$\begin{aligned}
\|w\|_{L^2}^2 &\le \exp^{\int_0^t \frac{2|t-1|}{a^2}\,d\tau}\left\|w(x,0)\right\|_{L^2}^2 \\
&= 0.
\end{aligned} \tag{5.11}$$

Thus we prove the uniqueness of the solution.

On taking a more in-depth look, such as considering $\Omega = \{(x_1,x_2)|(\sqrt{2}-1)^2x_2^2 \le x_1^2 \le x_2^2\}$, we have

$$\begin{aligned}
-(w\cdot\nabla u, w) &\le -(t-1)\int_\Omega\left(w_1^2\frac{x_1^2-x_2^2-2x_1x_2}{r^4} + w_2^2\frac{x_1^2-x_2^2+2x_2x_2}{r^4}\right)dx \\
&= -\frac{\sqrt{2}}{2}(t-1)\int_\Omega\left(w_1^2\frac{\sin(\frac{\pi}{4}-2\theta)}{r^2} + w_2^2\frac{\sin(\frac{\pi}{4}+2\theta)}{r^2}\right)dx \\
&\le 0, \quad \forall t\ge 1.
\end{aligned} \tag{5.12}$$

Thus we obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 \le 0. \tag{5.13}$$

We get

$$\begin{aligned}
\|w\|_{L^2}^2 &\le \left\|w(x,0)\right\|_{L^2}^2 \\
&= 0, \quad \text{uniformly for } t\ge 1.
\end{aligned} \tag{5.14}$$

It clarifies that the uniqueness of the solution is possible even as $t\to\infty$ if the right scope is chosen in Ω. □

6 Analysis of stability between the equations and its vortex equation

In this section we discuss the stability of the solution, respectively, in $C(\Omega)$ and $L^2(\Omega)$ for the problem (5.1).

Let v, p_1, and v_0 denote the solution pair of a slight disturbance and the initial boundary value of v. Let $w = u - v$ be the difference of u and v, with initial value $w_0(x) \to 0$ in the sense of $L^2(\Omega)$ and boundary value $u \to v$ in the sense of $L^2(\partial\Omega) \cap L^\infty(\partial\Omega)$, $\forall t \in (0, T]$, and let $\tilde{p}$ be the difference of the corresponding pressure. Then, subtracting one equation from the other, if

$$\begin{cases} w_t + u \cdot \nabla w + w \cdot \nabla u - w \cdot \nabla w + \nabla \tilde{p} = 0, \\ \operatorname{div} u = 0, \qquad \operatorname{div} w = 0, \\ w(x, 0) = w_0(x), \\ w|_{\partial\Omega} = \psi(x, t) \end{cases} \tag{6.1}$$

for $(x, t) \in \Omega \times (0, T)$, and $\Omega = \{r \geq a | a > 0\} \cap \{(x_1, x_2) | x_1^2 \geq x_2^2\}$. Multiplying this by w, integrating over Ω, and using the Gauss formula, we obtain

$$\frac{d}{dt}\|w\|_{L_2}^2 \leq 2\left|(u \cdot \nabla w, w)\right| + 2\left|-(w \cdot \nabla u, w)\right| + 2\left|(w \cdot \nabla w, w)\right|. \tag{6.2}$$

Using the Gauss formula and $\operatorname{div} u = 0$, $\operatorname{div} w = 0$,

$$\begin{aligned} (\nabla \tilde{p}, w) &= \sum_{i=1}^{2} \int_\Omega \frac{\partial \tilde{p}}{\partial x_i} w_i \, dx \\ &= -\int_\Omega \tilde{p} \operatorname{div} w \, dx \\ &= 0, \end{aligned}$$

$$\begin{aligned} \left|(u \cdot \nabla w, w)\right| &= \left| \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial w_i}{\partial x_j} w_i \, dx \right| \\ &= \left| \int_{\partial\Omega} \frac{|w|^2}{2} u \cdot n \, ds \right| \\ &\leq \frac{1}{2} \|u\|_{L^\infty(\partial\Omega)} \|w\|_{L^2(\partial\Omega)}^2, \end{aligned}$$

where n stands for the outward unit normal to Ω.

