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The mean ergodic theorem for nonexpansive mappings in multi-Banach spaces

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Abstract

In this paper, we prove a mean ergodic theorem for nonexpansive mappings in multi-Banach spaces.

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1 Introduction

Let X be a Banach space and C be a closed convex subset of X. For each $j \geq 1$, a mapping $T_j : C \to C$ is said to be *nonexpansive* on C if

$$\|T_j x - T_j y\| \leq \|x - y\|$$

for all $x, y \in C$. For each $j \geq 1$, let $F(T_j)$ be the set of fixed points of T_j. If X is a strictly convex Banach space, then $F(T_j)$ is closed and convex.

In [1], Baillon proved the first nonlinear ergodic theorem such that, if X is a real Hilbert space and $F(T_j) \neq \emptyset$ for each $j \geq 1$, then, for each $x \in C$, the sequence $\{S_{n,j}x\}$ defined by

$$S_{n,j}x = \frac{1}{n}\left(x + T_j x + \cdots + T_j^{n-1}x\right)$$

converges weakly to a fixed point of T_j. It was also shown by Pazy [2] that, if X is a real Hilbert space and $S_{n,j}x$ converges weakly to $y \in C$, then $y \in F(T)$. These results were extended by Baillon [3], Bruck [4] and Reich [5, 6] and [7].

2 Multi-Banach spaces

The notion of a multi-normed space was introduced by Dales and Polyakov in [8]. This concept is somewhat similar to an operator sequence space and has some connections with the operator spaces and Banach lattices. Observations on multi-normed spaces and examples are given in [8–10].

Let $(E, \|\cdot\|)$ be a complex normed space and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinate-wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1 A multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$ with $k \geq 2$ satisfying the following conditions:

(A1) $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$ $(\sigma \in \Sigma_k, x_1, \ldots, x_k \in E)$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|)\|(x_1, \ldots, x_k)\|_k$ $(\alpha_1, \ldots, \alpha_k \in \mathbb{C}, x_1, \ldots, x_k \in E)$;

(A3) $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ $(x_1, \ldots, x_{k-1} \in E)$;

(A4) $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ $(x_1, \ldots, x_{k-1} \in E)$.

In this case, we say that $\{(E^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ is a *multi-normed space*.

Lemma 2.2 ([10]) *Suppose that* $\{(E^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ *is a multi-normed space and take* $k \in \mathbb{N}$. *Then we have the following*:

(1) $\|(x, \ldots, x)\|_k = \|x\|$ $(x \in E)$;

(2) $\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|x_1, \ldots, x_k\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|$ $(x_1, \ldots, x_k \in E)$.

It follows from (2) that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case $\{(E^k, \|\cdot\|_k)\}_{k\in\mathbb{N}}$ is a multi-Banach space.

Now, we give two important examples of multi-norms for an arbitrary normed space E [8].

Example 2.3 The sequence $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ on $\{E^k : k \in \mathbb{N}\}$ defined by

$$\left\|(x_1, \ldots, x_k)\right\|_k := \max_{i\in\mathbb{N}_k} \|x_i\| \quad (x_1, \ldots, x_k \in E)$$

is a multi-norm, which is called the *minimum multi-norm*. The terminology 'minimum' is justified by the property (2).

Example 2.4 Let $\{(\|\cdot\|_k^\alpha : k \in \mathbb{N}) : \alpha \in A\}$ be the (nonempty) family of all multi-norms on $\{E^k : k \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, set

$$\left\|(x_1, \ldots, x_k)\right\|_k := \sup_{\alpha\in A} \left\|(x_1, \ldots, x_k)\right\|_k^\alpha \quad (x_1, \ldots, x_k \in E).$$

Then $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ is a multi-norm on $\{E^k : k \in \mathbb{N}\}$, which is called the *maximum multi-norm*.

We need the following observation, which can easily be deduced from the triangle inequality for the norm $\|\cdot\|_k$ and the property (2) of multi-norms.

