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L^p Estimates for Marcinkiewicz integral operators and extrapolation

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Abstract
In this article, we establish L^p estimates for parametric Marcinkiewicz integral operators with rough kernels. These estimates and extrapolation arguments improve and extend some known results on Marcinkiewicz integrals.
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1 Introduction

Throughout this article, let $\mathbf{S}^{n-1}$, $n \geq 2$ be the unit sphere in $\mathbf{R}^n$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, we let $u' = u/|u|$ for $u \in \mathbf{R}^n \setminus \{0\}$ and p' denote the exponent conjugate to p; that is $1/p + 1/p' = 1$.

Let $K_{\Omega,h} = \Omega(u')h(|u|)|u|^{\rho-n}$, where $\rho = a + ib$ ($a, b \in \mathbf{R}$ with $a > 0$), h is a measurable function on $\mathbf{R}^+$ and Ω is a function on $\mathbf{S}^{n-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.1}$$

For a suitable mapping $\phi : \mathbf{R}^+ \to \mathbf{R}$, a measurable function h on $\mathbf{S}^{n-1}$ and an Ω satisfying (1.1), we define the Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\Omega,h,\phi}$ for $f \in \mathcal{S}(\mathbf{R}^n)$ by

$$\mathcal{M}^{\rho}_{\Omega,\phi,h} f(x) = \left(\int_0^\infty \left| t^{-\rho} \int_{|u|\leq t} f\bigl(x - \phi(|u|)u'\bigr) K_{\Omega,h}\, du \right|^2 \frac{dt}{t} \right)^{1/2}.$$

If $\phi(t) = t$, we denote $\mathcal{M}^{\rho}_{\Omega,\phi,h}$ by $\mathcal{M}^{\rho}_{\Omega,h}$. The operators $\mathcal{M}^{\rho}_{\Omega,\phi,h}$ have their roots in the classical Marcinkiewicz integral operators $\mathcal{M}^{1}_{\Omega,1}$ which were introduced by Stein in [1] in which he studied the L^p Boundedness of $\mathcal{M}_{\Omega,1}$ when $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ ($0 < \alpha \leq 1$). More precisely, he proved that $\mathcal{M}_{\Omega,1}$ is of type (p,p) for $1 < p \leq 2$ and of weak type $(1,1)$.

The Marcinkiewicz integral operators play an important role in many fields in mathematics such as Poisson integrals, singular integrals and singular Radon transforms. They have received much attention from many authors (we refer the readers to [1–6], as well as [7], and the references therein).

Before introducing our results, let us recall the definition of the space $L(\log L)^\alpha(\mathbf{S}^{n-1})$ and the definition of the block space $B_q^{(0,\nu)}(\mathbf{S}^{n-1})$, which are related to our work. For $\alpha > 0$, let $L(\log L)^{\alpha}(\mathbf{S}^{n-1})$ denote the class of all measurable functions Ω on $\mathbf{S}^{n-1}$ that satisfy

$$\|\Omega\|_{L(\log L)^{\alpha}(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(x)| \log^{\alpha}\bigl(2 + |\Omega(x)|\bigr)\, d\sigma(x) < \infty.$$

The special class of block spaces $B_q^{(0,\nu)}(\mathbf{S}^{n-1})$ (for $\nu > -1$ and $q > 1$) was introduced by Jiang and Lu in the study of the singular integral operators (see [8]), and it is defined as follows: A q-block on $\mathbf{S}^{n-1}$ is an L^q function $b(x)$ that satisfies (i) $\operatorname{supp}(b) \subseteq I$, (ii) $\|b\|_{L^q(\mathbf{S}^{n-1})} \le |I|^{-1/q'}$, where $|I| = \sigma(I)$ and $I = B(x_0', \delta) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \delta\}$ is a cap on $\mathbf{S}^{n-1}$ for some $x_0' \in \mathbf{S}^{n-1}$ and $\delta \in (0,1]$. The block space $B_q^{(0,\nu)}(\mathbf{S}^{n-1})$ is defined by

$$B_q^{(0,\nu)}(\mathbf{S}^{n-1}) = \left\{ \Omega \in L^1(\mathbf{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} C_\mu b_\mu \text{ with } M_q^{(0,\nu)}(\{C_\mu\}) < \infty \right\},$$

where each C_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$, and

$$M_q^{(0,\nu)}(\{C_\mu\}) = \sum_{\mu=1}^{\infty} |C_\mu| \bigl(1 + \log^{(\nu+1)}(|I_\mu|^{-1})\bigr).$$

Define $\|\Omega\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1})} = \inf\{M_q^{(0,\nu)}(\{C_\mu\}) : \Omega = \sum_{\mu=1}^{\infty} C_\mu b_\mu\}$, where the infimum is taken over the whole q-block decomposition of Ω, then $\|\cdot\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\nu)}(\mathbf{S}^{n-1})$, and the space $(B_q^{(0,\nu)}(\mathbf{S}^{n-1}), \|\cdot\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1})})$ is a Banach space.

