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Weighted boundedness of multilinear operators associated to singular integral operators with non-smooth kernels

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Abstract

In this paper, we establish the weighted sharp maximal function inequalities for a multilinear operator associated to a singular integral operator with non-smooth kernel. As an application, we obtain the boundedness of the operator on weighted Lebesgue and Morrey spaces.

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1 Introduction and preliminaries

As the development of singular integral operators (see [1, 2]), their commutators and multilinear operators have been well studied. In [3–5], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [6]) proves a similar result when the singular integral operators are replaced by the fractional integral operators. In [7, 8], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces is obtained. In [9, 10], the boundedness for the commutators generated by the singular integral operators and the weighted *BMO* and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces is obtained (also see [11]). In [12, 13], some singular integral operators with non-smooth kernels are introduced, and the boundedness for the operators and their commutators is obtained (see [14–17]). Motivated by these, in this paper, we study multilinear operators generated by singular integral operators with non-smooth kernels and the weighted Lipschitz and *BMO* functions.

In this paper, we study some singular integral operators as follows (see [13]).

Definition 1 A family of operators D_t, $t > 0$, is said to be an 'approximation to the identity' if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x,y) f(y)\,dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies

$$|a_t(x,y)| \leq h_t(x,y) = Ct^{-n/2}\rho\left(|x-y|^2/t\right),$$

where ρ is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}\rho\bigl(r^2\bigr) = 0$$

for some $\epsilon > 0$.

Definition 2 A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with the kernel $K(x,y)$ so that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an 'approximation to the identity' $\{B_t, t > 0\}$ such that TB_t has the associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} \bigl|K(x,y) - k_t(x,y)\bigr|\,dx \le c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an 'approximation to the identity' $\{A_t, t > 0\}$ such that $A_t T$ has the associated kernel $K_t(x,y)$ which satisfies

$$\bigl|K_t(x,y)\bigr| \le c_4 t^{-n/2} \quad \text{if } |x-y| \le c_3 t^{1/2}$$

and

$$\bigl|K(x,y) - K_t(x,y)\bigr| \le c_4 t^{\delta/2}|x-y|^{-n-\delta} \quad \text{if } |x-y| \ge c_3 t^{1/2}$$

for some $\delta > 0$, $c_3, c_4 > 0$. Moreover, let m be a positive integer and b be a function on R^n. Set

$$R_{m+1}(b;x,y) = b(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha b(y)(x-y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y)f(y)\,dy.$$

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b is a non-trivial generalization of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [18–20]). The main purpose of this paper is to prove sharp maximal inequalities for the multilinear operator T^b. As an application, we obtain the weighted L^p-norm inequality and Morrey space boundedness for the multilinear operator T^b.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well known that (see [1, 2])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x} \inf_{c\in C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.$$

Let

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

For $\eta > 0$, let $M^{\#}_{\eta}(f)(x) = M^{\#}(|f|^{\eta})^{1/\eta}(x)$ and $M_{\eta}(f)(x) = M(|f|^{\eta})^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function w, set

$$M_{\eta,p,w}(f)(x) = \sup_{Q\ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y)\, dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The sharp maximal function $M_A(f)$ associated with the 'approximation to the identity' $\{A_t, t > 0\}$ is defined by

$$M^{\#}_A(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| \, dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $\eta > 0$, let $M^{\#}_{A,\eta}(f) = M^{\#}_A(|f|^{\eta})^{1/\eta}$.

The A_p weight is defined by (see [1]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \left\{ w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x), \text{a.e.} \right\}.$$

Given a non-negative weight function w. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $Lip_{\beta}(w)$ is the space of functions b such that

$$\|b\|_{Lip_{\beta}(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p}\, dy \right)^{1/p} < \infty,$$

and the weighted BMO space $BMO(w)$ is the space of functions b such that

$$\|b\|_{BMO(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q \left| b(y) - b_Q \right|^p w(x)^{1-p} \, dy \right)^{1/p} < \infty.$$

Remark (1) It has been known that (see [9, 21]), for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \le Ck \|b\|_{Lip_\beta(w)} w(x) w\left(2^k Q\right)^{\beta/n}.$$

(2) It has been known that (see [1, 21]), for $b \in BMO(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \le Ck \|b\|_{BMO(w)} w(x).$$

(3) Let $b \in Lip_\beta(w)$ or $b \in BMO(w)$ and $w \in A_1$. By [22], we know that spaces $Lip_\beta(w)$ or $BMO(w)$ coincide and the norms $\|b\|_{Lip_\beta(w)}$ or $\|b\|_{BMO(w)}$ are equivalent with respect to different values $1 \le p < \infty$.