Lemma 2.5 *Suppose that* $k \in \mathbb{N}$ *and* $(x_1, \ldots, x_k) \in E^k$. *For each* $j \in \{1, \ldots, k\}$, *let* $\{x_n^j\}_{n\geq 1}$ *be a sequence in* E *such that* $\lim_{n\to\infty} x_n^j = x_j$. *Then, for each* $(y_1, \ldots, y_k) \in E^k$, *we have*

$$\lim_{n\to\infty} \left(x_n^1 - y_1, \ldots, x_n^k - y_k\right) = (x_1 - y_1, \ldots, x_k - y_k).$$

Definition 2.6 Let $\{(E^k, \|\cdots\|_k)\}_{k\in\mathbb{N}}$ be a multi-normed space. A sequence $\{x_n\}_{n\geq 1}$ in E is called a *multi-null* sequence if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \left\|(x_n, \ldots, x_{n+k-1})\right\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in E$. We say that the sequence $\{x_n\}_{n\geq 1}$ is *multi-convergent* to a point $x \in E$ and write

$$\lim_{n\to\infty} x_n = x$$

if $\{x_n - x\}_n$ is a multi-null sequence.

3 Main results

To prove the main results in this paper, first, we introduce some lemmas.

Lemma 3.1 ([11]) *Let $\{(X^j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ be a uniformly convex multi-Banach space with modulus of the convexity δ. Let $x_j, y_j \in X$. If $\|(x_1,\dots,x_j)\|_j \leq r$, $\|(y_1,\dots,y_j)\|_j \leq r$, $r \leq R$ and $\|(x_1 - y_1,\dots,x_j - y_j)\|_j \geq \epsilon > 0$, then*

$$\big\|\big(\lambda x_1 + (1-\lambda)y_1,\dots,\lambda x_j + (1-\lambda)y_j\big)\big\|_j \leq r\big(1 - 2\lambda(1-\lambda)\delta_R(\epsilon)\big)$$

for all $\lambda \in [0,1]$, where $\delta_R(\epsilon) = \delta(\frac{\epsilon}{R})$.

To proceed, let $\{(X^j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ denote a uniformly convex multi-Banach space with modulus of the convexity δ.

Lemma 3.2 *Let C be a closed convex subset of X and for each $j \geq 1$, $T_j : C \to C$ be a nonexpansive mapping. Let $x \in C$, $f_j \in F(T_j)$ for each $j \geq 1$ and $0 < \alpha \leq \beta < 1$. Then, for any $\epsilon > 0$, there exists $N > 0$ such that, for all $n \geq N$,*

$$\begin{aligned}&\big\|\big(T_1^k\big(\lambda T_1^n x + (1-\lambda)f_1\big) - \big(\lambda T_1^{n+k}x + (1-\lambda)f_1\big),\\&\qquad \dots, T_j^k\big(\lambda T_j^n x + (1-\lambda)f_j\big) - \big(\lambda T_j^{n+k}x + (1-\lambda)f_j\big)\big)\big\|_j\\&\quad < \epsilon\end{aligned}$$

for all $k > 0$ and $\lambda \in [\alpha, \beta]$.

Proof Put

$$\begin{aligned}&r = \lim_n \big\|\big(T_1^n x - f_1,\dots,T_j^n x - f_j\big)\big\|_j, \qquad R = \big\|(x - f_1,\dots,x - f_j)\big\|_j,\\&c = \min\big\{2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta\big\}.\end{aligned}$$

For given $\epsilon > 0$, choose $d > 0$ such that $\frac{r}{r+d} > 1 - c\delta_R(\epsilon)$. Then there exists $N > 0$ such that, for all $n \geq N$,

$$\big\|\big(T_1^n x - f_1,\dots,T_j^n x - f_j\big)\big\|_j < r + d.$$

For each $n \geq N$, $k > 0$ and $\alpha \leq \lambda \leq \beta$, we put

$$u_j = (1-\lambda)\big(T_j^k z - f_j\big), \qquad v_j = \lambda\big(T_j^{n+k}x - T_j^k z\big),$$

where $z_j = \lambda T_j^n x + (1-\lambda)f_j$. Then we have

$$\big\|(u_1,\dots,u_j)\big\|_j \leq \lambda(1-\lambda)\big\|\big(T_1^n x - f_1,\dots,T_j^n x - f_j\big)\big\|_j$$

and

$$\|(v_1,\ldots,v_j)\|_j \le \lambda(1-\lambda)\|(T_1^n x - f_1,\ldots,T_j^n x - f_j)\|_j.$$

Suppose that

$$\begin{aligned}
&\|(u_1 - v_1,\ldots,u_j - v_j)\|_j \\
&\quad = \|(T_1^k z - (\lambda T_1^{n+k}x + (1-\lambda)f_1),\ldots,T_1^k z - (\lambda T_j^{n+k}x + (1-\lambda)f_j))\|_j \\
&\quad \ge \epsilon.
\end{aligned}$$