Employing the ideas of [9], Wu [10] pointed out that for $q > 1$ and for $\nu_2 > \nu_1 > -1$,

$$\bigcup_{r>1} L^r(\mathbf{S}^{n-1}) \subset B_q^{(0,\nu_2)}(\mathbf{S}^{n-1}) \subset B_q^{(0,\nu_1)}(\mathbf{S}^{n-1}).$$

The study of parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega,h}^{\rho}$ was initiated by Hörmander in [11] in which he showed that $\mathcal{M}_{\Omega,1}^{\rho}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $\rho > 0$ and $\Omega \in \operatorname{Lip}_\alpha(\mathbf{S}^{n-1})$ with $\alpha > 0$. However, the authors of [12] proved that $\mathcal{M}_{\Omega,1}^{\rho}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $\operatorname{Re}(\rho) > 0$ and $\Omega \in \operatorname{Lip}_\alpha(\mathbf{S}^{n-1})$ with $0 < \alpha \le 1$. This result was improved in [13] in which the authors established that $\mathcal{M}_{\Omega,h}^{\rho}$ is bounded on $L^2(\mathbf{R}^n)$ if $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$, where $\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ is the collection of all measurable functions $h : [0,\infty) \to \mathbf{C}$ satisfying $\|h\|_{\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})} = \sup_{R\in\mathbf{Z}} (\int_0^R |h(t)|^\gamma \frac{dt}{t})^{1/\gamma} < \infty$.

On the other hand, Al-Qassem and Al-Salman in [2] found that if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$, then $\mathcal{M}_{\Omega,1}^{1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Furthermore, they proved that $\nu = -1/2$ is sharp on $L^2(\mathbf{R}^n)$.

Walsh in [7] found that $\mathcal{M}_{\Omega,1}^{1}$ is bounded on $L^2(\mathbf{R}^n)$ if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, and the exponent 1/2 is the best possible. However, under the same conditions, Al-Salman *et al.* in [4] improved this result for any $1 < p < \infty$.

Recently, it was proved in [14] that if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \le 2$, then $\mathcal{M}_{\Omega,\phi,h}^{1}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, where ϕ is $C^2([0,\infty))$, a convex and increasing function with $\phi(0) = 0$. Very recently, Al-Qassem and Pan established in [15] that if $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1,2]$ and $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \le 2$, then $\mathcal{M}_{\Omega,\mathcal{P},h}^{\rho}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, where $\mathcal{P}(x) = (P_1(x), P_2(x), \ldots, P_m(x))$ is a polynomial mapping and each P_i is a real valued polynomial on $\mathbf{R}^n$.

Our main concern in this work is in dealing with Marcinkiewicz operators under very weak conditions on the singular kernels. In fact, we establish certain estimates for $\mathcal{M}^{\rho}_{\Omega,\phi,h}$, and then we apply an extrapolation argument to obtain and improve some results on Marcinkiewicz integrals. Our approach in this work provides an alternative way in dealing with such kind of operators. Our main result is described in the following theorem.

Theorem 1.1 *Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$, $h \in L^\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$. Suppose that ϕ is $C^2([0,\infty))$, a convex and increasing function with $\phi(0) = 0$. Then for any $f \in L^p(\mathbf{R}^m)$ with p satisfying $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$, there exists a constant C_p (independent of Ω, h, γ, and q) such that*

$$\|\mathcal{M}^{\rho}_{\Omega,\phi,h}f\|_{L^p(\mathbf{R}^m)} \le C_p A(\gamma)(q-1)^{-1/2}\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{L^p(\mathbf{R}^m)},$$

where $A(\gamma) = \begin{cases} \gamma^{1/2} & \text{if } \gamma > 2, \\ (\gamma-1)^{-1/2} & \text{if } 1 < \gamma \le 2. \end{cases}$

Throughout this paper, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2 Definitions and lemmas

In this section, we present and establish some lemmas used in the sequel. Let us start this section by introducing the following.