Definition 3 Let φ be a positive, increasing function on R^+, and let there exist a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for } t \ge 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $0 \le \eta < n$ and $1 \le p < n/\eta$,

$$\|f\|_{L^{p,\eta,\varphi}(w)} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} \left| f(y) \right|^p w(y) \, dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x - y| < d\}$. The generalized fractional weighted Morrey space is defined by

$$L^{p,\eta,\varphi}\left(R^n, w\right) = \left\{ f \in L^1_{\text{loc}}\left(R^n\right) : \|f\|_{L^{p,\eta,\varphi}(w)} < \infty \right\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized weighted Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey space (see [23, 24]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue space (see [1]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [22, 25–27]).

2 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be a singular integral operator with non-smooth kernel as given in Definition 2, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then there* *exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}_{A,\eta}\big(T^b(f)\big)(\tilde{x}) \le C \sum_{|\alpha|=m} \big\|D^\alpha b\big\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Theorem 2 *Let T be a singular integral operator with non-smooth kernel as given in Definition 2, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}_{A,\eta}\big(T^b(f)\big)(\tilde{x}) \le C \sum_{|\alpha|=m} \big\|D^\alpha b\big\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Theorem 3 *Let T be a singular integral operator with non-smooth kernel as given in Definition 3, $w \in A_1$, $1 < p < \infty$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, w)$ to $L^p(R^n, w^{1-p})$.*

Theorem 4 *Let T be a singular integral operator with non-smooth kernel as given in Definition 3, $w \in A_1$, $1 < p < \infty$, $0 < D < 2^n$ and $D^\alpha b \in BMO(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{p,\varphi}(R^n, w)$ to $L^{p,\varphi}(R^n, w^{1-p})$.*

Theorem 5 *Let T be a singular integral operator with non-smooth kernel as given in Definition 3, $w \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, w)$ to $L^q(R^n, w^{1-q})$.*

Theorem 6 *Let T be a singular integral operator with non-smooth kernel as given in Definition 3, $w \in A_1$, $0 < \beta < 1$, $0 < D < 2^n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha b \in Lip_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{p,\beta,\varphi}(R^n, w)$ to $L^{q,\varphi}(R^n, w^{1-q})$.*

To prove the theorems, we need the following lemmas.

Lemma 1 (see [1, p.485]) *Let $0 < p < q < \infty$, and for any function $f \ge 0$, we define that, for $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \big|\big\{x \in R^n : f(x) > \lambda\big\}\big|^{1/q}, \qquad N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup *is taken for all measurable sets Q with $0 < |Q| < \infty$. Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \big(q/(q-p)\big)^{1/p} \|f\|_{WL^q}.$$

Lemma 2 (see [12, 13]) *Let T be a singular integral operator with non-smooth kernel as given in Definition 2. Then T is bounded on $L^p(R^n, w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

Lemma 3 ([12, 13]) *Let $\{A_t, t > 0\}$ be an 'approximation to the identity'. For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that*

$$\big|\big\{x \in R^n : M(f)(x) > D\lambda, M^{\#}_A(f)(x) \le \gamma\lambda\big\}\big| \le C\gamma \big|\big\{x \in R^n : M(f)(x) > \lambda\big\}\big|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus, for $f \in L^p(R^n)$, $1 < p < \infty$, $0 < \eta < \infty$ and $w \in A_1$,

$$\left\|M_\eta(f)\right\|_{L^p(w)} \le C\left\|M^{\#}_{A,\eta}(f)\right\|_{L^p(w)}.$$

Lemma 4 (see [1, 6]) *Let $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then*

$$\left\|M_{\eta,s,w}(f)\right\|_{L^q(w)} \le C\|f\|_{L^p(w)}.$$

Lemma 5 (see [12, 13]) *Let $\{A_t, t > 0\}$ be an 'approximation to the identity', $0 < D < 2^n$, $1 < p < \infty$, $0 < \eta < \infty$, $w \in A_1$ and $w \in A_1$. Then*

$$\left\|M_\eta(f)\right\|_{L^{p,\varphi}(w)} \le C\left\|M^{\#}_{A,\eta}(f)\right\|_{L^{p,\varphi}(w)}.$$

Lemma 6 (see [22, 25]) *Let $0 \le \eta < n$, $0 < D < 2^n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then*

$$\left\|M_{\eta,s,w}(f)\right\|_{L^{q,\varphi}(w)} \le C\|f\|_{L^{p,\eta,\varphi}(w)}.$$