Then, by Lemma 3.1, we have

$$\begin{aligned}
&\|(\lambda u_1 + (1-\lambda)v_1,\ldots,\lambda u_j + (1-\lambda)v_j)\|_j \\
&\quad = \lambda(1-\lambda)\|(T_1^{n+k}x - f_1,\ldots,T_j^{n+k}x - f_j)\|_j \\
&\quad \le \lambda(1-\lambda)\|(T_1^n x - f_1,\ldots,T_j^n x - f_j)\|_j\big(1 - 2\lambda(1-\lambda)\delta_R(\epsilon)\big) \\
&\quad \le \lambda(1-\lambda)\|(T_1^n x - f_1,\ldots,T_j^n x - f_j)\|_j\big(1 - c\delta_R(\epsilon)\big).
\end{aligned}$$

Hence we have

$$(r+d)\big(1 - c\delta_R(\epsilon)\big) < r \le (r+d)\big(1 - C\delta_R(\epsilon)\big),$$

which is a contradiction. This completes the proof. □

Lemma 3.3 (Browder [12]) *Let C be a closed convex subset of X and $T_j : C \to C$ be a nonexpansive mapping. If $\{u_i\}$ is a weakly convergent sequence in C with the weak limit u_0 and $\lim_i \|u_i - T_j u_i\| = 0$, then u_0 is a fixed point of T_j.*

Lemma 3.4 *Let C be a closed convex subset of X and, for each $j \ge 1$, $T_j : C \to C$ be a nonexpansive mapping. Then, for all $x \in C$ and $n > 0$,*

$$\lim_{i\to\infty}\sup_{j\to\infty}\|(T_1^k S_{n,1} T_1^i x - S_{n,1} T_1^k T_1^i x,\ldots,T_j^k S_{n,j} T_j^i x - S_{n,j} T_j^k T_j^i x)\|_j = 0 \tag{1}$$

uniformly for each $k \ge 1$.

Proof By induction on n, we prove this lemma. First, we prove the conclusion in the case $n = 2$. Put

$$\begin{aligned}
&r = \lim_{n\to\infty}\sup_{j\ge 1}\|(T_1^{n+1}x - T_1^n x,\ldots,T_j^{n+1}x - T_j^n x)\|_j, \\
&R = \|(x - T_1 x,\ldots,x - T_j x)\|_j, \qquad x_{i,j} = T_j^i x
\end{aligned}$$

for each $i \ge 1$.

If $r \neq 0$, then, for any $\epsilon > 0$, choose $c > 0$ such that $\frac{r}{r+c} > 1 - \delta_R(\epsilon)/2$. Then there exists $N > 0$ such that, for all $i \geq N$,

$$\left\| \left(T_1^k x_{i,1} - T_1^{k+1} x_{i,1}, \ldots, T_j^k x_{i,j} - T_j^{k+1} x_{i,j} \right) \right\|_j \leq r + c$$

for each $k \geq 1$. If we put

$$u_j = \frac{1}{2}\left(T_j^k z - T_j^k x_{i,j} \right), \qquad v_j = \frac{1}{2}\left(T_j^{k+1} x_{i,j} - T_j^k z_j \right),$$

where $i \geq N$, $k > 0$ and $z_j = \frac{1}{2}(x_{i,j} + T_j x_{i,j})$, then we have

$$\begin{aligned}
\left\| (u_1, \ldots, u_j) \right\|_j &\leq \frac{1}{2} \left\| (z_1 - x_{i,1} - z_j - x_{i,j}) \right\|_j \\
&= \frac{1}{4} \left\| (T_1 x_{i,1} - x_{i,1}, \ldots, T_j x_{i,j} - x_{i,j}) \right\|_j \\
&\leq \frac{1}{4}(r + c).
\end{aligned}$$

Similarly, we have $\|(v_1, \ldots, v_j)\|_j \leq \frac{1}{4}(r + c)$. Suppose that

$$\begin{aligned}
&\left\| (u_1 - v_1, \ldots, u_j - v_j) \right\|_j \\
&\quad = \left\| \left(T_1^k z_1 - \frac{1}{2}\left(T_1^{k+1} x_{i,1} + T_1^k x_{i,1} \right), \ldots, T_j^k z_j - \frac{1}{2}\left(T_j^{k+1} x_{i,j} + T_j^k x_{i,j} \right) \right) \right\|_j \\
&\quad \geq \epsilon.
\end{aligned}$$

Then, by Lemma 3.1, we have

$$\begin{aligned}
\left\| \frac{1}{2}(u_1 + v_1, \ldots, u_j + v_j) \right\|_j &= \frac{1}{4} \left\| \left(T_1^{k+1} x_{i,1} - T_1^k x_{i,1}, \ldots, T_j^{k+1} x_{i,j} - T_j^k x_{i,j} \right) \right\|_j \\
&\leq \frac{1}{4}(r + c)\left(1 - \frac{1}{2}\delta_R(\epsilon) \right),
\end{aligned}$$

which contradicts $r > (r + c)(1 - \frac{1}{2}\delta_R(\epsilon))$.