Definition 2.1 Let $\theta \ge 2$. For a suitable function ϕ defined on $\mathbf{R}^+$, a measurable function $h : \mathbf{R}^+ \to \mathbf{C}$ and $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$, we define the family of measures $\{\sigma_{\Omega,\phi,h,t} : t \in \mathbf{R}^+\}$ and the corresponding maximal operators $\sigma^*_{\Omega,\phi,h}$ and $M_{\Omega,\phi,h,\theta}$ on $\mathbf{R}^m$ by

$$\int_{\mathbf{R}^m} f\,d\sigma_{\Omega,\phi,h,t} = t^{-\rho}\int_{1/2t\le|u|\le t} f\big(\phi(|u|)u'\big)h(|u|)\frac{\Omega(u')}{|u|^{n-\rho}}\,du,$$

$$\sigma^*_{\Omega,\phi,h}f(x) = \sup_{t\in\mathbf{R}^+}\big||\sigma_{\Omega,\phi,h,t}| * f(x)\big|,$$

$$M_{\Omega,\phi,h,\theta}f(x) = \sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\big||\sigma_{\Omega,\phi,h,t}| * |f(x)|\big|\frac{dt}{t},$$

where $|\sigma_{\Omega,\phi,h,t}|$ is defined in the same way as $\sigma_{\Omega,\phi,h,t}$, but with replacing Ω, h by $|\Omega|$, $|h|$, respectively. We write $\|\sigma\|$ for the total variation of σ.

In order to prove Theorem 1.1, it suffices to prove the following lemmas.

Lemma 2.2 *Let $\theta \ge 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in L^\gamma(\mathbf{R}^+, \frac{ds}{s})$ for some $\gamma > 1$. Suppose that ϕ is $C^2([0,\infty))$, a convex and increasing function with $\phi(0) = 0$. Then there are constants C and α with $0 < \alpha < \frac{1}{2q'}$ such that*

$$\|\sigma_{\Omega,\phi,h,t}\| \le C; \tag{2.1}$$

$$\int_{\theta^k}^{\theta^{k+1}}\big|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\big|^2\frac{dt}{t} \le C(\ln\theta)\big|\xi\big(\phi(\theta)\big)^{k-1}\big|^{\frac{-2\alpha}{q'\gamma'}}\|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}; \tag{2.2}$$

$$\int_{\theta^k}^{\theta^{k+1}}\big|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\big|^2\frac{dt}{t} \le C(\ln\theta)\big|\xi\big(\phi(\theta)\big)^{k+1}\big|^{\frac{2\alpha}{q'\gamma'}}\|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \tag{2.3}$$

hold for all $k \in \mathbf{Z}$. The constant C is independent of k, ξ and ϕ.

Proof As $L^q(\mathbf{S}^{n-1}) \subseteq L^2(\mathbf{S}^{n-1})$ for $q \geq 2$, it is enough to prove this lemma for $1 < q \leq 2$. By Hölder's inequality, we get

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\leq \int_{\frac{1}{2}t}^{t} \left|h(s)\right| \left| \int_{\mathbf{S}^{n-1}} e^{-i\phi(s)x\cdot\xi}\,\Omega(x)\,d\sigma(x)\right| \frac{ds}{s} \\ &\leq \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})} \left(\int_{\frac{1}{2}t}^{t} \left|\int_{\mathbf{S}^{n-1}} e^{-i\phi(s)\xi\cdot x}\Omega(x)\,d\sigma(x)\right|^{\gamma'} \frac{ds}{s}\right)^{1/\gamma'}.\end{aligned}$$

Let us first consider the case $1 < \gamma \leq 2$. By a change of variable, we obtain

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\leq \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \left(\int_{\frac{1}{2}t}^{t} \left|\int_{\mathbf{S}^{n-1}} e^{-i\phi(s)\xi\cdot x}\Omega(x)\,d\sigma(x)\right|^{2} \frac{ds}{s}\right)^{1/\gamma'} \\ &\leq \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)}J(\xi,x,y)\,d\sigma(x)\,d\sigma(y)\right)^{1/\gamma'},\end{aligned}$$

where $J(\xi,x,y) = \int_{1/2}^{1} e^{-i\phi(ts)\xi\cdot(x-y)}\frac{ds}{s}$. Write $J(\xi,x,y) = \int_{1/2}^{1} Y_t'(s)\frac{ds}{s}$, where

$$Y_t(s) = \int_{1/2}^{s} e^{-i\phi(tw)\xi\cdot(x-y)}\,dw, \quad 1/2 \leq w \leq s \leq 1.$$