Lemma 7 (see [19]) *Let b be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and any $q > n$. Then*

$$\left|R_m(b;x,y)\right| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha b(z)\right|^q\,dz\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 8 *Let $\{A_t, t > 0\}$ be an 'approximation to the identity', $w \in A_1$ and $b \in BMO(w)$. Then, for every $f \in L^p(w)$, $p > 1$, $1 < r < \infty$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|}\int_Q \left|A_{t_Q}\left((b-b_Q)f\right)(y)\right|\,dy \le C\|b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Proof We write, for any cube Q with $\tilde{x} \in Q$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|A_{t_Q}\left((b-b_Q)f\right)(x)\right|\,dx \\
&\quad\le \frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x,y)\left|\left(b(y)-b_Q\right)f(y)\right|\,dy\,dx \\
&\quad\le \frac{1}{|Q|}\int_Q\int_Q h_{t_Q}(x,y)\left|\left(b(y)-b_Q\right)f(y)\right|\,dy\,dx \\
&\qquad + \sum_{k=0}^{\infty}\frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)\left|\left(b(y)-b_Q\right)f(y)\right|\,dy\,dx \\
&\quad= I + II.
\end{aligned}$$

We have, by Hölder's inequality,

$$\begin{aligned}
I &\le \frac{C}{|Q||Q|}\int_Q\int_Q\big|\big(b(y)-b_Q\big)f(y)\big|\,dy\,dx \\
&\le \frac{C}{|Q|}\int_Q\big|b(y)-b_Q\big|w(y)^{-1/r}\big|f(y)\big|w(y)^{1/r}\,dy \\
&\le \frac{C}{|Q|}\left(\int_Q\big|b(y)-b_Q\big|^{r'}w(y)^{1-r'}\,dy\right)^{1/r'}\left(\int_Q\big|f(y)\big|^{r}w(y)\,dy\right)^{1/r} \\
&\le \frac{C}{|Q|}\|b\|_{BMO(w)}w(Q)^{1/r'}w(Q)^{1/r}\left(\frac{1}{w(Q)}\int_Q\big|f(y)\big|^{r}w(y)\,dy\right)^{1/r} \\
&\le C\|b\|_{BMO(w)}\frac{w(Q)}{|Q|}M_{r,w}(f)(\tilde{x}) \\
&\le C\|b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For II, notice for $x\in Q$ and $y\in 2^{k+1}Q\setminus 2^kQ$, then $|x-y|\ge 2^{k-1}t_Q$ and $h_{t_Q}(x,y)\le C\frac{s(2^{2(k-1)})}{|Q|}$, then

$$\begin{aligned}
II &\le C\sum_{k=0}^{\infty}s\big(2^{2(k-1)}\big)\frac{1}{|Q||Q|}\int_Q\int_{2^{k+1}Q}\big|\big(b(y)-b_Q\big)f(y)\big|\,dy\,dx \\
&\le C\sum_{k=0}^{\infty}2^{kn}s\big(2^{2(k-1)}\big)\frac{1}{|2^{k+1}Q|} \\
&\quad\times\int_{2^{k+1}Q}\big|\big(b(y)-b_{2^{k+1}Q}\big)+(b_{2^{k+1}Q}-b_Q)\big|\big|f(y)\big|\,dy \\
&\le C\sum_{k=0}^{\infty}2^{kn}s\big(2^{2(k-1)}\big)\big|2^{k+1}Q\big|^{-1}\left(\int_{2^{k+1}Q}\big|b(y)-b_{2^{k+1}Q}\big|^{r'}w(y)^{1-r'}\,dy\right)^{1/r'} \\
&\quad\times\left(\int_{2^{k+1}Q}\big|f(y)\big|^{r}w(y)\,dy\right)^{1/r} \\
&\quad+C\sum_{k=0}^{\infty}2^{kn}s\big(2^{2(k-1)}\big)\big|2^{k+1}Q\big|^{-1}k\|b\|_{BMO(w)}w(\tilde{x})\left(\int_{2^{k+1}Q}\big|f(y)\big|^{r}w(y)\,dy\right)^{1/r} \\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r} \\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}w(y)\,dy\right)^{1/r}\big|2^{k+1}Q\big|w\big(2^{k+1}Q\big)^{-1/r} \\
&\le C\|b\|_{BMO(w)}\sum_{k=0}^{\infty}k2^{kn}s\big(2^{2(k-1)}\big)\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}+w(\tilde{x})\right) \\
&\quad\times\left(\frac{1}{w(2^{k+1}Q)}\int_{2^{k+1}Q}\big|f(y)\big|^{r}w(y)\,dy\right)^{1/r} \\
&\le C\|b\|_{BMO(w)}\sum_{k=0}^{\infty}k2^{kn}s\big(2^{2(k-1)}\big)w(\tilde{x})M_{r,w}(f)(\tilde{x}) \\
&\le C\|b\|_{BMO(w)}w(\tilde{x})M_{r,w}(f)(\tilde{x}),
\end{aligned}$$