If $r = 0$, then, for any $\epsilon > 0$, choose $i > 0$ so large that $\sup_j \|(u_1, \ldots, u_j)\|_j < \frac{\epsilon}{2}$. Hence we have

$$\begin{aligned}
&\sup_{j \geq 1} \left\| \left(T_1^k z_1 - \frac{1}{2}\left(T_1^{k+1} x_{i,1} + T_1^k x_{i,1} \right), \ldots, T_j^k z_j - \frac{1}{2}\left(T_j^{k+1} x_{i,j} + T_j^k x_{i,j} \right) \right) \right\|_j \\
&\quad = \sup_{j \geq 1} \left\| (u_1 - v_1, \ldots, u_j - v_j) \right\|_j \\
&\quad \leq \sup_{j \geq 1} \left\| (u_1, \ldots, u_j) \right\|_j + \sup_{j \geq 1} \left\| (v_1, \ldots, v_j) \right\|_j \\
&\quad < \epsilon.
\end{aligned}$$

This completes the proof of the case $n = 2$.

Now, suppose that

$$\lim_{i\to\infty}\sup_{j\geq 1}\bigl\|\bigl(T_1^kS_{n-1,1}x_{i,1}-S_{n-1,1}T_1^kx_{i,1},\dots,T_j^kS_{n-1,j}x_{i,j}-S_{n-1,j}T_j^kx_{i,j}\bigr)\bigr\|_j=0$$

uniformly for each $k\geq 1$. We claim that

$$\lim_{i\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\bigr\|_j$$

exists. Put

$$r=\liminf_{i\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\bigr\|_j.$$

For any $\epsilon>0$, choose $i>0$ such that

$$\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\bigr\|_j<r+\frac{\epsilon}{2}$$

and

$$\sup_{j\geq 1}\bigl\|\bigl(S_{n-1,1}T_1^kx_{i+1,1}-T_1^kS_{n-1,1}x_{i+1,1},\dots,S_{n-1,j}T_j^kx_{i+1,j}-T_j^kS_{n-1,j}x_{i+1,j}\bigr)\bigr\|_j<\frac{\epsilon}{2}.$$

Then we have

$$\begin{aligned}
&\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i+k,1}-x_{i+k,1},\dots,S_{n-1,j}T_jx_{i+k,j}-x_{i+k,j})\bigr\|_j\\
&\quad\leq\sup_{j\geq 1}\bigl\|\bigl(S_{n-1,1}T_1^kx_{i+1,1}-T_1^kS_{n-1,1}x_{i+1,1},\dots,S_{n-1,j}T_j^kx_{i+1,j}-T_j^kS_{n-1,j}x_{i+1,j}\bigr)\bigr\|_j\\
&\qquad+\sup_{j\geq 1}\bigl\|\bigl(T_1^kS_{n-1,1}x_{i+1,1}-T_1^kx_{i,1},\dots,T_j^kS_{n-1,j}x_{i+1,j}-T_j^kx_{i,j}\bigr)\bigr\|_j\\
&\quad<\frac{\epsilon}{2}+r+\frac{\epsilon}{2}\\
&\quad=r+\epsilon
\end{aligned}$$

for all $k\geq 1$. Therefore, we have

$$\begin{aligned}
&\limsup_{i\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\bigr\|_j\\
&\quad=\limsup_{k\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i+k,1}-x_{i+k,1},\dots,S_{n-1,j}T_jx_{i+k,j}-x_{i+k,j})\bigr\|_j\\
&\quad<r+\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, we have

$$\begin{aligned}
&\limsup_{i\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,1}-x_{i,1})\bigr\|_j\\
&\quad\leq\liminf_{i\to\infty}\sup_{j\geq 1}\bigl\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\bigr\|_j,
\end{aligned}$$

i.e., $\lim_{i\to\infty}\sup_{j\geq 1}\|(S_{n-1,1}T_1x_{i,1}-x_{i,1},\dots,S_{n-1,j}T_jx_{i,j}-x_{i,j})\|_j$ exists.