By the conditions on ϕ and the mean value theorem we have

$$\frac{d}{dw}\big(\phi(tw)\big) = t\phi'(tw) \geq \frac{\phi(tw)}{w} \geq \frac{\phi(t/2)}{s} \quad \text{for } 1/2 \leq w \leq s \leq 1.$$

Hence, by Van der Corput's lemma, $|Y_t(s)| \leq s|\phi(t/2)\xi|^{-1}|\xi'\cdot(x-y)|^{-1}$, and then by integration by parts, we conclude

$$\left|J(\xi,x,y)\right| \leq C\left|\phi(t/2)\xi\right|^{-1}\left|\xi'\cdot(x-y)\right|^{-1}.$$

Combining the last estimate with the trivial estimate $|J(\xi,x,y)| \leq C$, and choosing $0 < 2\alpha q' < 1$, we get

$$\left|J(\xi,x,y)\right| \leq C\left|\phi(t/2)\xi\right|^{-\alpha}\left|\xi'\cdot(x-y)\right|^{-\alpha},$$

which leads to

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\leq C\left|\phi(t/2)\xi\right|^{\frac{-\alpha}{\gamma'}} \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{(2/\gamma')} \\ &\quad \times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \left|\xi'\cdot(x-y)\right|^{-\alpha q'}\,d\sigma(x)\,d\sigma(y)\right)^{1/q'\gamma'}.\end{aligned}$$

By the assumption of ϕ, and since the last integral is finite, we obtain

$$\int_{\theta^k}^{\theta^{k+1}} \left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right|^2 \frac{dt}{t} \leq C(\ln\theta)\left|\xi\left(\phi(\theta)\right)^{k-1}\right|^{\frac{-2\alpha}{\gamma'}} \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2.$$

For the case $\gamma > 2$, we use Hölder's inequality to obtain

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\le \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\left(\int_{\frac{1}{2}t}^{t}\left|\int_{\mathbf{S}^{n-1}} e^{-i\phi(s)\xi\cdot x}\Omega(x)\,d\sigma(x)\right|^2\frac{ds}{s}\right)^{1/2}\\ &\le \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x)\overline{\Omega(y)}\right.\\ &\quad\left.\times\left(\int_{\frac{1}{2}}^{1} e^{-i\phi(st)\xi\cdot x}e^{i\phi(st)\xi\cdot y}\frac{ds}{s}\right)d\sigma(x)\,d\sigma(y)\right)^{1/2}.\end{aligned}$$

By this, Van der Corput's lemma, and the above procedure, we obtain

$$\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| \le C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\left|\xi\phi(t/2)\right|^{\frac{-\alpha}{q'\gamma'}},$$

and therefore

$$\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right|^2\frac{dt}{t} \le C(\ln\theta)\left|\xi\left(\phi(\theta)\right)^{k-1}\right|^{\frac{-2\alpha}{q'\gamma'}}\|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}.$$

The estimate in (2.3) can be proved by using the cancellation property of Ω. By a change of variable, we have

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\le \int_{\frac{1}{2}}^{1}\int_{\mathbf{S}^{n-1}}\left|e^{-i\phi(ts)\xi\cdot x}-1\right|\left|\Omega(x)\right|\left|h(st)\right|d\sigma(x)\frac{ds}{s}\\ &\le |\xi|\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\int_{\frac{1}{2}}^{1}\left|h(st)\right|\left|\phi(ts)\right|\frac{ds}{s}.\end{aligned}$$

Since $\phi(t)$ is increasing and $\frac{1}{2} < s < 1$, we obtain

$$\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| \le C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\left|\xi\phi(t)\right|,$$

which when combined with the trivial estimate $|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| \le (\ln 2)$, we derive

$$\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right|^2\frac{dt}{t} \le C(\ln\theta)\left|\xi\phi(\theta)^{k+1}\right|^{\frac{2\alpha}{q'\gamma'}}\|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}.$$

The proof is complete. □

Following a similar argument to the one used in [16, Lemma 2.7], we achieve the following lemma.