where the last inequality follows from

$$\sum_{k=1}^{\infty} k 2^{(k-1)n} s\big(2^{2(k-1)}\big) \le C \sum_{k=1}^{\infty} k 2^{-(k-1)\epsilon} < \infty$$

for some $\epsilon > 0$. This completes the proof. □

Lemma 9 *Let $\{A_t, t > 0\}$ be an 'approximation to the identity', $w \in A_1$, $0 < \beta < 1$, $1 < r < \infty$ and $b \in Lip_\beta(w)$. Then, for every $f \in L^p(w)$, $p > 1$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q \big|A_{t_Q}\big((b - b_Q)f\big)(y)\big|\, dy \le C\|b\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,r}(f)(\tilde{x}).$$

The same argument as in the proof of Lemma 8 will give the proof of Lemma 9, we omit the details.

3 Proofs of theorems

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q \big|T^b(f)(x) - A_{t_Q}\big(T^b(f)\big)(x)\big|^\eta\, dx\right)^{1/\eta} \le C \sum_{|\alpha|=m} \big\|D^\alpha b\big\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}),$$

where $t_Q = d^2$ and d denotes the side length of Q. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned}
T^b(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x-y|^m} K(x,y) f_1(y)\, dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K(x,y) f_1(y)\, dy \\
&\quad + \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^m} K(x,y) f_2(y)\, dy \\
&= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x)
\end{aligned}$$

and

$$\begin{aligned}
A_{t_Q} T^b(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}_j; x, y)}{|x-y|^m} K_t(x,y) f_1(y)\, dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K_t(x,y) f_1(y)\, dy \\
&\quad + \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^m} K_t(x,y) f_2(y)\, dy \\
&= A_{t_Q} T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - A_{t_Q} T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + A_{t_Q} T^{\tilde{b}}(f_2)(x),
\end{aligned}$$

then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - A_{t_Q}T^b(f)(x)\right|^\eta\,dx\right)^{1/\eta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right)(x)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|A_{t_Q}T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|A_{t_Q}T\left(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right)(x)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - A_{t_Q}T^{\tilde{b}}(f_2)(x)\right|^\eta dx\right)^{1/\eta} \\
&\quad= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For I_1, noting that $w \in A_1$, w satisfies the reverse of Hölder's inequality

$$\left(\frac{1}{|Q|}\int_Q w(x)^{p_0}\,dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q w(x)\,dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [1]). We take $q = rp_0/(r + p_0 - 1)$ in Lemma 7 and have $1 < q < r$ and $p_0 = q(r-1)/(r-q)$, then by Lemma 7 and Hölder's inequality, we get

$$\begin{aligned}
\left|R_m(\tilde{b};x,y)\right| &\le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha\tilde{b}(z)\right|^q dz\right)^{1/q} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha\tilde{b}(z)\right|^q w(z)^{q(1-r)/r}w(z)^{q(r-1)/r}\,dz\right)^{1/q} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha\tilde{b}(z)\right|^r w(z)^{1-r}\,dz\right)^{1/r} \\
&\quad\times\left(\int_{\tilde{Q}(x,y)} w(z)^{q(r-1)/(r-q)}\,dz\right)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\left\|D^\alpha b\right\|_{BMO(w)}w(\tilde{Q})^{1/r}|\tilde{Q}|^{(r-q)/rq} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} w(z)^{p_0}\,dz\right)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(w)}|\tilde{Q}|^{-1/q}w(\tilde{Q})^{1/r}|\tilde{Q}|^{1/q-1/r} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} w(z)\,dz\right)^{(r-1)/r}
\end{aligned}$$

$$
\begin{aligned}
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/q-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).
\end{aligned}
$$

Thus, by the L^s-boundedness of T (see Lemma 2) for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$
\begin{aligned}
I_1 &\le \frac{C}{|Q|} \int_Q \left| T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1 \right)(x) \right| dx \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \left(\frac{1}{|Q|} \int_{R^n} \left| T(f_1)(x) \right|^s dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} \left| f_1(x) \right|^s dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} \left| f(x) \right|^s w(x)^{s/r} w(x)^{-s/r} dx \right)^{1/s} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} \left| f(x) \right|^r w(x) dx \right)^{1/r} \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) |Q|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} \left| f(x) \right|^r w(x) dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}
$$