Now, we put

$$r = \lim_{i\to\infty} \sup_{j\geq 1} \bigl\| (S_{n-1,1}T_1x_{i,1} - x_{i,1}, \dots, S_{n-1,j}T_jx_{i,j} - x_{i,j}) \bigr\|_j.$$

If $r \neq 0$, then, for any ϵ, choose $c > 0$ such that

$$\frac{r-c}{r+2c} > 1 - \left(2\frac{(n-1)}{n^2}\right)\delta_{3r}(\epsilon).$$

Then there exists $N > 0$ such that, if, for all $i \geq N$, we put

$$u_j = \frac{n}{(n-1)}\bigl(T_j^kS_{n,j}x_{i,j} - T_j^kx_{i,j}\bigr), \qquad v_j = n\bigl(S_{n-1,j}T_j^kx_{i+1,j} - T_j^kS_{n,j}x_{i,j}\bigr),$$

so

$$\begin{aligned}
&\bigl\|(u_1,\dots,u_j)\bigr\|_j \leq \bigl\|(S_{n-1,1}T_1x_{i,1} - x_{i,1},\dots,S_{n-1,j}T_jx_{i,j} - x_{i,j})\bigr\|_j \leq r + c, \\
&\bigl\|(v_1,\dots,v_2)\bigr\|_j \\
&\quad \leq n\bigl\|\bigl(S_{n-1,1}T_1^kx_{i+1,1} - T_1^kS_{n-1,1}x_{i+1,1},\dots,S_{n-1,j}T_j^kx_{i+1,j} - T_j^kS_{n-1,j}x_{i+1,j}\bigr)\bigr\|_j \\
&\qquad + \bigl\|(S_{n-1,1}T_1x_{i,1} - x_{i,1},\dots,S_{n-1,j}T_jx_{i,j} - x_{i,j})\bigr\|_j \\
&\quad \leq r + 2c
\end{aligned}$$

and

$$\begin{aligned}
&\bigl\|(u_1 - v_1,\dots,u_j - v_j)\bigr\|_j \\
&\quad = \frac{n}{n-1}\bigl\|\bigl(T_1^kS_{n,1}x_{i,1} - S_{n,1}T_1^kx_{i,1},\dots,T_j^kS_{n,j}x_{i,j} - S_{n,j}T_j^kx_{i,j}\bigr)\bigr\|_j.
\end{aligned}$$

Hence, by the method in the proof of the case $n = 2$, we have

$$\sup_{j\geq 1}\bigl\|\bigl(T_1^kS_{n,1}x_{i,1} - S_{n,1}T_1^kx_{i,1},\dots,T_j^kS_{n,j}x_{i,j} - S_{n,j}T_j^kx_{i,j}\bigr)\bigr\|_j < \epsilon$$

for all $k \geq 1$ and $i \geq N$.

If $r = 0$, then, as in the proof of the case $n = 2$, there exists N' such that, for each $i \geq N'$,

$$\sup_{j\geq 1}\bigl\|(u_1,\dots,u_j)\bigr\|_j < \frac{\epsilon}{2}, \qquad \sup_{j\geq 1}\bigl\|(v_1,\dots,v_j)\bigr\|_j < \frac{\epsilon}{2}.$$

Therefore, we have

$$\sup_{j\geq 1}\bigl\|\bigl(T_1^kS_{n,1}x_{i,1} - S_{n,1}T_1^kx_{i,1},\dots,T_j^kS_{n,j}x_{i,j} - S_{n,j}T_j^kx_{i,j}\bigr)\bigr\|_j < \epsilon.$$

This completes the proof. □

Now, assume that the norm of X is Frechet differentiable and then we have the following.

Proposition 3.5 ([4, 6, 13]) *Let C be a closed convex subset of X and, for each $j \geq 1$, $T_j : C \to C$ be a nonexpansive mapping. If we put $W_j(x) = \cap_m \overline{co}\{T_j^k x : k \geq m\}$ for all $x \in C$, then $W_j(x) \cap F(T_j)$ is at most one point.*

In this paper, we give a new proof of the following theorem, which is due to Reich [6].