Lemma 2.3 *Suppose that ϕ is given as in Lemma* 2.2. *Let $\mathcal{M}_{\phi,y}f$ be the maximal function of f in the direction y defined by*

$$\mathcal{M}_{\phi,y}f(x) = \sup_{t\in\mathbf{R}^+}\frac{1}{t}\left|\int_{t/2}^{t} f\left(x-\phi(r)y\right)dr\right|.$$

Then there exists a constant C_p such that

$$\left\|\mathcal{M}_{\phi,y}(f)\right\|_{L^p(\mathbf{R}^m)} \le C_p\|f\|_{L^p(\mathbf{R}^m)}$$

for any $f \in L^p(\mathbf{R}^m)$ with $1 < p \le \infty$.

Proof By a change of variable, we get

$$\mathcal{M}_{\phi,y}f(x) \le \sup_{t\in\mathbf{R}^+}\left(\int_{\phi(t/2)}^{\phi(t)} \left|f(x-ry)\right| \frac{dr}{\phi^{-1}(r)\phi'(\phi^{-1}(r))}\right).$$

Since the function $\frac{1}{\phi^{-1}(r)\phi'(\phi^{-1}(r))}$ is non-negative, decreasing and its integral over $[\phi(t/2), \phi(t)]$ is equal to $\ln(2)$, then by [16, Lemma 2.6] we obtain

$$\mathcal{M}_{\phi,y}(f) \le C\mathcal{M}_y f(x),$$

where $\mathcal{M}_y f(x) = \sup_{L\in\mathbf{R}} \frac{1}{L}\int_0^L |f(x-ry)|\,dr$ is the Hardy-Littlewood maximal function of f in the direction of y. By this, and since $\mathcal{M}_y(f)$ is bounded in $L^p(\mathbf{R}^m)$ with bounded independent of y, we obtain our desired result. □

Lemma 2.4 *Let $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $h \in L^\gamma(\mathbf{R}^+, \frac{ds}{s})$ for some $\gamma > 1$. Assume that $\sigma^*_{\Omega,\phi,h}$ and ϕ are given as in Definition 2.1 and Lemma 2.2, respectively. Then for any $f \in L^p(\mathbf{R}^m)$ with $\gamma' < p \le \infty$, there exists a constant C_p (independent of Ω, h and f) such that*

$$\left\|\sigma^*_{\Omega,\phi,h}f(x)\right\|_{L^p(\mathbf{R}^m)} \le C_p\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{L^p(\mathbf{R}^m)}. \tag{2.4}$$

Proof By Hölder's inequality, we have

$$\begin{aligned}\left||\sigma_{\Omega,\phi,h}| * f(x)\right| &\le \|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1})} \\ &\quad\times \sup_{t\in\mathbf{R}^+}\left(\int_{\frac{t}{2}}^{t}\int_{\mathbf{S}^{n-1}}\left|\Omega(y)\right|\left|f\left(x-\phi(s)y\right)\right|^{\gamma'} d\sigma(y)\frac{ds}{s}\right)^{1/\gamma'}.\end{aligned}$$

Using Minkowski's inequality for integrals gives

$$\begin{aligned}\left\|\sigma^*_{\Omega,\phi,h}f(x)\right\|_{L^p(\mathbf{R}^m)} &\le C\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1})} \\ &\quad\times\left(\int_{\mathbf{S}^{n-1}}\left|\Omega(y)\right|\left(\left\|\mathcal{M}_{\phi,y}\left(|f|^{\gamma'}\right)\right\|_{L^{p/\gamma'}(\mathbf{R}^m)}\right)d\sigma(y)\right)^{1/\gamma'}.\end{aligned}$$

By using Hölder's inequality plus Lemma 2.3, we finish the proof. □

Lemma 2.5 *Let $h \in L^\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \le 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $\theta = 2^{q'\gamma'}$. Assume that $\{\sigma_{\Omega,\phi,h,t}, t \in \mathbf{R}^+\}$ and ϕ are given as in Definition 2.1 and Lemma 2.2, respectively. Then for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there is a positive constant C_p such that*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t} * g_k|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^m)} \\ &\quad\le C_p\frac{\|h\|_{L^\gamma(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}\end{aligned}$$

holds for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^m$.