For I_2, by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogorov's inequality (see Lemma 1), we obtain

$$
\begin{aligned}
I_2 &\le C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q \left| T\left(D^\alpha \tilde{b} f_1\right)(x) \right|^\eta dx \right)^{1/\eta} \\
&\le C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\| T(D^\alpha \tilde{b} f_1) \chi_Q \|_{L^\eta}}{\| \chi_Q \|_{L^{\eta/(1-\eta)}}} \\
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \left\| T\left(D^\alpha \tilde{b} f_1\right) \right\|_{WL^1} \\
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} \left| D^\alpha \tilde{b}(x) f_1(x) \right| dx \\
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} \left| D^\alpha b(x) - \left(D^\alpha b\right)_{\tilde{Q}} \right| w(x)^{-1/r} \left| f(x) \right| w(x)^{1/r} dx \\
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} \left| \left(D^\alpha b(x) - \left(D^\alpha b\right)_{\tilde{Q}}\right) \right|^{r'} w(x)^{1-r'} dx \right)^{1/r'} \left(\int_{\tilde{Q}} \left| f(x) \right|^r w(x) dx \right)^{1/r}
\end{aligned}
$$

$$\begin{aligned}
&\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{BMO(w)} w(\tilde{Q})^{1/r'} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x)\,dx \right)^{1/r} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{r,w}(f)(\tilde{x}) \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_3 and I_4, by Lemma 8 and similar to the proof of I_1 and I_2, we get

$$\begin{aligned}
I_3 &\le \frac{C}{|Q|} \int_Q \left| T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right)(x) \right| dx \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}), \\
I_4 &\le C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q \left| T\left(D^\alpha \tilde{b} f_1\right)(x)\right|^\eta dx \right)^{1/\eta} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_5, note that $|x-y| \approx |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus Q$. We have, by Lemma 7 and similar to the proof of I_1,

$$\left| R_m(\tilde{b}; x, y) \right| \le C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}).$$

Thus, by the conditions on K and K_t, and $w \in A_1 \subseteq A_r$,

$$\begin{aligned}
&\left| T^{\tilde{b}}(f_2)(x) - A_{t_Q} T^{\tilde{b}}(f_2)(x_0) \right| \\
&\quad\le \int_{R^n} \frac{|R_m(\tilde{b}; x, y)|}{|x-y|^m} \left|K(x,y) - K_t(x,y)\right| \left|f_2(y)\right| dy \\
&\qquad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{|D^\alpha \tilde{b}_1(y)||(x-y)^{\alpha_1}|}{|x-y|^m} \left|K(x,y) - K_t(x,y)\right| \left|f_2(y)\right| dy \\
&\quad\le \sum_{k=0}^\infty \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{d^\delta}{|x_0 - y|^{n+\delta}} \left|f(y)\right| w(y)^{1/r} w(y)^{-1/r}\,dy \\
&\qquad + C \sum_{|\alpha|=m} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}} \left| \left(D^\alpha b\right)_{2^{k+1}\tilde{Q}} - \left(D^\alpha b\right)_{\tilde{Q}} \right| \frac{d^\delta}{|x_0 - y|^{n+\delta}} \left|f(y)\right| w(y)^{1/r} w(y)^{-1/r}\,dy \\
&\qquad + C \sum_{|\alpha|=m} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}} \left| D^\alpha b(y) - \left(D^\alpha b\right)_{2^{k+1}\tilde{Q}} \right| \frac{d^\delta}{|x_0 - y|^{n+\delta}} \left|f(y)\right| w(y)^{1/r} w(y)^{-1/r}\,dy \\
&\quad\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^\infty k \frac{d^\delta}{(2^k d)^{n+\delta}} \left(\int_{2^k \tilde{Q}} \left|f(y)\right|^r w(y)\,dx \right)^{1/r} \\
&\qquad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)}\,dy \right)^{(r-1)/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)\,dy \right)^{1/r} \left|2^k \tilde{Q}\right| w\left(2^k \tilde{Q}\right)^{-1/r}
\end{aligned}$$

$$
\begin{aligned}
&+ C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \frac{d^{\delta}}{(2^k d)^{n+\delta}} \left(\int_{2^k \tilde{Q}} \left| D^{\alpha} b(y) - \left(D^{\alpha} b \right)_{2^k \tilde{Q}} \right|^{r'} w(y)^{1-r'} \, dy \right)^{1/r'} \\
&\times \left(\int_{2^k \tilde{Q}} \left| f(y) \right|^{r} w(y) \, dy \right)^{1/r} \\
\leq{}& C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{BMO(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k\delta} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} \left| f(y) \right|^{r} w(y) \, dx \right)^{1/r} \\
&+ C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{BMO(w)} \sum_{k=1}^{\infty} 2^{-k\delta} \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} \left| f(y) \right|^{r} w(y) \, dx \right)^{1/r} \\
\leq{}& C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k\delta} \\
\leq{}& C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}
$$