Similarly,

$$\begin{aligned} \left|-(w \cdot \nabla u, w)\right| &\leq \left((2 + \sqrt{2}) \frac{|c(t)|}{a^2} + M_1 + M_2 \right) \|w\|_{L^2}^2, \\ \left|(w \cdot \nabla w, w)\right| &= \left| \int_{\partial\Omega} \frac{|w|^2}{2} w \cdot n \, ds \right| \\ &\leq \frac{1}{2} \|w\|_{L^\infty(\partial\Omega)} \|w\|_{L^2(\partial\Omega)}^2. \end{aligned}$$

Combining the formulas above, we have

$$\begin{aligned} \frac{d}{dt}\|w\|_{L^2}^2 \leq{}& 2\left((2 + \sqrt{2}) \frac{|c(t)|}{a^2} + M_1 + M_2 \right) \|w\|_{L^2}^2 \\ &+ 2\left(\|u\|_{L^\infty(\partial\Omega)} + \|w\|_{L^\infty(\partial\Omega)} \right) \|w\|_{L^2(\partial\Omega)}^2. \end{aligned} \tag{6.3}$$

Since $u \to v$ in the sense of $L^2(\partial\Omega) \cap L^\infty(\partial\Omega)$, we can make, for every given $\varepsilon > 0$,

$$\left(\|u\|_{L^\infty(\partial\Omega)} + \|w\|_{L^\infty(\partial\Omega)}\right)\|w\|^2_{L^2(\partial\Omega)} \leq \varepsilon.$$

Using the Gronwall inequality in (6.3), for every $t \in [0, T]$,

$$\begin{aligned} \|w\|^2_{L^2} &\leq \exp^{\int_0^t 2((2+\sqrt{2})\frac{|c(\tau)|}{a^2}+M_1+M_2)\,d\tau} \left\|w_0(x)\right\|^2_{L^2} \\ &\quad + 2\varepsilon \int_0^t \exp^{\int_\tau^t 2((2+\sqrt{2})\frac{|c(s)|}{a^2}+M_1+M_2)\,ds}\,d\tau \\ &\to 0, \end{aligned} \tag{6.4}$$

as $u \to v$ in the sense of $L^2(\partial\Omega) \cap L^\infty(\partial\Omega)$ and $\|w_0\|^2_{L^2(\Omega)} \to 0$. So we reach the stability of the solution in finite time.

Remark 6.1 Adopting the same method as we use in the proof of Theorem 5.1, we obtain

$$\frac{d}{dt}\|w\|^2_{L^2} \leq 2\left(\|u\|_{L^\infty(\partial\Omega)} + \|w\|_{L^\infty(\partial\Omega)}\right)\|w\|^2_{L^2(\partial\Omega)}, \tag{6.5}$$

$$\begin{aligned} \|w\|^2_{L^2} &\leq 2t\varepsilon + 2\left\|w(x,0)\right\|^2_{L^2} \\ &\to 0, \quad \text{as } u \to v \text{ in the sense of } L^2(\partial\Omega) \cap L^\infty(\partial\Omega), \end{aligned}$$

for $t \in [0, T]$.

7 Explicit solution to (3 + 1)-dimensional Navier-Stokes equation

We now give an improvement of the example in reference [13].