Proof We employ some ideas from [2, 15], and [17]. By Schwarz's inequality, we obtain

$$\begin{aligned} |\sigma_{\Omega,\phi,h,t} * g_k|^2 \le{}& C\|h\|^{\gamma}_{L^{\gamma}(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})} \\ &\times \left(\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}\bigl|g_k\bigl(x-\phi(s)y\bigr)\bigr|^2\bigl|\Omega(y)\bigr|\bigl|h(s)\bigr|^{2-\gamma}\,d\sigma(y)\frac{ds}{s}\right). \end{aligned} \tag{2.5}$$

Let us first prove this lemma for the case $2 \le p < \frac{2\gamma}{2-\gamma}$. By duality, there is a non-negative function $\psi \in L^{(p/2)'}(\mathbf{R}^m)$ with $\|\psi\|_{L^{(p/2)'}(\mathbf{R}^m)} \le 1$ such that

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t} * g_k|^2\frac{dt}{t}\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)} = \int_{\mathbf{R}^m}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\bigl|\sigma_{\Omega,\phi,h,t} * g_k(x)\bigr|^2\frac{dt}{t}\psi(x)\,dx.$$

By this, (2.5), and a change of variable we derive

$$\begin{aligned} &\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t} * g_k|^2\frac{dt}{t}\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)} \\ &\quad\le C\|h\|^{\gamma}_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\int_{\mathbf{R}^m}\left(\sum_{k\in\mathbf{Z}}\bigl|g_k(x)\bigr|^2\right)M_{\Omega,\phi,|h|^{2-\gamma},\theta}\psi(-x)\,dx. \end{aligned}$$

Since $h \in L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})$, then $|h(\cdot)|^{2-\gamma} \in L^{\gamma/(2-\gamma)}(\mathbf{R}^+,\frac{dt}{t})$, and since $(\frac{p}{2})' > (\frac{\gamma}{2-\gamma})'$, then by Lemma 2.4, Hölder's inequality, and the same arguments that Stein and Wainger used in [18], we obtain

$$\begin{aligned} &\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t} * g_k|^2\frac{dt}{t}\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)} \\ &\quad\le C\|h\|^{\gamma}_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}\bigl\|M_{\Omega,\phi,|h|^{2-\gamma},\theta}\psi(-x)\bigr\|_{L^{(p/2)'}(\mathbf{R}^m)} \\ &\quad\le C\ln(\theta)\|h\|^{\gamma}_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}\bigl\|\sigma^*{}_{\Omega,\phi,|h|^{2-\gamma},\theta}\psi(-x)\bigr\|_{L^{(p/2)'}(\mathbf{R}^m)} \\ &\quad\le C_p\frac{\|h\|^2_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}}{(q-1)(\gamma-1)}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}. \end{aligned}$$

For the case $\frac{2\gamma}{3\gamma-2} < p < 2$, by the duality, there are functions $\zeta = \zeta_k(x,t)$ defined on $\mathbf{R}^m \times \mathbf{R}^+$ with $\|\|\|\zeta_k\|_{L^2([\theta^k,\theta^{k+1}],\frac{dt}{t})}\|_{l^2}\|_{L^{p'}(\mathbf{R}^m)} \le 1$ such that

$$\begin{aligned} &\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t} * g_k|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^m)} \\ &\quad\le C_p\frac{\|(\Upsilon(\zeta))^{1/2}\|_{L^{p'}(\mathbf{R}^m)}}{[(q-1)(\gamma-1)]^{1/2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}, \end{aligned} \tag{2.6}$$

where

$$\Upsilon(\zeta) = \sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\bigl|\sigma_{\Omega,\phi,h,t} * \zeta_k(x,t)\bigr|^2\frac{dt}{t}.$$

As $\frac{p'}{2} > 1$, we obtain, by applying the above procedure,

$$\begin{aligned}\left\|\Upsilon(\zeta)\right\|_{L^{(p'/2)}(\mathbf{R}^m)} &\le C\|h\|^{\gamma}_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\zeta_k(\cdot,t)\right|^2\frac{dt}{t}\right)\right\|_{L^{(p'/2)}(\mathbf{R}^m)} \\ &\quad\times\left\|{\sigma^*}_{\Omega,\phi,|h|^{2-\gamma},\theta}(\varsigma)\right\|_{L^{(p'/2)'}(\mathbf{R}^m)} \\ &\le C\|h\|^2_{L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})},\end{aligned} \tag{2.7}$$

where ς is a function in $L^{(p'/2)'}(\mathbf{R}^m)$ with $\|\varsigma\|_{L^{(p'/2)'}(\mathbf{R}^m)} \le 1$. Thus, by (2.6) and (2.7), our estimate holds for $\frac{2\gamma}{3\gamma-2} \le p < 2$; and therefore the proof of Lemma 2.5 is complete. □

In the same manner, we prove the following lemma.