Thus

$$
I_5 \leq C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{BMO(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^{\infty}(R^n)$ and some constant C_0 that the following inequality holds:

$$
\left(\frac{1}{|Q|} \int_{Q} \left| T^{b}(f)(x) - A_{t_Q}\left(T^{b}(f) \right)(x) \right|^{\eta} dx \right)^{1/\eta} \leq C \sum_{|\alpha|=m} \left\| D^{\alpha} b \right\|_{Lip_{\beta}(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}),
$$

where $t_Q = d^2$ and d denotes the side length of Q. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$
\begin{aligned}
&\left(\frac{1}{|Q|} \int_{Q} \left| T^{b}(f)(x) - A_{t_Q} T^{b}(f)(x) \right|^{\eta} dx \right)^{1/\eta} \\
\leq{}& \left(\frac{1}{|Q|} \int_{Q} \left| T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right)(x) \right|^{\eta} dx \right)^{1/\eta} \\
&+ \left(\frac{1}{|Q|} \int_{Q} \left| T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x - \cdot)^{\alpha} D^{\alpha} \tilde{b}}{|x - \cdot|^m} f_1 \right)(x) \right|^{\eta} dx \right)^{1/\eta} \\
&+ \left(\frac{1}{|Q|} \int_{Q} \left| A_{t_Q} T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right)(x) \right|^{\eta} dx \right)^{1/\eta} \\
&+ \left(\frac{1}{|Q|} \int_{Q} \left| A_{t_Q} T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x - \cdot)^{\alpha} D^{\alpha} \tilde{b}}{|x - \cdot|^m} f_1 \right)(x) \right|^{\eta} dx \right)^{1/\eta} \\
&+ \left(\frac{1}{|Q|} \int_{Q} \left| T^{\tilde{b}}(f_2)(x) - A_{t_Q} T^{\tilde{b}}(f_2)(x) \right|^{\eta} dx \right)^{1/\eta} \\
={}& J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
$$

For J_1 and J_2, by using the same argument as in the proof of Theorem 1, we get

$$
\begin{aligned}
\bigl|R_m(\tilde{b};x,y)\bigr| &\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\biggl(\int_{\tilde{Q}(x,y)} \bigl|D^\alpha \tilde{b}(z)\bigr|^q w(z)^{q(1-r)/r} w(z)^{q(r-1)/r}\,dz\biggr)^{1/q} \\
&\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\biggl(\int_{\tilde{Q}(x,y)} \bigl|D^\alpha \tilde{b}(z)\bigr|^r w(z)^{1-r}\,dz\biggr)^{1/r} \\
&\quad \times \biggl(\int_{\tilde{Q}(x,y)} w(z)^{q(r-1)/(r-q)}\,dz\biggr)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{(r-q)/rq} \\
&\quad \times \biggl(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} w(z)^{p_0}\,dz\biggr)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r} \\
&\quad \times \biggl(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} w(z)\,dz\biggr)^{(r-1)/r} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r} w(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} \frac{w(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_1 &\le C\sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x})|Q|^{-1/s}\biggl(\int_{R^n} \bigl|f_1(x)\bigr|^s\,dx\biggr)^{1/s} \\
&\le C\sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x})|Q|^{-1/s}\biggl(\int_{\tilde{Q}} \bigl|f(x)\bigr|^r w(x)\,dx\biggr)^{1/r} \\
&\quad \times \biggl(\int_{\tilde{Q}} w(x)^{-s/(r-s)}\,dx\biggr)^{(r-s)/rs} \\
&\le C\sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{x})|\tilde{Q}|^{-1/s} w(\tilde{Q})^{1/r}\biggl(\frac{1}{w(\tilde{Q})^{1-r\beta/n}}\int_{\tilde{Q}} \bigl|f(x)\bigr|^r w(x)\,dx\biggr)^{1/r} \\
&\quad \times \biggl(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} w(x)^{-s/(r-s)}\,dx\biggr)^{(r-s)/rs}\biggl(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} w(x)\,dx\biggr)^{1/r}|\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
&\le C\sum_{|\alpha|=m} \bigl\|D^\alpha b\bigr\|_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}), \\
J_2 &\le C\sum_{|\alpha|=m} \frac{1}{|Q|}\int_{\tilde{Q}} \bigl|D^\alpha b(x) - \bigl(D^\alpha b\bigr)_{\tilde{Q}}\bigr| w(x)^{-1/r}\bigl|f(x)\bigr| w(x)^{1/r}\,dx \\
&\le C\sum_{|\alpha|=m} \frac{1}{|Q|}\biggl(\int_{\tilde{Q}} \bigl|\bigl(D^\alpha b(x) - \bigl(D^\alpha b\bigr)_{\tilde{Q}}\bigr)\bigr|^{r'} w(x)^{1-r'}\,dx\biggr)^{1/r'}\biggl(\int_{\tilde{Q}} \bigl|f(x)\bigr|^r w(x)\,dx\biggr)^{1/r}
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{Q})^{\beta/n+1/r'}w(\tilde{Q})^{1/r-\beta/n}\left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}}\int_{\tilde{Q}}|f(x)|^r w(x)\,dx\right)^{1/r}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}\frac{w(\tilde{Q})}{|\tilde{Q}|}M_{\beta,r,w}(f)(\tilde{x})\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}$$