Theorem 3.6 *Let $\{(X^j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ be a uniformly convex multi-Banach space which has the Fréchet differentiable norm. Let C be a closed convex subset of X and, for each $j \geq 1$, $T_j : C \to C$ be a nonexpansive mapping. Then the following statements are equivalent*:

(1) $F(T_j) \neq \emptyset$.
(2) $\{T_j^n x\}$ *is bounded for all* $x \in C$.
(3) *For all $x \in C$, $\{S_n T_j^i x\}$ converges weakly to a point $(y_1, \ldots, y_j) \in C^j$ uniformly for each $i \geq 1$.*

Proof (1) $\Longleftrightarrow$ (2) is well known in [12].

(3) $\Longleftrightarrow$ (2) Suppose that, for some $x \in C$, there exists an unbounded subsequence $\{T_j^{n_i} x\}$ of $\{T_j^n x\}$. For each $j \geq 1$, since T_j is a nonexpansive mapping, it follows that, for each $m > 0$, the sequence $\{S_{m<j} T_j^{n_i} x\}$ is also unbounded, which contradicts the condition (3).

(2) $\Longleftrightarrow$ (3) Since $\{T_j^n x\}$ is bounded and

$$\begin{aligned}
&\big\|\big(T_1 S_{n,1} T_1^i x - S_{n,1} T_1^i x, \ldots, T_j S_{n,j} T_j^i x - S_{n,j} T_j^i x\big)\big\|_j \\
&\quad\leq \big\|\big(T_1 S_{n,1} T_1^i x - S_{n,1} T_1 T_1^i x, \ldots, T_j S_{n,j} T_j^i x - S_{n,j} T_j T_j^i x\big)\big\|_j \\
&\qquad + \big\|\big(S_{n,1} T_1 T_1^i x - S_{n,1} T_1^i x, \ldots, S_{n,j} T_j T_j^i x - S_{n,j} T_j^i x\big)\big\|_j \\
&\quad\leq \big\|\big(T_1 S_{n,1} T_1^i x - S_{n,1} T_1 T_1^i x, \ldots, T_j S_{n,j} T_j^i x - S_{n,j} T_j T_j^i x\big)\big\|_j \\
&\qquad + \frac{1}{n}\big\|\big(T_1^{i+1+n} x - T_1^i x, \ldots, T_j^{i+1+n} x - T_j^i x\big)\big\|_j,
\end{aligned}$$

there exists a sequence $\{S_{n,j} T_j^{i_n} x\}$ such that

$$\lim_{n\to\infty} \sup_{j\geq 1} \big\|\big(T_1 S_{n,1} T_1^{i_n} x - S_{n,1} T_1^{i_n} x, \ldots, T_j S_{n,j} T_j^{i_n} x - S_{n,j} T_j^{i_n} x\big)\big\|_j = 0.$$

Then, by Lemma 3.3 and Proposition 3.5, it follows that any weakly multi-convergent subsequence of $\{S_{n,j} T_j^{i_n} x\}$ multi-converges weakly to a point y_j, *i.e.*, $S_{n,j} T_j^{i_n} x \rightharpoonup y_j$, where $y_j = W_j(x) \cap F(T_j)$. Also, by Lemma 3.4, it follows that

$$\lim_{n\to\infty} \sup_{j\geq 1} \big\|\big(T_1 S_{n,1} T_1^{i_n+kn+i} x - S_{n,1} T_1^{i_n+kn+i} x, \ldots, T_j S_{n,j} T_j^{i_n+kn+i} x - S_{n,j} T_j^{i_n+kn+i} x\big)\big\|_j = 0$$

for all $i, k \geq 1$. Therefore, $S_{n,j} T_j^{i_n+kn} x_i \rightharpoonup y_j$ uniformly for each $k \geq 1$.

On the other hand, for each $n \geq 1$ with $m \geq i_n$, we have

$$\begin{aligned}
S_{m,j} T_j^i x &= \frac{1}{m} \sum_{k=0}^{m-1} T_j^k x_i \\
&= \frac{1}{m}\left(\sum_{k=i_n+tn}^{m-1} T_j^k x_i + n\left(\sum_{k=0}^{t} S_n T_j^{i_n+kn} x_i\right) + \sum_{k=0}^{i_n} T_j^k x_i\right),
\end{aligned}$$

where $m = tn + i_n + r$, $r < n$. Since $\{S_{n,j}T_j^{i_n+kn}x_i\}$ multi-converges to y_j uniformly for each $k \geq 1$, it follows that $\{S_{m,j}T_j^i x\}$ converges weakly to y_j uniformly for each $i \geq 1$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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