Lemma 2.6 *Let $h \in L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})$ for some $\gamma \ge 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $\theta = 2^{q'\gamma'}$. Assume that $\{\sigma_{\Omega,\phi,h,t}, t \in \mathbf{R}^+\}$ and ϕ are given as in Definition 2.1 and Lemma 2.2, respectively. Then for any p satisfying $1 < p < \infty$, there exists a constant C_p such that*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{\Omega,\phi,h,t}*g_k|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^m)} \\ &\quad\le C_p\frac{\gamma^{1/2}\|h\|_{L^{\gamma}(\mathbf{R}^+,\frac{ds}{s})}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{(q-1)^{1/2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^m)}\end{aligned}$$

holds for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^m$.

3 Proof of the main result

We prove Theorem 1.1 by applying the same approaches that Al-Qassem and Al-Salman [2] as well as Fan and Pan [17] used. Let us first assume that $h \in L^{\gamma}(\mathbf{R}^+,\frac{dt}{t})$ for some $1 < \gamma \le 2$; and ϕ is $C^2([0,\infty))$, a convex and increasing function with $\phi(0) = 0$. By Minkowski's inequality, we get

$$\begin{aligned}\mathcal{M}^{\rho}_{\Omega,\phi,h}f(x) &= \left(\int_0^{\infty}\left|\sum_{k=0}^{\infty}t^{-\rho}\int_{2^{-k-1}t<|u|\le 2^{-k}t}f\big(x-\phi(|u|)u'\big)K_{\Omega,h}\,du\right|^2\frac{dt}{t}\right)^{1/2} \\ &\le\sum_{k=0}^{\infty}\left(\int_0^{\infty}\left|t^{-\rho}\int_{2^{-k-1}t<|u|\le 2^{-k}t}f\big(x-\phi(|u|)u'\big)K_{\Omega,h}\,du\right|^2\frac{dt}{t}\right)^{1/2} \\ &= \frac{2^a}{2^a-1}\left(\int_0^{\infty}\left|\sigma_{\Omega,\phi,h,t}*f(x)\right|^2\frac{dt}{t}\right)^{1/2}.\end{aligned} \tag{3.1}$$

Take $\theta = 2^{q'\gamma'}$; and for $k \in \mathbf{Z}$, let $\{\Lambda_{k,\theta}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $\mathcal{I}_{k,\theta} = [\theta^{-k-1},\theta^{-k+1}]$. More precisely, we require the following:

$$\begin{aligned}&\Lambda_{k,\theta} \in C^{\infty}, \qquad 0 \le \Lambda_{k,\theta} \le 1, \qquad \sum_k\Lambda_{k,\theta}(t) = 1, \\ &\operatorname{supp}\Lambda_{k,\theta} \subseteq \mathcal{I}_{k,\theta} \quad\text{and}\quad \left|\frac{d^s\Lambda_{k,\theta}(t)}{dt^s}\right| \le \frac{C_s}{t^s},\end{aligned}$$

where C_s is independent of θ. Let $\widehat{\Psi_{k,\theta}}(\xi) = \Lambda_{k,\theta}(|\xi|)$. Decompose $\sigma_{\Omega,\phi,h,t} * f(x) = \sum_{j\in\mathbf{Z}} Y_{\Omega,\phi,h,j,\theta}(x,t)$, where

$$Y_{\Omega,\phi,h,j,\theta}(x,t) = \sum_{k\in\mathbf{Z}} \sigma_{\Omega,\phi,h,t} * \Psi_{k+j,\theta} f(x) \chi_{[\theta^k,\theta^{k+1})}(t).$$

Define $S_{\Omega,\phi,h,j,\theta} f(x) = (\int_0^\infty |Y_{\Omega,\phi,h,j,\theta}(x,t)|^2 \frac{dt}{t})^{1/2}$. Then for any $f \in \mathcal{S}(\mathbf{R}^m)$,

$$\mathcal{M}^\rho_{\Omega,\phi,h} f(x) \le \frac{2^a}{2^a - 1} \sum_{j\in\mathbf{Z}} S_{\Omega,\phi,h,j,\theta}(f). \tag{3.2}$$