For J_3 and J_4, by Lemma 9 and similar to the proof of J_1 and J_2, we get

$$\begin{aligned}
J_3 &\le \frac{C}{|Q|}\int_Q\left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|dx\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})M_{\beta,r,w}(f)(\tilde{x}),\\
J_4 &\le C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q\left|T\left(D^\alpha\tilde{b}f_1\right)(x)\right|^\eta dx\right)^{1/\eta}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}$$

For J_5, by Lemma 7 and similar to the proof of J_1, for $k\ge 0$, we have

$$\left|R_m(\tilde{b};x,y)\right|\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w\left(2^k\tilde{Q}\right)^{\beta/n}w(\tilde{x}).$$

Thus

$$\begin{aligned}
&\left|T^{\tilde{b}}(f_2)(x)-A_{t_Q}T^{\tilde{b}}(f_2)(x_0)\right|\\
&\quad\le\int_{R^n}\frac{|R_m(\tilde{b};x,y)|}{|x-y|^m}\left|K(x,y)-K_t(x,y)\right|\left|f_2(y)\right|dy\\
&\qquad+\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{|D^\alpha\tilde{b}_1(y)||(x-y)^{\alpha_1}|}{|x-y|^m}\left|K(x,y)-K_t(x,y)\right|\left|f_2(y)\right|dy\\
&\quad\le\sum_{k=0}^\infty\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})w\left(2^k\tilde{Q}\right)^{\beta/n}\\
&\qquad\times\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{d^\delta}{|x_0-y|^{n+\delta}}\left|f(y)\right|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\qquad+C\sum_{|\alpha|=m}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}}\left|\left(D^\alpha b\right)_{2^{k+1}\tilde{Q}}-\left(D^\alpha b\right)_{\tilde{Q}}\right|\frac{d^\delta}{|x_0-y|^{n+\delta}}\left|f(y)\right|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\qquad+C\sum_{|\alpha|=m}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}}\left|D^\alpha b(y)-\left(D^\alpha b\right)_{2^{k+1}\tilde{Q}}\right|\frac{d^\delta}{|x_0-y|^{n+\delta}}\left|f(y)\right|w(y)^{1/r}w(y)^{-1/r}\,dy\\
&\quad\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{Lip_\beta(w)}w(\tilde{x})\sum_{k=1}^\infty k\frac{d^\delta}{(2^kd)^{n+\delta}}w\left(2^k\tilde{Q}\right)^{\beta/n}\left(\int_{2^k\tilde{Q}}\left|f(y)\right|^r w(y)\,dx\right)^{1/r}\\
&\qquad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-1/(r-1)}\,dy\right)^{(r-1)/r}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\,dy\right)^{1/r}\left|2^k\tilde{Q}\right|w\left(2^k\tilde{Q}\right)^{-1/r}
\end{aligned}$$

$$
\begin{aligned}
&\quad + C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\frac{d^{\delta}}{(2^k d)^{n+\delta}}\left(\int_{2^k\tilde{Q}}\left|D^{\alpha}b(y)-\left(D^{\alpha}b\right)_{2^k\tilde{Q}}\right|^{r'}w(y)^{1-r'}\,dy\right)^{1/r'}\\
&\quad \times\left(\int_{2^k\tilde{Q}}\left|f(y)\right|^{r}w(y)\,dy\right)^{1/r}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{Lip_{\beta}(w)}w(\tilde{x})\sum_{k=1}^{\infty}k2^{-k\delta}\left(\frac{1}{w(2^k\tilde{Q})^{1-r\beta/n}}\int_{2^k\tilde{Q}}\left|f(y)\right|^{r}w(y)\,dx\right)^{1/r}\\
&\quad + C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{Lip_{\beta}(w)}\sum_{k=1}^{\infty}2^{-k\delta}\frac{w(2^k\tilde{Q})}{|2^k\tilde{Q}|}\left(\frac{1}{w(2^k\tilde{Q})^{1-r\beta/n}}\int_{2^k\tilde{Q}}\left|f(y)\right|^{r}w(y)\,dx\right)^{1/r}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{Lip_{\beta}(w)}w(\tilde{x})M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Choose $1 < r < p$ in Theorem 1 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 3 and 4,