Example 7.1 Let $0 < T \leq \infty$, and consider the initial problem for the Navier-Stokes equation,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \sigma \triangle u, & \text{in } R^3 \times (0, T), \\ \operatorname{div} u = 0, & \text{in } R^3 \times (0, T), \\ u(x,0) = u_0 = (u_{10}, u_{20}, u_{30}), \qquad p(x,0) = p_0, \\ \quad u_{i0}, p_0 \in C^\infty([0, T_1] \times B_M) \\ \quad \text{for any } 0 < T_1 < T, 0 < M < +\infty, \end{cases} \tag{7.1}$$

where $B_M = \{x \in R^3 \,|\, |x| < M\}$. Suppose

$$\begin{aligned} u_{10} &= \frac{1}{\sqrt{T}}\left(-1 + c_1 \exp\left(\frac{1}{12\sigma T}s^2 - \frac{1}{\sigma\sqrt{T}}s + c\right)\right), \\ u_{20} &= \frac{1}{\sqrt{T}}\left(-1 + c_2 \exp\left(\frac{1}{12\sigma T}s^2 - \frac{1}{\sigma\sqrt{T}}s + c\right)\right), \\ u_{30} &= -\frac{1}{\sqrt{T}}\left(1 + (c_1 + c_2) \exp\left(\frac{1}{12\sigma T}s^2 - \frac{1}{\sigma\sqrt{T}}s + c\right)\right), \end{aligned} \tag{7.2}$$

$$p_0 = -\frac{1}{T}\left(\frac{1}{2\sqrt{T}}s + c_3\right),$$
$$s = \sum_{i=1}^{3}(x_i - x_{i0}).$$

Then for arbitrary constants c_1, c_2, c_3, and c, (7.1) has a class of smooth blow-up solutions at finite time T,

$$u = (u_1, u_2, u_3), p, \tag{7.3}$$

$u_i, p \in C^\infty([0, T_1] \times B_M)$ for any $0 < T_1 < T$, $0 < M < \infty$. We have

$$\begin{cases} u_1 = \frac{1}{\sqrt{T-t}}(-1 + c_1 \exp(\frac{1}{12\sigma(T-t)}s^2 - \frac{1}{\sigma\sqrt{T-t}}s + c)), \\ u_2 = \frac{1}{\sqrt{T-t}}(-1 + c_2 \exp(\frac{1}{12\sigma(T-t)}s^2 - \frac{1}{\sigma\sqrt{T-t}}s + c)), \\ u_3 = -\frac{1}{\sqrt{T-t}}(1 + (c_1 + c_2)\exp(\frac{1}{12\sigma(T-t)}s^2 - \frac{1}{\sigma\sqrt{T-t}}s + c)), \\ p = \frac{1}{T-t}(\frac{1}{2\sqrt{T-t}}s + c_3). \end{cases} \tag{7.4}$$

Moreover, the initial function satisfies the second equation,

$$\operatorname{div} u_0 = 0, \quad \text{in } R^3. \tag{7.5}$$

Here a solution of (7.1) is called a smooth blow-up solution at finite time T, if $u_i, p \in C^\infty([0, T_1] \times B_M) \cap W^{m_1,q_1}(0, T_1; W^{m_2,q_2}(B_M))$ for any $T_1 \in (0, T)$, any nonnegative integer numbers m_1, m_2, any positive real numbers q_1, q_2, but

$$\lim_{t\to T^-} \|u\|_{W^{m,q}(B_M)} = +\infty, \qquad \lim_{t\to T^-} \|p\|_{W^{m,q}(B_M)} = +\infty, \tag{7.6}$$

for some nonnegative integer numbers m and positive real numbers q, M.

Remark 7.2 This example indicates that the C^∞ solution of the Navier-Stokes equation does not always tend to a solution of the Euler equation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The third author is grateful to Professor Boling Guo and Professor Zhouping Xin for their support, and he also thanks Dr Jihui Wu for help in writing this paper. The work is supported by National Natural Science Foundation of China (No. 11161057).

Received: 11 February 2014 Accepted: 27 June 2014 Published: 01 Aug 2014

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10.1186/1029-242X-2014-277
Cite this article as: Song et al.: **Nonhomogeneous boundary value problem for** (I,J) **similar solutions of incompressible two-dimensional Euler equations.** *Journal of Inequalities and Applications* 2014, 2014:277