Let us first compute the L^2-norm of $S_{\Omega,\phi,h,\theta,j}(f)$. By using Plancherel's theorem and Lemma 2.2, we obtain

$$\begin{aligned}\|S_{\Omega,\phi,h,\theta,j}(f)\|^2_{L^2(\mathbf{R}^m)} &\le \sum_{k\in\mathbf{Z}} \int_{\Gamma_{k+j,\theta}} \left(\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,\phi,h,\theta,t}(\xi)|^2 \frac{dt}{t}\right) |\hat{f}(\xi)|^2 \, d\xi \\ &\le C_p (\ln\theta) \|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} 2^{-\alpha|j|} \sum_{k\in\mathbf{Z}} \int_{\Gamma_{k+j,\theta}} |\hat{f}(\xi)|^2 \, d\xi \\ &\le C_p (\ln\theta) \|h\|^2_{L^\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} 2^{-\alpha|j|} \|f\|^2_{L^2(\mathbf{R}^m)},\end{aligned}$$

where $\Gamma_{k,\theta} = \{\xi \in \mathbf{R}^m : |\xi| \in \mathcal{I}_{k,\theta}\}$. Thus,

$$\|S_{\Omega,\phi,h,\theta,j}(f)\|_{L^2(\mathbf{R}^m)} \le C_p \frac{\|h\|_{L^\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}} 2^{\frac{-\alpha|j|}{2}} \|f\|_{L^2(\mathbf{R}^m)}. \tag{3.3}$$

Applying the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [19, p.96], plus using Lemma 2.5, we see that

$$\|S_{\Omega,\phi,h,\theta,j}(f)\|_{L^p(\mathbf{R}^m)} \le C_p \frac{\|h\|_{L^\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}} \|f\|_{L^p(\mathbf{R}^m)} \tag{3.4}$$

holds for $|1/p - 1/2| < 1/\gamma'$. By interpolation between (3.3) and (3.4) we obtain

$$\|S_{\Omega,\phi,h,\theta,j}(f)\|_{L^p(\mathbf{R}^m)} \le C_p 2^{\frac{-\alpha|j|}{2}} \frac{\|h\|_{L^\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}} \|f\|_{L^p(\mathbf{R}^m)}. \tag{3.5}$$

Consequently, by (3.2) and (3.5), we get our result for the case $h \in L^\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \le 2$.

The proof of our theorem for the case $h \in L^\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma \ge 2$ is obtained by following the above argument, except that we need to invoke Lemma 2.6 instead of Lemma 2.5. Therefore, the proof of Theorem 1.1 is complete.

4 Further results

The power of our theorem is in applying the extrapolation method on it (see [16]). In particular, Theorem 1.1 and extrapolation lead to the following theorem.

Theorem 4.1 *Suppose that* $h \in L^\gamma(\mathbf{R}^+, \frac{dt}{t})$ *for some* $\gamma > 1$ *and* Ω *satisfies* (1.1). *Let* ϕ *be* $C^2([0,\infty))$, *a convex and increasing function with* $\phi(0) = 0$.

(i) *If* $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\| \mathcal{M}^{\rho}_{\Omega,\phi,h} f \right\|_{L^p(\mathbf{R}^m)} \le C_p A(\gamma) \|h\|_{L^\gamma(\mathbf{R}^+, \frac{dt}{t})} \|f\|_{L^p(\mathbf{R}^m)} \left(1 + \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}\right)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

(ii) *If* $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, *then*

$$\left\| \mathcal{M}^{\rho}_{\Omega,\phi,h} f \right\|_{L^p(\mathbf{R}^m)} \le C_p A(\gamma) \|h\|_{L^\gamma(\mathbf{R}^+, \frac{dt}{t})} \|f\|_{L^p(\mathbf{R}^m)} \left(1 + \|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}\right)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, *where* $A(\gamma) = \begin{cases} \gamma^{1/2} & \text{if } \gamma > 2, \\ (\gamma - 1)^{-1/2} & \text{if } 1 < \gamma \le 2. \end{cases}$

We point out that the L^p boundedness of $\mathcal{M}^1_{\Omega,\phi,h}$ was obtained in [14] if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$, and the L^p boundedness ($1 < p < \infty$) of $\mathcal{M}^1_{\Omega,1}$ was investigated in [4] if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$.

Competing interests
The author declares that they have no competing interests.

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