$$
\begin{aligned}
\left\|T^b(f)\right\|_{L^p(w^{1-p})} &\le \left\|M_{\eta}\left(T^b(f)\right)\right\|_{L^p(w^{1-p})} \le C\left\|M^{\#}_{A,\eta}\left(T^b(f)\right)\right\|_{L^p(w^{1-p})}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\left\|wM_{r,w}(f)\right\|_{L^p(w^{1-p})}\\
&= C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\left\|M_{r,w}(f)\right\|_{L^p(w)}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $1 < r < p$ in Theorem 1 and notice $w^{1-p} \in A_1$, then we have, by Lemmas 5 and 6,

$$
\begin{aligned}
\left\|T^b(f)\right\|_{L^{p,\varphi}(w^{1-p})} &\le \left\|M_{\eta}\left(T^b(f)\right)\right\|_{L^{p,\varphi}(w^{1-p})} \le C\left\|M^{\#}_{A,\eta}\left(T^b(f)\right)\right\|_{L^{p,\varphi}(w^{1-p})}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\left\|wM_{r,w}(f)\right\|_{L^{p,\varphi}(w^{1-p})}\\
&= C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\left\|M_{r,w}(f)\right\|_{L^{p,\varphi}(w)}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{BMO(w)}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}
$$

This completes the proof of Theorem 4. □

Proof of Theorem 5 Choose $1 < r < p$ in Theorem 2 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 3 and 4,

$$
\begin{aligned}
\left\|T^b(f)\right\|_{L^q(w^{1-q})} &\le \left\|M_{\eta}\left(T^b(f)\right)\right\|_{L^q(w^{1-q})} \le C\left\|M^{\#}_{A,\eta}\left(T^b(f)\right)\right\|_{L^q(w^{1-q})}\\
&\le C\sum_{|\alpha|=m}\left\|D^{\alpha}b\right\|_{Lip_{\beta}(w)}\left\|wM_{\beta,r,w}(f)\right\|_{L^q(w^{1-q})}
\end{aligned}
$$

$$
\begin{aligned}
&= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^q(w)} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof of Theorem 5. □

Proof of Theorem 6 Choose $1 < r < p$ in Theorem 2 and notice $w^{1-q} \in A_1$, then we have, by Lemmas 5 and 6,

$$
\begin{aligned}
\|T^b(f)\|_{L^{q,\varphi}(w^{1-q})} &\leq \|M_\eta(T^b(f))\|_{L^{q,\varphi}(w^{1-q})} \leq C\|M^\#_{A,\eta}(T^b(f))\|_{L^{q,\varphi}(w^{1-q})} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^{q,\varphi}(w^{1-q})} \\
&= C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^{q,\varphi}(w)} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{Lip_\beta(w)} \|f\|_{L^{p,\beta,\varphi}(w)}.
\end{aligned}
$$

This completes the proof of Theorem 6. □

4 Applications

In this section we shall apply the theorems of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [13]). Given $0 \leq \theta < \pi$. Define

$$S_\theta = \{z \in C : |\arg(z)| \leq \theta\} \cup \{0\}$$

and its interior by S^0_θ. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta| \|(\eta I - L)^{-1}\| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S^0_\mu) = \{f : S^0_\theta \to C : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. Set

$$\Psi(S^0_\mu) = \left\{g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c\frac{|z|^s}{1 + |z|^{2s}}\right\}.$$

If L is of type θ and $g \in H_\infty(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta)\, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-to-one and has a dense range, then, for $f \in H_\infty(S^0_\mu)$,

$$f(L) = \left[h(L)\right]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ if

$$\left\|g(L)\right\| \leq N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S^0_\mu)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [12] and Theorems 1-6, we get the following.

Corollary *Assume that the following conditions are satisfied*:

(i) *The holomorphic semigroup* e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$ *is represented by the kernels* $a_z(x,y)$ *which satisfy, for all* $\nu > \theta$, *an upper bound*

$$\left|a_z(x,y)\right| \leq c_\nu h_{|z|}(x,y)$$

for $x, y \in R^n$, *and* $0 \leq |\arg(z)| < \pi/2 - \theta$, *where* $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ *and s is a positive, bounded and decreasing function satisfying*

$$\lim_{r\to\infty} r^{n+\epsilon}s\left(r^2\right) = 0.$$

(ii) *The operator L has a bounded holomorphic functional calculus in* $L^2(R^n)$; *that is, for all* $\nu > \theta$ *and* $g \in H_\infty(S^0_\mu)$, *the operator* $g(L)$ *satisfies*

$$\left\|g(L)(f)\right\|_{L^2} \leq c_\nu \|g\|_{L^\infty}\|f\|_{L^2}.$$

Then Theorems 1-6 *hold for the multilinear operator* $g(L)^b$ *associated to* $g(L)$ *and* b.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper, and read and approved the final manuscript.

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